Jónsson Jónsson-Tarski Algebras

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Outline

Part 1: Introduction.

Plan for Introduction

We have two main concepts to introduce:

1. Jónsson Algebra
2. Jónsson-Tarski Algebra

Jónsson Algebras: Part 1

Definition

A *Jónsson algebra* is an infinite algebra J, in a countable algebraic language, which has no proper subalgebras of the same cardinality as J.

Jónsson Algebras: Part 2

Some fairly trivial examples of countable Jónsson algebras exist, but uncountable Jónsson algebras are more difficult to construct.

Examples (Countable Jónsson Algebras)

- $\langle \omega; f \rangle$ is Jónsson, where $f(n) = n - 1$ for $n \neq 0$, and $f(0) = 0$.
- The unital ring $\langle \mathbb{Z}; +, -, \cdot, 0, 1 \rangle$ is Jónsson.

Jónsson Algebras: Part 3

Originally the motivation for studying Jónsson algebras was set-theoretic, e.g.,

Which cardinalities can a Jónsson algebra have?

Some results:

(ZFC): $\aleph_0$, $\aleph_1$, $\aleph_2$, $\aleph_3, \ldots$ (P. Erdős, A. Hajnal)

(ZFC): $\aleph_{\omega+1}$ (S. Shelah)

(ZFC): Any successor of a regular cardinal. (J. Tryba, W. H. Woodin)

(ZFC + GCH): Any successor cardinal. (P. Erdős, A. Hajnal, R. Rado)

(ZFC + $V = L$): Any cardinal. (J. Keisler, F. Rowbottom)

Jónsson Algebras: Part 4

Some varieties are known to contain uncountable Jónsson algebras,
(groups, semigroups, loops, ...)
while others do not.
(semilattices, boolean algebras, ...)

In some varieties the existence of uncountable Jónsson algebras is an open question.
(rings, lattices, ...)

Some varieties have uncountable Jónsson algebras, but only in certain cardinalities.
(any unary variety: only $\aleph_1$.)

Jónsson-Tarski Algebras: Part 1

Definition

A *Jónsson-Tarski algebra* is an algebra $\langle A; \cdot, \ell, r\rangle$ with one binary operation $\cdot$ and two unary operations ℓ and r, satisfying the identities

1. $\ell(x \cdot y) = x$,
2. $r(x \cdot y) = y$, and
3. $\ell(z) \cdot r(z) = z$.

These algebras capture the situation of a bijection $A \times A \to A$:

- In every Jónsson-Tarski algebra, $\cdot$ is a bijection $A \times A \to A$,
- For every bijection $A \times A \to A$ one can define a corresponding Jónsson-Tarski algebra.

Jónsson-Tarski Algebras: Part 2

Jónsson-Tarski algebras were introduced by B. Jónsson and A. Tarski in 1961.

They were an example of a variety $\mathcal{V}$ in which $F_{\mathcal{V}}(m) \cong F_{\mathcal{V}}(n)$ for any finite m, n.

The authors proved that, if $\mathcal{V}$ contains a finite algebra with more than one element, then $F_{\mathcal{V}}(m) \ncong F_{\mathcal{V}}(n)$ when $m \neq n$.

But the variety of Jónsson-Tarski algebras does not contain a finite algebra with more than one element.

Why Jónsson Jónsson-Tarski algebras?

We now discuss the connection between Jónsson algebras and Jónsson-Tarski algebras.

Together with K. Kearnes, we briefly conjectured that there cannot be an uncountable Jónsson algebra in a minimal variety.

However, the variety of Jónsson-Tarski algebras is a minimal variety.

We managed to construct a Jónsson algebra in this variety, of cardinality $\aleph_1$. (We will show this construction soon!)

So by constructing a Jónsson Jónsson-Tarski algebra, we proved that minimal varieties can contain uncountable Jónsson algebras.

Further Questions

After constructing a Jónsson Jónsson-Tarski algebra of cardinality $\aleph_1$, we still had some unanswered questions:

Can Jónsson Jónsson-Tarski algebras exist in other cardinalities?

If not, what is the obstacle that prevents them from being larger?

How many Jónsson Jónsson-Tarski algebras are there, up to isomorphism?

We will answer all of these questions today!

Part 2: The Existence of Jónsson Jónsson-Tarski Algebras.

Constructing Jónsson-Tarski Algebras

To construct a Jónsson-Tarski algebra, it is enough to specify its multiplication table. The other operations (ℓ and r) can be deduced.

Example

$\cdot$	0	1	2	3	4	$\cdots$
0	1	2	5	9	14	$\cdots$
1	0	4	8	13	19	$\cdots$
2	3	7	12	18	25	$\cdots$
3	6	11	17	24	32	$\cdots$
4	10	16	23	31	40	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Sample calculations:

$$\ell(17) = \ell(3 \cdot 2) = 3$$
$$r(10) = r(4 \cdot 0) = 0$$

Takeaway:

$\ell(x) =$ the row in which x appears.
$r(x) =$ the col in which x appears.

A Countable Jónsson Jónsson-Tarski Algebra

Theorem (K. Kearnes, DuBeau (2020))

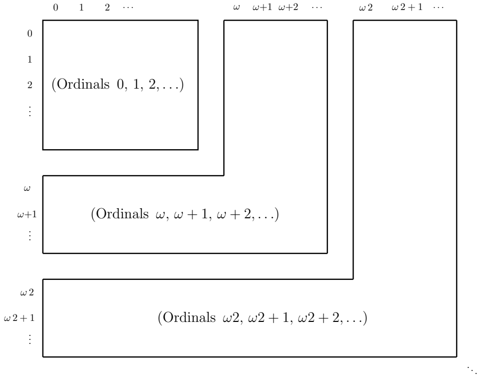
The Jónsson-Tarski algebra with the following multiplication table is Jónsson:

$\cdot$	0	1	2	3	4	$\cdots$
0	1	2	5	9	14	$\cdots$
1	0	4	8	13	19	$\cdots$
2	3	7	12	18	25	$\cdots$
3	6	11	17	24	32	$\cdots$
4	10	16	23	31	40	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

In fact it has no proper subalgebras.

Construction of Uncountable JJT: Part 1

1. Start with the countable JT algebra from the previous slide. Call this J_ω. It has universe ω.
2. Given a JT algebra J_λ with universe λ, extend to a JT algebra $J_{\lambda+\omega}$ with universe $\lambda+\omega$.

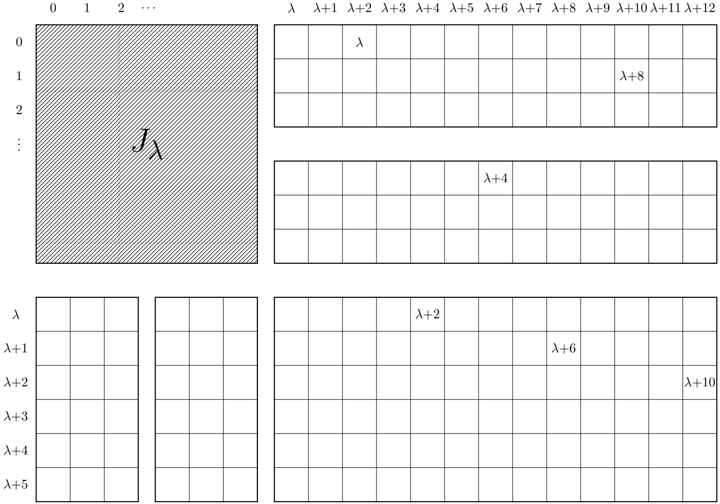


When we extend from J_λ to $J_{\lambda+\omega}$, we call it adding a new "layer."

Each time, the ordinals that must be placed are the ordinals $\{\lambda + n : n \in \omega\}$.

Adding a layer for each countable limit ordinal λ gives us a Jónsson-Tarski algebra of size $\aleph_1$, where the limit ordinals are subalgebras.

Construction of Uncountable JJT: Part 2

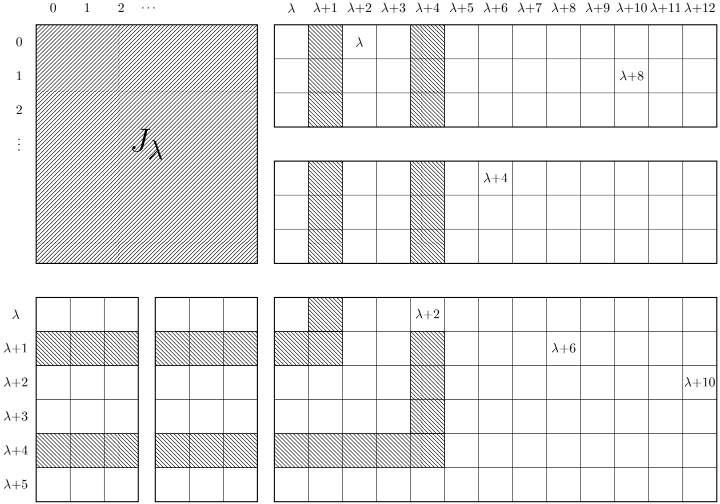


This figure shows how to place $\lambda + n$ in the table when n is even.

We always have $r(\lambda + n) = \lambda + n + 2$.

There is one $\lambda + n$, n even, in every row.

Construction of Uncountable JJT: Part 3



This figure shows where the ordinals $\lambda + n$, n odd, are placed in the table. We fill one L-shaped region at a time.

All the ordinals in the region corresponding to row/column $\lambda + m$ are greater than $\lambda + m$.

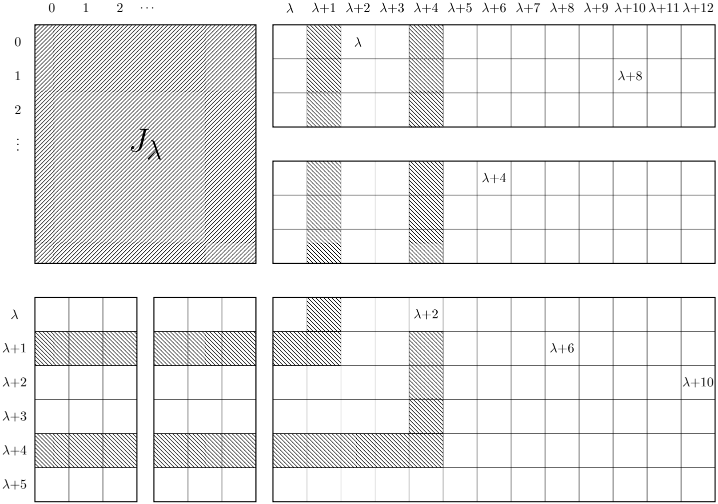
Construction of Uncountable JJT: Part 4

Now we must argue J is Jónsson. These are the main ideas:

Fix a nonzero countable limit ordinal λ. We will show:

1. Every $\lambda + n$, $n \in \omega$, generates the element λ.
2. The element λ generates the entire set $\lambda + \omega$.
3. Therefore, every $\lambda + n$ generates the entire set $\lambda + \omega$.

Why does every $\lambda + n$ generate λ?



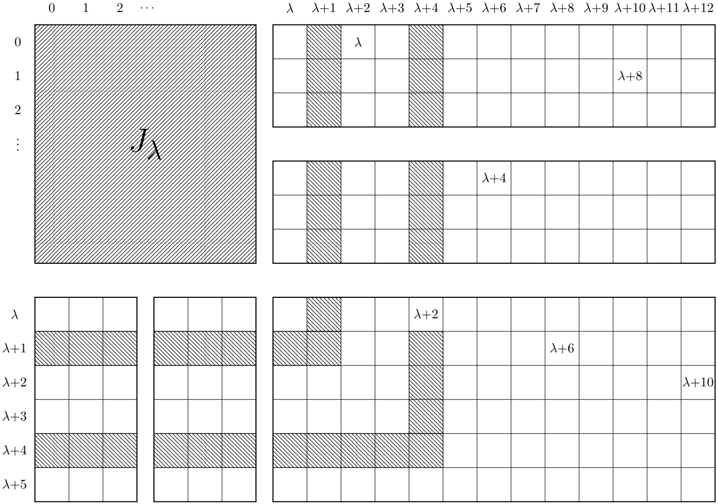
Suffices to show every $\lambda + n$, $n \neq 0$, generates a smaller $\lambda + m$.

For $n \equiv 0 \pmod 4$, use $\ell \circ r$.

For $n \equiv 2 \pmod 4$, use ℓ.

For n odd, at least one of ℓ, r will work.

Why does λ generate the entire set $\lambda + \omega$?



By repeatedly applying r, λ can generate any $\lambda + 2k$, $k \in \omega$.

There is an ordinal of the form $\lambda + 2k$ in every row of the table.

Thus any element of $\lambda + \omega$ is $\ell(r^k(\lambda))$ for some k.

Construction of Uncountable JJT: Conclusion

We have proven, for each λ,

1. Every $\lambda + n$, $n \in \omega$, generates the element λ.
2. The element λ generates the entire set $\lambda + \omega$.
3. Therefore, every $\lambda + n$ generates the entire set $\lambda + \omega$.

So, any subset $S \subseteq J$ generates the set $\displaystyle\bigcup_{\alpha \in S} \alpha + \omega$.

Conclusion: the subalgebras of J are the countable limit ordinals, and the set ω_1 (which is J itself).

Every proper subalgebra of J is countable, so J is Jónsson.

Part 3: The Sizes of Jónsson Jónsson-Tarski Algebras.

What About Larger JJT Algebras? Part 1

We have produced Jónsson Jónsson-Tarski algebras of cardinality $\aleph_0$ and $\aleph_1$. What about, say, $\aleph_2$?

In other varieties, authors have been able to use similar constructions for higher cardinalities. For example:

Theorem (P. Erdős, A. Hajnal (1965))

For each finite n there exists a Jónsson algebra of cardinality $\aleph_n$.

Proof: Inductively uses Jónsson algebras of cardinality $\aleph_k$ to construct a Jónsson algebra of cardinality $\aleph_{k+1}$.

What About Larger JJT Algebras? Part 2

Another relevant result:

Theorem (S. Shelah (1980))

There exists a Jónsson group of cardinality $\aleph_1$.

Later in the same paper:

Remark (S. Shelah (1980))

"The proof of [the above theorem] works also for $\aleph_2$ without any CH but for any $\aleph_n$, we need more complicated amalgamations, and the situation is not clear."

No JJT Algebra Of Size $> \aleph_1$: Part 1

In our case, however:

Corollary (DuBeau)

If J is a Jónsson Jónsson-Tarski algebra, then $|J| \leq \aleph_1$.

It follows from a more general theorem:

Theorem (DuBeau)

If J is an algebra in a language of size λ, where $|J| > \lambda^+$, and the subalgebra lattice of J is distributive, then J has a proper subalgebra of size $|J|$.

We will use a lemma about algebras with distributive subalgebra lattices:

No JJT Algebra Of Size $> \aleph_1$: Part 2

Lemma (DuBeau)

Let J be an algebra whose subalgebra lattice is distributive and $A \leq B \leq J$. If $S \subseteq J$, and for all $s \in S$, $\langle s \rangle \cap (B \setminus A) = \emptyset$, then it follows that $\langle S \rangle \cap (B \setminus A) = \emptyset$.

Proof.

The subalgebra lattice of J also satisfies this infinite version of the dist. law:

$$H \wedge \Big(\bigvee_{i \in I} K_i \Big) = \bigvee_{i \in I} (H \wedge K_i).$$

Now with A, B, and S as in the lemma, we get

$$B \wedge \langle S \rangle = B \wedge \Big(\bigvee_{s \in S} \langle s \rangle \Big) = \bigvee_{s \in S} (B \wedge \langle s \rangle) \leq A.$$

$\square$

No JJT Algebra Of Size $> \aleph_1$: Part 3

Now suppose J is an algebra of cardinality κ in a language of size λ, whose subalgebra lattice is distributive, and suppose $\kappa > \lambda^+$.

We will only prove the case where κ is regular.

Find a sequence of subalgebras $\{J_\alpha\}_{\alpha \leq \lambda^+}$ which is

- strictly increasing ($J_\alpha \lneqq J_\beta$ when $\alpha < \beta$)
- continuous. ($J_\gamma = \bigcup_{\alpha<\gamma} J_\alpha$ when γ is a limit ordinal).

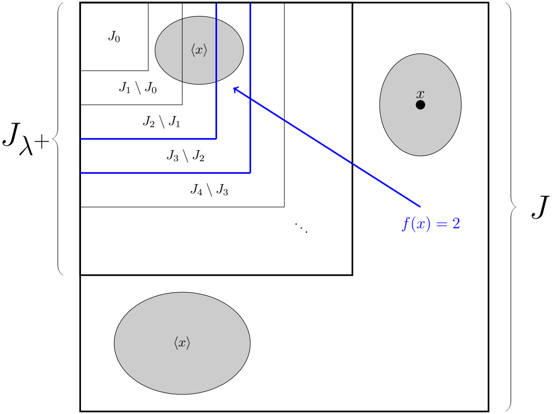
One way to do this is: let

- $J_0 = \langle x_0 \rangle$ for some x_0
- $J_1 = \langle \{x_0, x_1\} \rangle$ for some $x_1 \notin J_0$,
- etc.

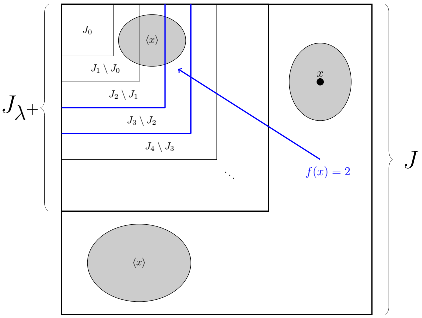
No JJT Algebra Of Size $> \aleph_1$: Part 4

Now define $f : J \to \lambda^+$ by

$$x \mapsto \sup\{\beta < \lambda^+ : \langle x \rangle \cap (J_{\beta+1} \setminus J_\beta) \neq \emptyset\}.$$



No JJT Algebra Of Size $> \aleph_1$: Part 5



Now f maps $J \to \lambda^+$.

Since $|J| = \kappa > \lambda^+$ and κ is regular, there must exist a subset $S \subseteq J$, $|S| = \kappa$, where $f(s_1) = f(s_2) =: \gamma$ for all $s_1, s_2 \in S$.

But then $\langle s \rangle \cap (J_{\lambda^+} \setminus J_{\gamma+1}) = \emptyset$ for all $s \in S$. So our earlier lemma implies $\langle S \rangle \cap (J_{\lambda^+} \setminus J_{\gamma+1}) = \emptyset$.

Therefore $\langle S \rangle$ is a proper subalgebra of J of size κ. $\square$

No JJT Algebra Of Size $> \aleph_1$: Part 6

We only showed the case where κ was regular, but with a similar argument for the singular case, we can prove:

Theorem

If J is an algebra in a language of size λ, where $|J| > \lambda^+$, and the subalgebra lattice of J is distributive, then J has a proper subalgebra of size $|J|$.

To show that Jónsson Jónsson-Tarski algebras cannot have cardinality greater than $\aleph_1$, we just need to prove:

Lemma (DuBeau)

The variety of Jónsson-Tarski algebras is subalgebra distributive: that is, every member of the variety has a distributive subalgebra lattice.

No JJT Algebra Of Size $> \aleph_1$: Part 7

A paper of Shapiro (1988) gave a result attributed to R. McKenzie, of the form "$\mathcal{V}$ is subalgebra distributive if and only if..."

It's a Klukovits-type condition on the terms of the variety: "For every term p there exist terms $s, u_1, \dots u_k, v_1, \dots, v_\ell$ such that..."

We had already developed a normal form for terms in the variety of Jónsson-Tarski algebras. (K. Kearnes, DuBeau (2020))

So it was fairly straightforward to show that the condition from Shapiro (1988) was satisfied in the variety of Jónsson-Tarski algebras, meaning that all Jónsson-Tarski algebras have distributive subalgebra lattices.

Part 3: The Number of Jónsson Jónsson-Tarski Algebras.

How Many Jónsson Jónsson-Tarski Algebras Are There?

Having constructed one Jónsson Jónsson-Tarski algebra of size $\aleph_1$, we now show:

Theorem (DuBeau)

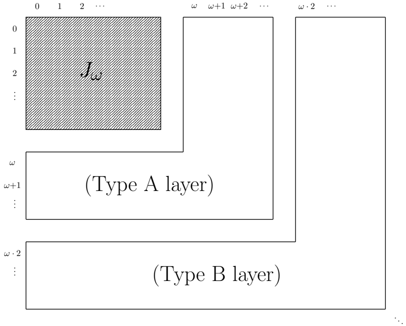
There exist $2^{\aleph_1}$ many pairwise nonisomorphic Jónsson Jónsson-Tarski algebras of cardinality $\aleph_1$.

This requires only minor modifications to our construction!

$2^{\aleph_1}$ Many JJT Algebras: Part 1

Original construction: we followed essentially the **same pattern every time** when extending from universe λ to universe $\lambda + \omega$.

New construction: we can choose from one of **two patterns** when extending from universe λ to universe $\lambda + \omega$. That is, we add either a "type A" layer or a "type B" layer each time.

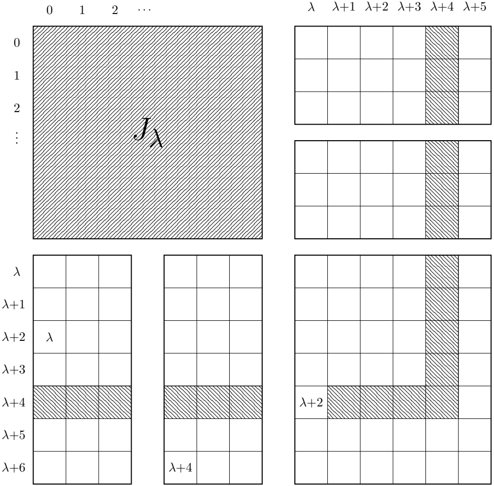


$2^{\aleph_1}$ Many JJT Algebras: Part 2

Type A construction: exactly the same construction shown in the previous proof.

Type B construction: the transpose / mirror image of the type A construction (exactly the same, but with ℓ and r exchanged.)

A type B layer is shown here.



$2^{\aleph_1}$ Many JJT Algebras: Part 3

Let $J_\omega =$ the countable JJT shown earlier in the talk.

Lemma

Let J be a Jónsson-Tarski algebra of size $\aleph_1$ formed by extending J_ω with any ω_1-sequence of type A and type B extensions. Then J is Jónsson.

Proof. Essentially the same as in the all-type-A construction:

1. Every $\lambda + n$, $n \in \omega$, generates the element λ.
2. The element λ generates the entire set $\lambda + \omega$.
3. Therefore, every $\lambda + n$ generates the entire set $\lambda + \omega$. $\square$

$2^{\aleph_1}$ Many JJT Algebras: Part 4

Lemma

Let J_{λ_1} and J_{λ_2} be Jónsson-Tarski algebras whose universes λ_1 and λ_2 are nonzero countable limit ordinals. Let $J^A_{\lambda_1}$ denote J_{λ_1} extended with a type A layer, and $J^B_{\lambda_2}$ denote J_{λ_2} extended with a type B layer. Then $J^A_{\lambda_1}$ is not isomorphic to $J^B_{\lambda_2}$.

Proof.

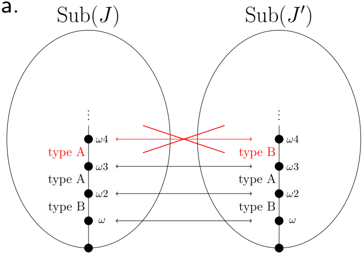
1. $J^A_{\lambda_1}$ has a "*type A generator:*" an element g such that every $x \in J^A_{\lambda_1}$ can be written as $x = \ell(r^k(g))$ for some $k \in \omega$. (For example, λ_1.)
2. $J^B_{\lambda_2}$ does not have a type A generator. (Requires careful checking.)
3. The property of having a type A generator would be preserved under isomorphism, so the two are not isomorphic.

$2^{\aleph_1}$ Many JJT Algebras: Part 5

Theorem (DuBeau)

Let J and J' be Jónsson-Tarski algebras of cardinality $\aleph_1$ formed by extending J_ω with two different ω_1-sequences of type A and type B extensions. Then J is not isomorphic to J'.

Proof. In both J and J', the proper subalgebras are the countable limit ordinals. An isomorphism $J \to J'$ would induce an isomorphism of subalgebra lattices. This causes a contradiction at the first index where the two sequences differ, by the previous lemma. □



$2^{\aleph_1}$ Many JJT Algebras: Conclusion

- There are $2^{\aleph_1}$ many sequences of A's and B's.
- All these sequences produce pairwise nonisomorphic JJT algebras.
- So, we have constructed $2^{\aleph_1}$ many pairwise nonisomorphic JJT algebras of cardinality $\aleph_1$.
- Conclusion: there are as many Jónsson algebras of cardinality $\aleph_1$ in this variety as there are algebras of cardinality $\aleph_1$ in this variety!

$2^{\aleph_1}$ Many JJT Algebras: Addendum

We actually have a slightly stronger theorem:

Theorem (DuBeau)

Let J_ω be **any** *Jónsson-Tarski algebra with universe ω. Then there are $2^{\aleph_1}$ many pairwise nonisomorphic extensions of J_ω into a Jónsson Jónsson-Tarski algebra of size $\aleph_1$.*

The proof is similar, but we cannot guarantee that **any** two sequences of A's and B's will produce nonisomorphic algebras.

Instead we argue that **there exist** $2^{\aleph_1}$ many sequences of A's and B's that produce pairwise nonisomorphic algebras.

The End

Thanks for coming!