Mal'tsev Condition Satisfaction Problems: Lattices are easier than semilattices

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Aims of this talk:

1. Introduce Mal'tsev Condition Satisfaction Problems (MCSPs)
2. Compare and contrast lattices vs. semilattices
3. Review FNV result on semilattice condition (2019)
4. Review my result on lattice condition (2020)

Definitions

Definition

- A strong Mal'tsev condition Σ is a finite set of equations. An algebra $\mathbf{A}$ satisfies the strong Mal'tsev condition Σ if there are term operations of $\mathbf{A}$ which model the equations of Σ.
- A Maltsev condition $\mathcal{M}$ consists of an increasing sequence Σ_i, $i \geq 1$, of strong Maltsev conditions. An algebra satisfies $\mathcal{M}$ if it satisfies Σ_i for some i.
- A Maltsev condition is linear if none of the equations used to define it involve compositions.

Example

Example (Existence of a binary commutative term)

$$C_2 := \{b(x, y) \approx b(y, x)\}$$

Example

The group $\langle \mathbb{Z}, +, -, 0 \rangle$ satisfies C_2 since $x + y$ is commutative.

Example

Example (Existence of a k-ary nu term)

$$NU(k) :=$$
$$\{m(x, y, y, \dots, y) \approx y,$$
$$m(y, x, y, \dots, y) \approx y$$
$$\vdots$$
$$m(y, y, \dots, x) \approx y)\}$$

Example

The lattice $\langle \mathbb{Z}, \wedge, \vee \rangle$ satisfies $NU(3)$ since $(x \wedge y) \vee (y \wedge z) \vee (x \wedge z)$ is a ternary nu term.

$NU(i) \implies NU(i+1)$ so $NU := (NU(i))_{i \in \mathbb{N}}$ is a Mal'tsev condition.

Mal'tsev Condition Satisfaction Problem

Definition

The (idempotent) Σ-satisfaction problem, $\mathrm{Sat}^{\mathrm{Id}}_{\Sigma}$, is the decision problem with input a finite (idempotent) algebra $\mathbf{A}$ and question "Does $\mathbf{A}$ satisfy Σ?"

Theorem (Horowitz)

$Sat^{Id}_{NU(k)} \in P$.

Proof.

It is enough to check that we can satisfy the equations over small subsets of A^k. $\square$

Lattices and Semilattices

Example (Existence of a semilattice term)

$$\mathit{Semilattice} :=$$
$$\{x \vee y \approx y \vee x,$$
$$x \vee x \approx x,$$
$$x \vee (y \vee z) \approx (x \vee y) \vee z\}$$

Example (Existence of lattice terms)

$$\mathit{Lattice} :=$$
$$\{x \vee y \approx y \vee x, \quad x \wedge y \approx y \wedge x$$
$$x \vee (y \vee z) \approx (x \vee y) \vee z, \quad x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$$
$$x \vee (x \wedge y) \approx x, \quad x \wedge (x \vee y) \approx x\}$$

Lattices and Semilattices compared

Example

- Both define a partial order
- Clearly *Lattice* $\implies$ *Semilattice*
- Both are nonlinear

Example

- *Lattice* $\implies$ $NU(3)$

$$(x \vee y) \wedge (y \vee z) \wedge (z \vee x)$$

- *Semilattice* does not imply NU

$\mathrm{Sat}^{\mathrm{Id}}_{Lattice}$ and $\mathrm{Sat}^{\mathrm{Id}}_{Semilattice}$

Theorem (Freese, Nation, Valeriote 2019)

$Sat^{Id}_{Semilattice}$ *is EXPTIME-complete.*

Theorem (Rooney 2020)

$Sat^{Id}_{Lattice} \in NP$.

$\mathrm{Sat}^{\mathrm{Id}}_{Semilattice}$ is EXPTIME-complete (FNV)

First Ingredient (Friedman c.1985 + Bergman, Juedes, Slutzki 1999 + Freese, Valeriote 2009)

GEN-CLO' is EXPTIME-complete.

GEN-CLO'

Given:

- a finite algebra $\mathbf{A}$, and
- a unary function $h : A \to A$.

Determine: Is h a term operation of $\mathbf{A}$?

$\mathrm{Sat}^{\mathrm{Id}}_{\mathit{Semilattice}}$ is EXPTIME-complete (FNV)

Second ingredient (FNV 2019)

BOUNDED-SEMILATTICE is EXPTIME-complete.

BOUNDED-SEMILATTICE

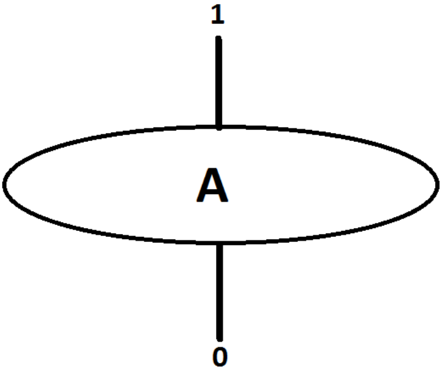
Given:

- a finite *idempotent* algebra $\mathbf{A}$, and
- an element $1 \in A$

Determine: Is there a term operation $\wedge$ of $\mathbf{A}$ such that $\langle A, \wedge, 1 \rangle$ is a bounded semilattice?

BOUNDED-SEMILATTICE is EXPTIME-complete

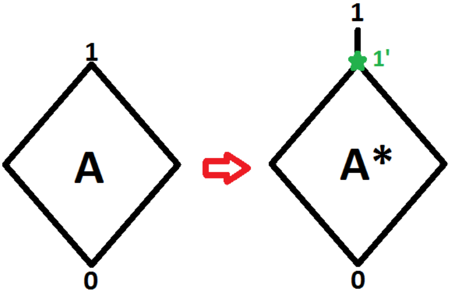
$\mathbf{A} = \langle A, \mathcal{F} \rangle$ is an instance of GEN-CLO'.



$g \in \mathcal{F} \mapsto g' \in \mathcal{G}$
$t_h(x, y, z) := x \wedge y$ if and only if $z = h'(x, y)$.

$\mathsf{Sat}^{\mathsf{Id}}_{Semilattice}$ is EXPTIME-complete

$\mathbf{A} = \langle A, \mathcal{F} \rangle, 1$ is an instance of BOUNDED-SEMILATTICE.



The only way for $\mathbf{A}^*$ to have a semilattice operation is with $1 > 1' > A \backslash \{1\}$. By construction that happens if and only if $\mathbf{A}$ was already a bounded semilattice with maximum element 1.

$\mathrm{Sat}^{\mathrm{Id}}_{Semilattice}$ is EXPTIME-complete

Proof Summary.

GEN-CLO' $\rightarrow$ BOUNDED-SEMILATTICE $\rightarrow$ $Sat^{\mathrm{Id}}_{Semilattice}$ $\square$

$\mathrm{Sat}^{\mathrm{Id}}_{Lattice} \in \mathrm{NP}$

Theorem (Rooney 2020)

If Σ is a strong Mal'tsev condition which implies NU, then $Sat^{Id}_{\Sigma} \in NP$.

Example

$Lattice \implies NU(3)$

$$(x \vee y) \wedge (y \vee z) \wedge (z \vee x)$$

$\mathrm{Sat}^{\mathrm{Id}}_{Lattice} \in \mathrm{NP}$

First Ingredient (FV 2009)

Use the algorithm of Freese and Valeriote to check whether **A** has a ternary nu term.

If **A** does not satisfty $NU(3)$, then **A** certainly does not satisfy $Lattice$.

Polynomial-time verifier for $\mathrm{Sat}^{\mathrm{Id}}_{Lattice}$

Verifier

INPUT:

- The instance $\mathbf{A}$ of $\mathrm{Sat}^{\mathrm{Id}}_{Lattice}$, and
- operations $\vee, \wedge$ satisfying the lattice equations on A

PROCEDURE:

1. Verify that $\vee$ and $\wedge$ satisfy the lattice equations
2. Verify that $\vee$ and $\wedge$ are term operations of $\mathbf{A}$

Baker and Pixley

Theorem (Baker, Pixley 1975)

Let $\mathbf{A}$ be an algebra with a ternary near unanimity term. A function $f : A^n \to A$ is a term operation of $\mathbf{A}$ if and only if every subalgebra of $\mathbf{A}^2$ is closed under f.

The result in Baker and Pixley's article is much more general.

Back to the verifier

Verifying that $\vee$ is a term operation

Using the theorem of Baker and Pixley (and the fact that $\mathbf{A}$ has a ternary nu term) we need only verify that for each $a, b, c, d \in A$ we have

$$\begin{pmatrix} a \vee c \\ b \vee d \end{pmatrix} \in \left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle_{\mathbf{A}^2}$$

$\mathrm{Sat}^{\mathrm{Id}}_{Lattice} \in \mathrm{NP}$

Proof Summary.

- First check that the instance **A** has a ternary nu term, then
- given the operation tables for lattice term operations of **A** one can verify in polynomial time that **A** satisfies *Lattice*.

□

The second item is achievable in polynomial time only because of the existence of an nu term for **A**.
The first item is only known to be achievable in polynomial time when **A** is idempotent.

Can we remove idempotence?

An open question

What is the complexity of $\mathrm{Sat}_{NU(k)}$?

Do we need NU?

No!

Theorem (Rooney 2021)

Let Σ be a strong Mal'tsev condition which implies the existence of an edge term. Then $Sat_{\Sigma}^{Id} \in NP$.

The polynomial-time verifier in this case relies on a result of Bulatov, Mayr and Szendrei from 2018 on subpower membership problems for finite algebras with cube terms.

What if we only look at linear conditions?

Theorem (Rooney 2020)

If Σ is a strong linear Mal'tsev condition which implies NU, then $Sat_{\Sigma}^{Id} \in P$.

This proof encodes the MCSP as a CSP over a template of bounded width using Baker and Pixley's result to prove correctness.

Thank you

Thanks!

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