Minimal (clones with a Taylor operation)

Marcin Kozik
joint work with: L. Barto, Z. Brady, A. Bulatov and D. Zhuk

Notation and conventions

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Operations satsfying these equations are called Taylor operations.

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... and we know that, for a finite A, the definitions coincide.

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- and I will argue that MTCs are easier.

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Overview

The plan of the talk:

- Some basic properties of MTCs.

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- Absorption (Zhuk) in MTCs.
- Edges (Bulatov) in MTCs.
- Random properties of MTCs.
- Open problems.

Basic properties of MTCs

Fact

Let $\mathcal{C}$ be an MTC on A, $B \subseteq A$ and $f \in \mathcal{C}$:

- $f(B, \ldots, B) = B$,

Then B is a subuniverse of $\mathcal{C}$.

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Proof.

$$c_A(c_B(x_1, \ldots, x_p), c_B(x_2, \ldots, x_p, x_1), \ldots, c_B(x_p, x_1, \ldots, x_{p-1}))$$

is cyclic on A and preserves B. $\square$

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In fact if $\mathcal{C}$ is an MTC, then so is every member of $\mathrm{HSP}_{\mathrm{fin}}(\mathcal{C})$.

Absorption in MTCs

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Theorem (Zhuk)

Let $\mathcal{C}$ be a Taylor clone, then

1. *$\mathcal{C}$ has a 2-absorbing subuniverse, or*
2. *$\mathcal{C}$ has a 3-absorbing subuniverse, or*
3. *$\mathcal{C}/\alpha$ is abelian for some congruence α, or*
4. *$\mathcal{C}/\alpha$ is polynomially complete for some congruence α.*

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I.e."Because we do not know anything better."

2-absorption

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Let $\mathcal{C}$ be an MTC on A and $B \subseteq A$. TFAE:

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The clone $\mathcal{C}$ acts on B "as max on $\{0, 1\}$".

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Let $\mathcal{C}$ be an MTC on A and $B \subseteq A$. TFAE:

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Center = "3-absorbing subuniverse" in MTCs.

Absorption is strong in MTCs

Fact

Let $\mathcal{C}$ be a MTC and B a 2-absorbing subuniverse, then

- $x = y \vee (x \in B \wedge y \in B)$ *is a congruence of* $\mathcal{C}$;

There exist a unique, smallest 2-absorbing subuniverse of A.

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Let $\mathcal{C}$ be a MTC and B, D be 3-absorbing subuniverses, then

- $B \cup D$ *is a subuniverse;*
- *if $B \cap D = \emptyset$ then $B^2 \cup D^2$ is a congruence on $B \cup D$.*

Edges in MTCs

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A graph of an algebra is connected.

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Theorem

In an MTC a subset is 2-absorbing if and only if it is stable.

Random properties of MTCs

Fact

Let $\mathcal{C}$ be a MTC on A.

- *$\mathcal{C}$ is generated by a cyclic operation of prime arity $> |A|$.*
- *$\mathcal{C}$ is generated by a Siggers operation: $s(a, r, e, a) = s(r, a, r, e)$.*
- *$\mathcal{C}$ is generated by a ternary operation (any operation witnessing all the edges generates $\mathcal{C}$).*

Open problems

Some (embarrassing) open questions:

- Does every $\{a, b\}$ form an edge of an MTC?

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- How about TCT in MTCs?
- Can a structural characterization of MTCs be proposed?
- and extended to Taylor clones, or general algebras?

Under the hood

Some proofs are nice, for example:

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Let R be an irredundant proper relation

- *R pp-defines binary irredundant and proper, or*
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implies (with some effort)

Theorem (Zhuk)

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4. *$\mathcal{C}/\alpha$ is polynomially complete for some congruence α.*

Some content is fragile, for example:

A clone on $\{0,1,2\}$ is generated by $m(a,b,c) =$

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- m is cyclic;
- clone is simple;
- at least two majority edges;
- $\mathrm{Clo}(m)$ is not an MTC.