Relative Maltsev definability (of some commutator properties)

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PALS

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This theorem has the structure $\mathscr{P} + \Gamma = \Sigma$, i.e. the class of varieties having property $\mathscr{P}$ and satisfying the weak ground Maltsev condition Γ is definable by the Maltsev condition Σ.

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Answer for specialists. In 2013, K+Sz+W proved Park's Conjecture for varieties with a difference term. (*A finitely generated variety with a finite residual bound is finitely based.*) The proof uses properties of the TC-commutator to establish a version the Freese-McKenzie property (C1). To extend the result, it would help to understand TC-commutator arithmetic for varieties that have a Taylor term but not a difference term.

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$$\begin{aligned} \exists t \qquad & t(x,x,x,x,x,x) \approx x \quad \& \\ & t(x,y,y,y,x,x) \approx t(y,x,y,x,y,x) \approx t(y,y,x,x,x,y). \end{aligned}$$

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An *(ordinary) Maltsev condition* is a sequence $\Sigma = (\sigma_n)_{n\in\omega}$ of successively weaker strong Maltsev conditions ($\forall n(\sigma_n \vdash \sigma_{n+1})$). A variety $\mathcal{V}$ *satisfies* Σ if it satisfies σ_n for some n. Σ *defines* the class of those varieties which satisfy Σ.

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$$\langle t \mid t(xxxxxx) \approx x, t(xyyyxx) \approx t(yxyxyx) \approx t(yyxxxy)\rangle.$$

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The interpretability relation $\leq$ is a lattice order on bi-interpretability classes of varieties.

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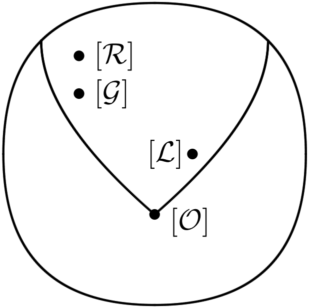

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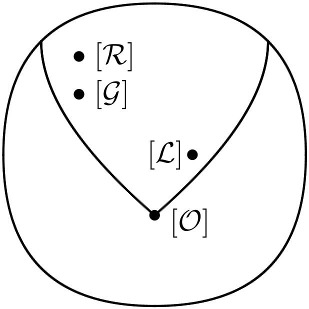

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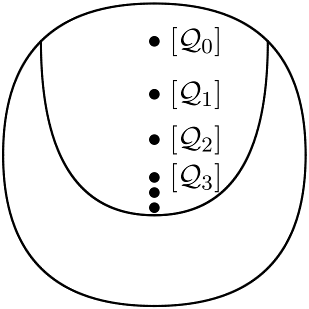

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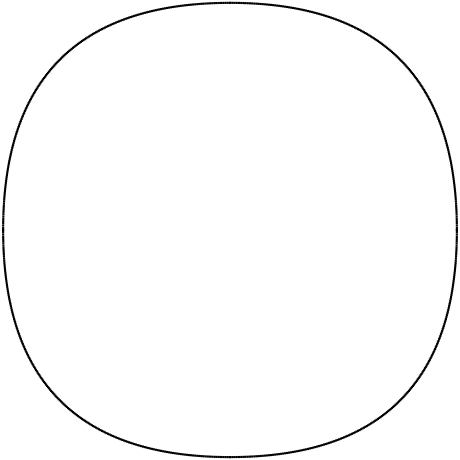

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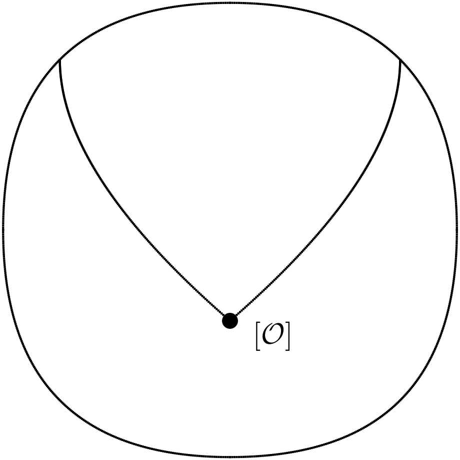

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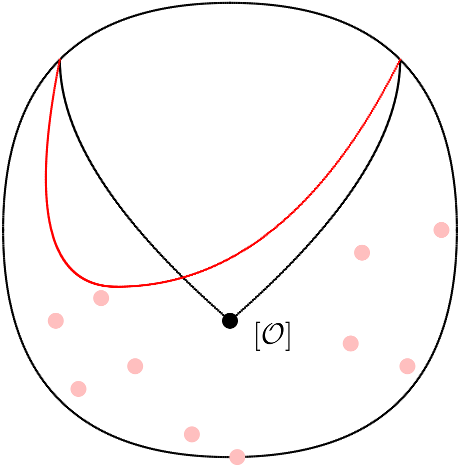

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Definition. (α, β-matrices) If $\mathbf{A}$ is an algebra and $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$, then an α, β**-matrix** is a 2×2 matrix of elements of $\mathbf{A}$ of the form

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} t(\mathbf{a}, \mathbf{u}) & t(\mathbf{a}, \mathbf{v}) \\ t(\mathbf{b}, \mathbf{u}) & t(\mathbf{b}, \mathbf{v}) \end{bmatrix}$$

where $t(\mathbf{x}, \mathbf{y})$ is an $(m+n)$-ary term operation of $\mathbf{A}$, $\mathbf{a} \; \alpha \; \mathbf{b}$, and $\mathbf{u} \; \beta \; \mathbf{v}$.

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Definition. (The centralizer) α centralizes β modulo δ if $p \equiv q \pmod{\delta}$ implies $r \equiv s \pmod{\delta}$ whenever $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ is an α, β-matrix. Write $\mathbf{C}(\alpha, \beta; \delta)$ to denote this.

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Remark. The implication

$$\mathbf{C}(\alpha, \beta; \delta) \implies [\alpha, \beta] \leq \delta$$

follows from the definition of the commutator.

The (TC-)commutator, Part 2

Definition. (The commutator) Let $[\alpha, \beta]$ be the least δ for which $\mathbf{C}(\alpha, \beta; \delta)$ holds.

Remark. The implication

$$\mathbf{C}(\alpha, \beta; \delta) \implies [\alpha, \beta] \leq \delta$$

follows from the definition of the commutator. For groups, the converse implication also holds. (The centralizer and the commutator carry the same information.)

Some properties involving the centralizer & commutator

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All properties on this page hold for the variety of groups and fail for the variety of semigroups.

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None of these commutator properties are Maltsev definable

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The variety of sets in the empty language has the property that for all $\mathbf{A} \in \mathcal{SET}$ and all $\alpha, \beta, \delta \in \mathrm{Con}(\mathbf{A})$ it is the case that $\mathbf{C}(\alpha, \beta; \delta)$ holds and $[\alpha, \beta] = 0$ holds.

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- each of the commutator properties on the preceding slide fails in some variety. (E.g. the variety of semigroups.)

Results #1

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Theorem. The following properties are equivalent for a variety $\mathcal{V}$ with a Taylor term.

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2. The commutator is left distributive:

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2. $\mathcal{V}$ has commutative commutator.

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