Free-algebra functors as coalgebraic signatures

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Intro

State based systems

Functors and Coalgebras

Fuctor properties

Weak Pullback Preservation

Functors parameterized by algebras

Free-algebra functor

Preimage preservation

Weak kernel preservation

Conclusion

- and Breaking News

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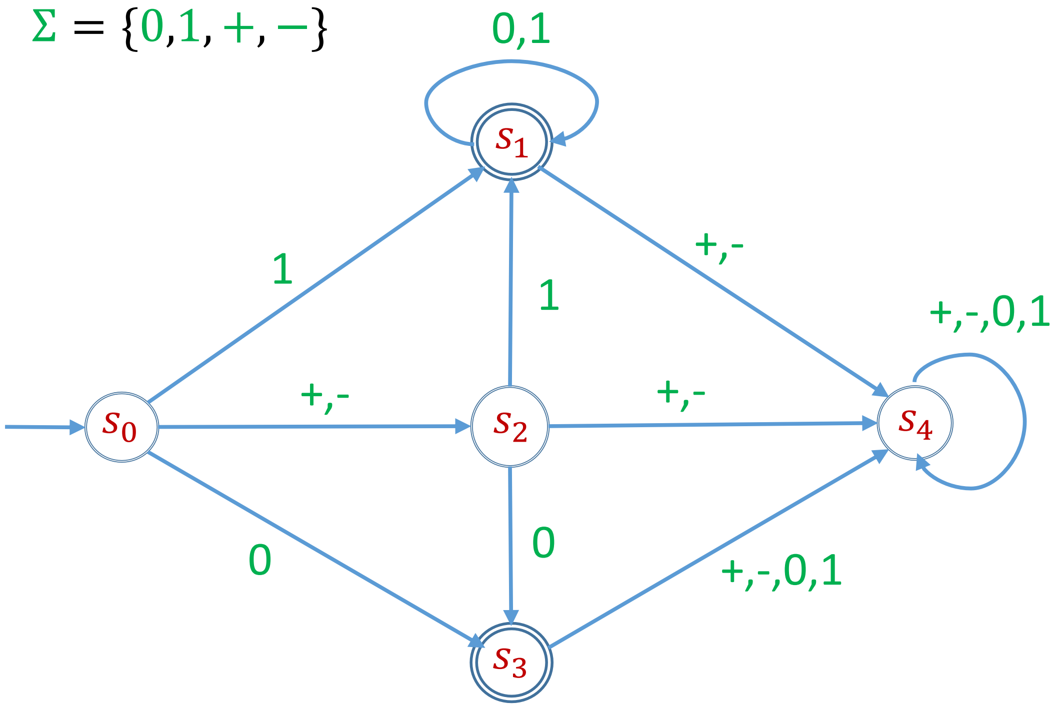
Automata

S set of states,

Σ input alphabet

- Acceptor:
  - $\delta: S \times \Sigma \rightarrow S$
  - $T \subseteq S$

$\Sigma = \{0,1,+,-\}$



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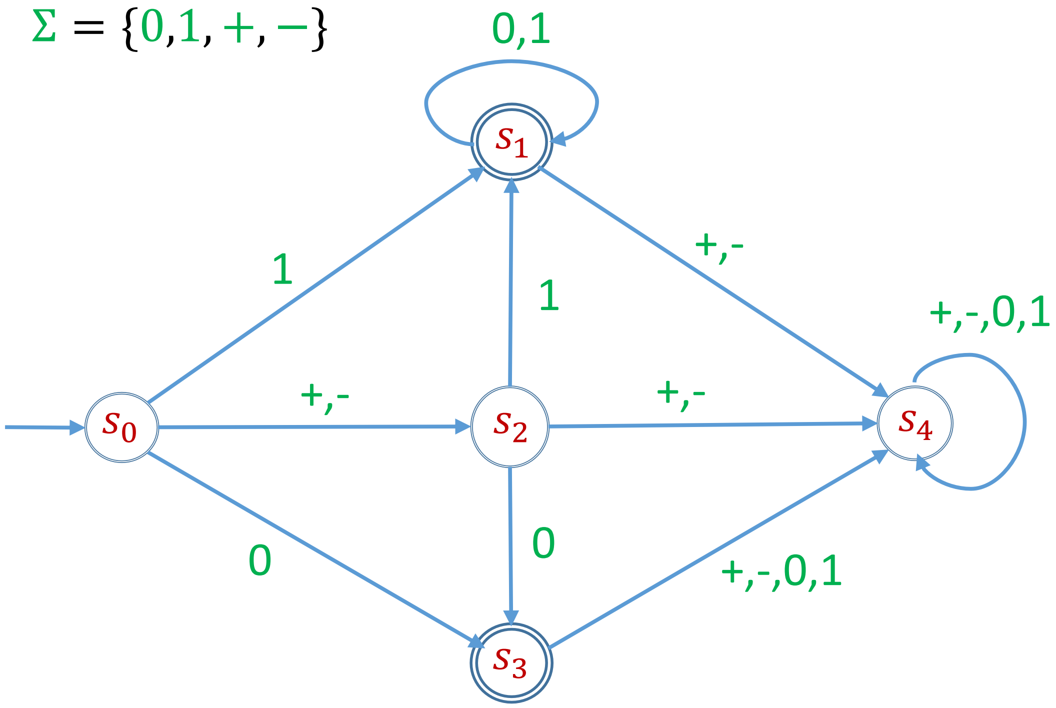

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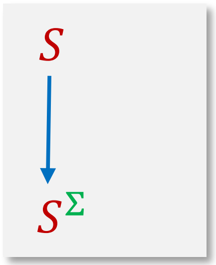

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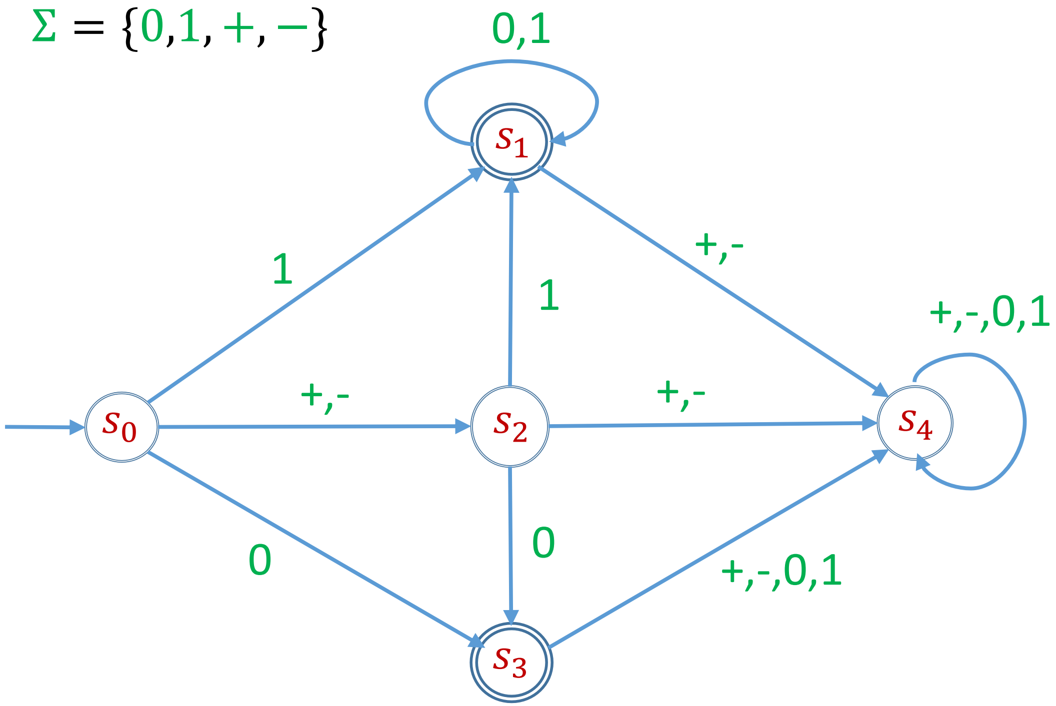

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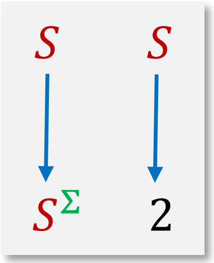

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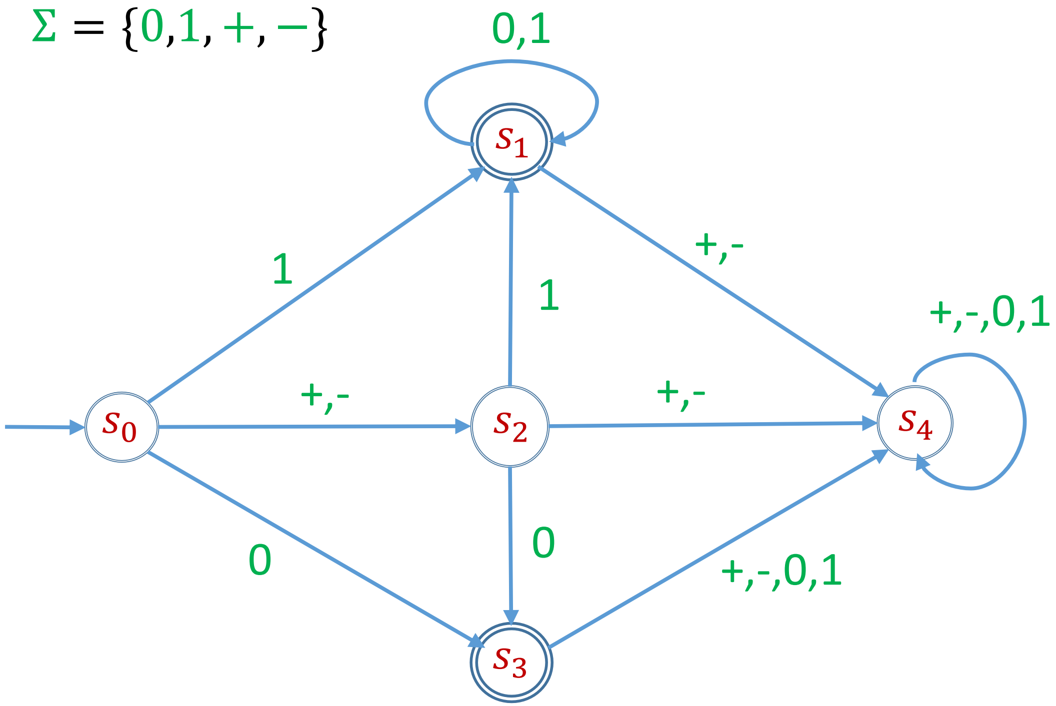

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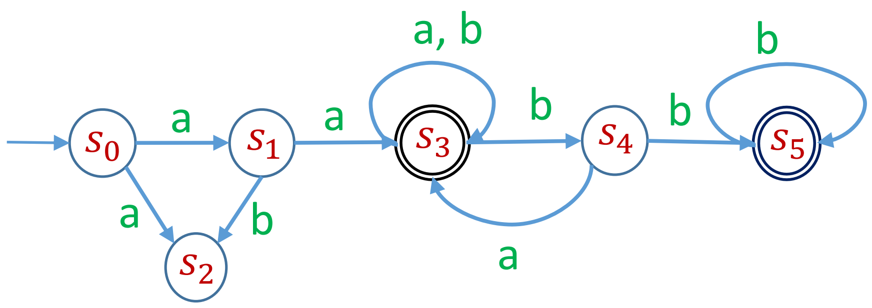
$$S \to S^{\Sigma} \times 2$$

Nondeterminism

- NFA
  - $\delta: S \times \Sigma \to \mathbb{P}(S)$
  - $T \subseteq S$



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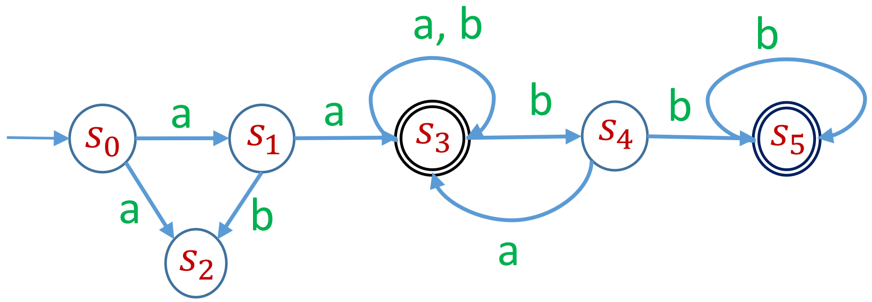

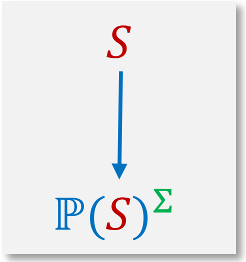

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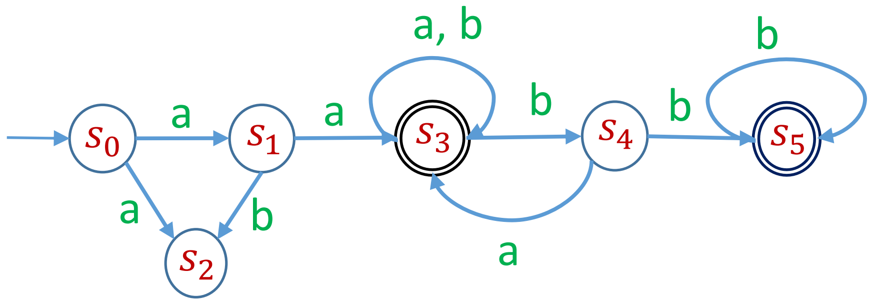

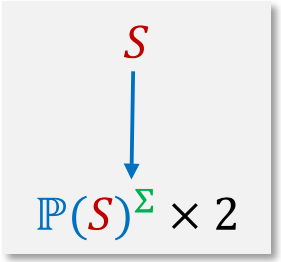

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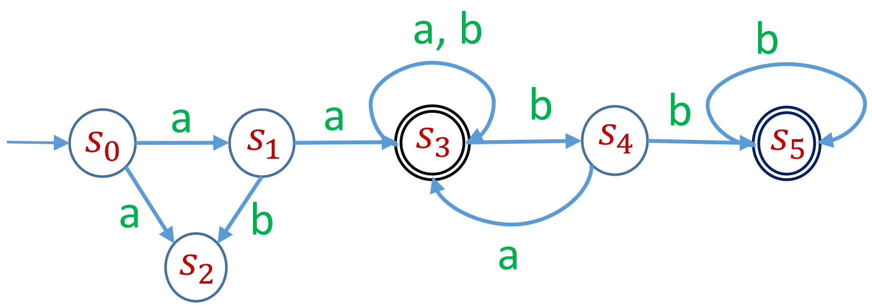
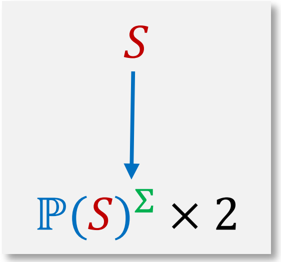
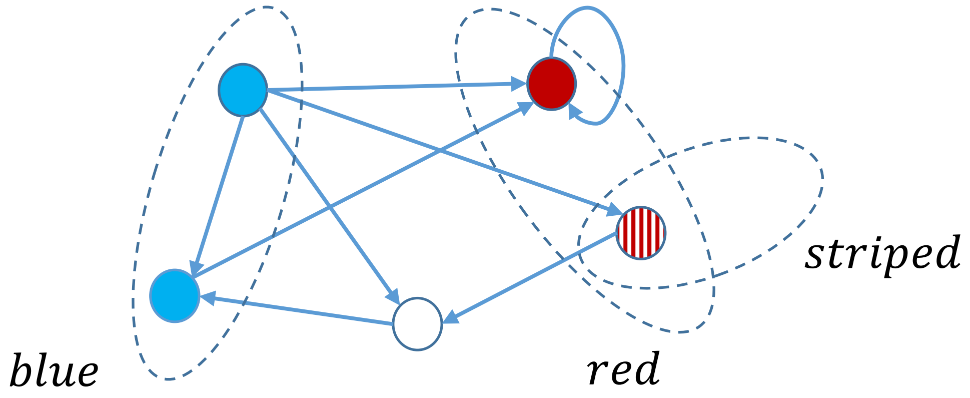




- Kripke structure
  - $R \subseteq S \times S$
  - $v : S \to \mathbb{P}(C)$



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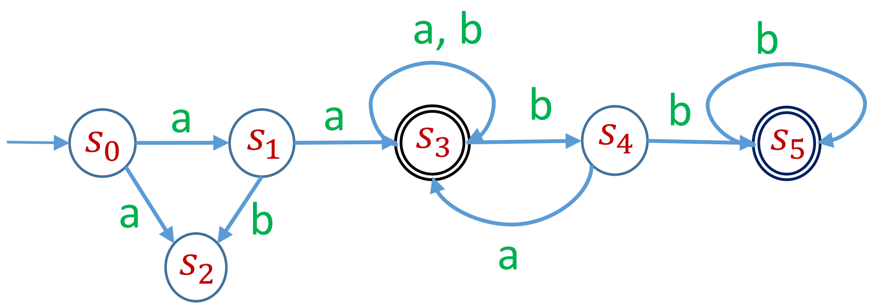

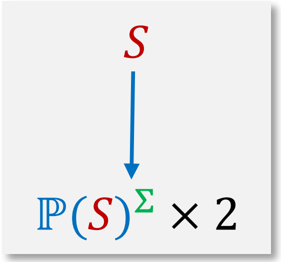

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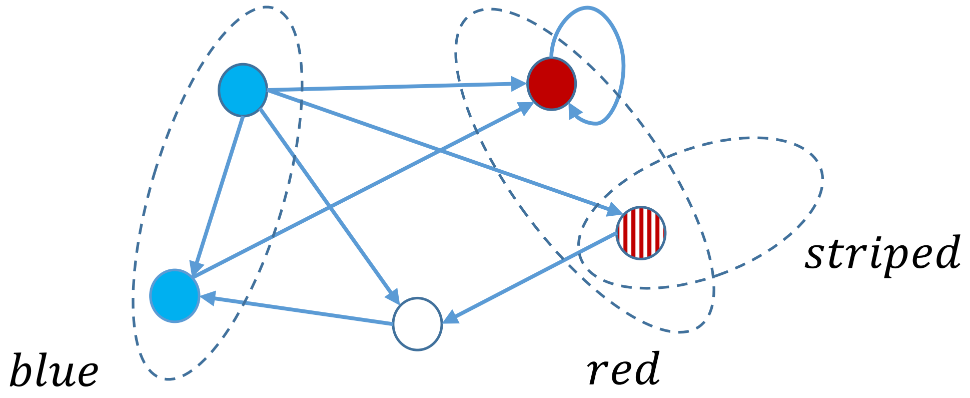
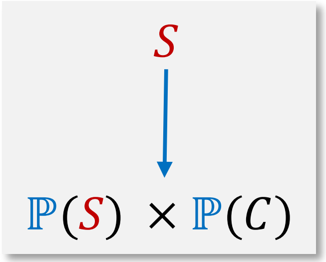
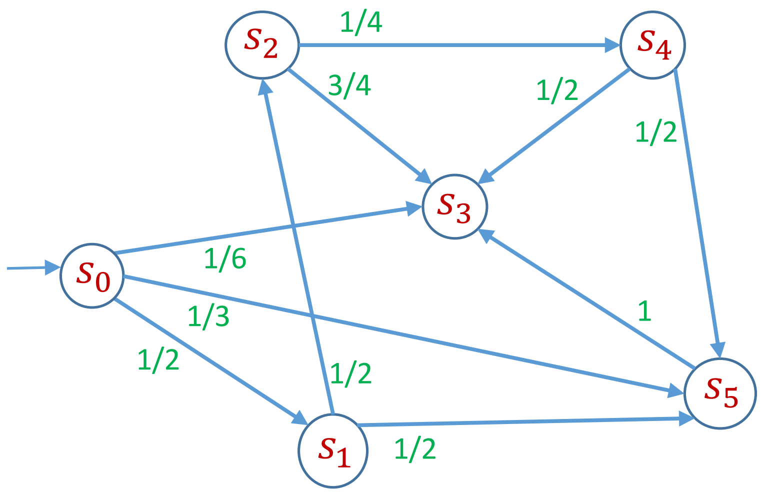




Probabilistic

- Probabilistic systems
  - $\delta: S \times S \to [0,1]_{\mathbb{R}}$
  - $\sum_{x \in S} \delta(s, x) = 1$



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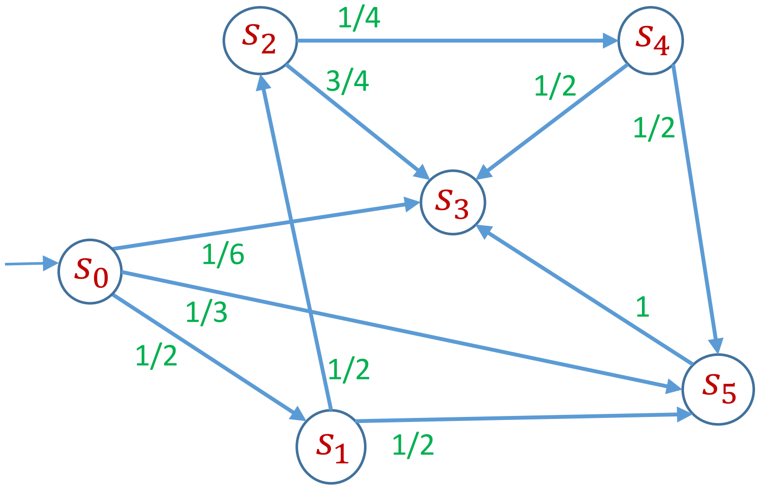
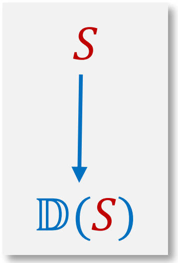
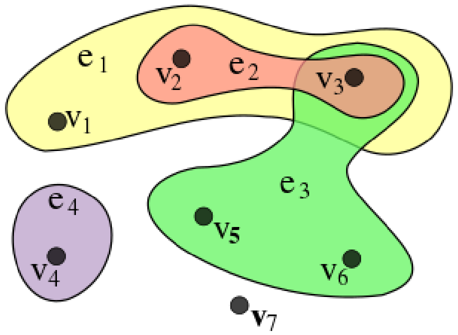




Higher order

- Neighbourhood structure
  - $R \subseteq S \times 2^S$
  - $c: S \rightarrow \mathbb{P}(C)$



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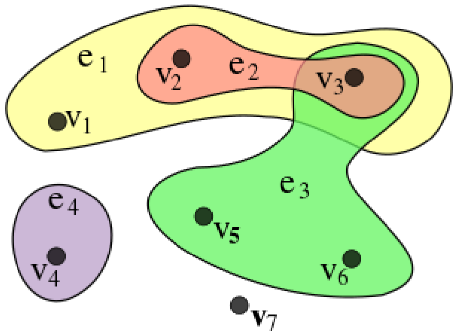

$$S \downarrow 2^{2^{S}} \times \mathbb{P}(C)$$

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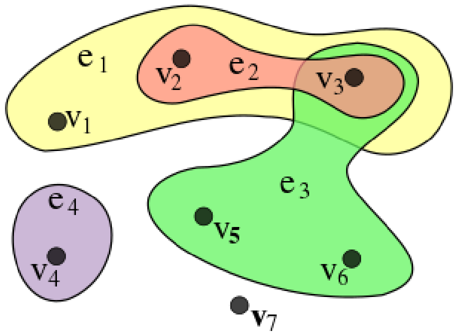
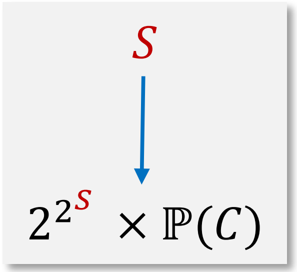
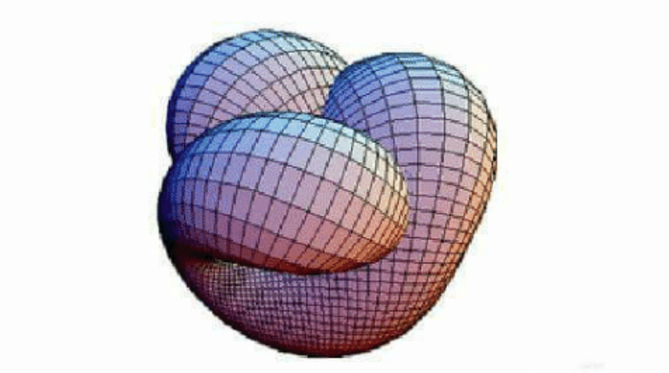




- Topological space
  - $\tau \subseteq \mathbb{P}(\mathbb{P}(S))$
    - closed under
      - unions
      - finite intersections



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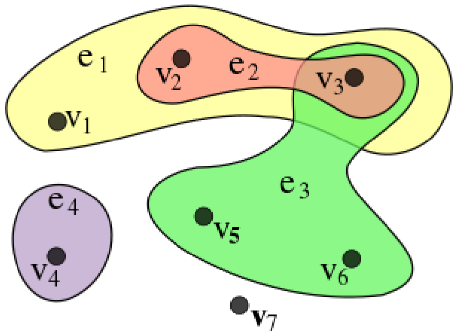

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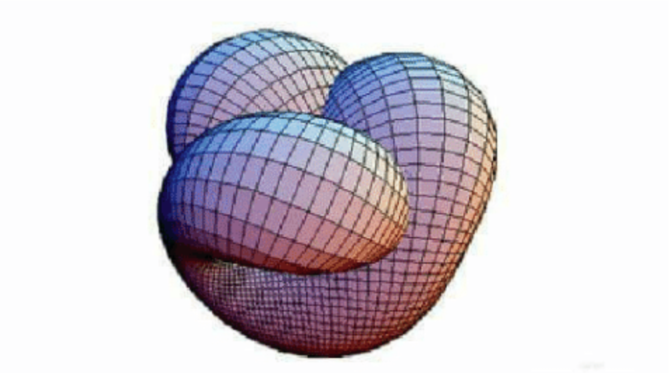

S
$\downarrow$
$\mathbb{F}il(S)$

Coalgebras and Algebras

$$S \xrightarrow{\alpha} F(S)$$

$$S \to S^{\Sigma} \times 2 \qquad S \to \mathbb{P}(S)^{\Sigma} \times 2 \qquad S \to \mathbb{D}(S) \qquad S \to 2^{2^{S}} \times \mathbb{P}(C) \qquad S \to \mathbb{F}il(S)$$

- Set functors fix the „signature“ of coalgebras

Coalgebras and Algebras

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- Set functors fix the „signature“ of coalgebras
- Algebras are upside-down coalgebras

$$A \times A \rightarrow A \qquad A^2 + A^2 \rightarrow A \qquad A^2 + A + 1 \rightarrow A \qquad \ldots \qquad \mathbb{P}(A) \rightarrow A$$

$$F(A) \xrightarrow{\alpha} A$$

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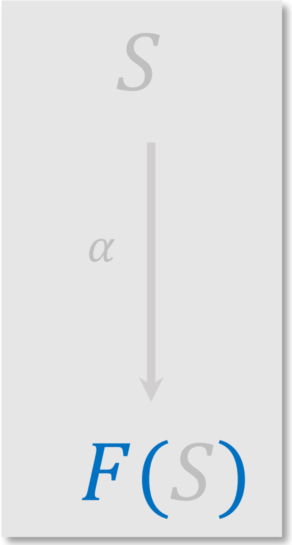
Functors

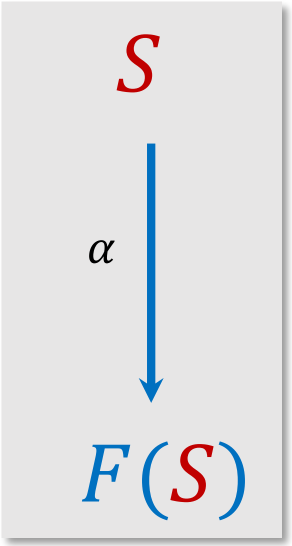


- $F(X)$ a „*natural set theoretical construction*“
- F should be a *functor*:
  - for each X construct new set $F(X)$
  - for each $f: X \to Y$ provide map $Ff: F(X) \to F(Y)$

    such that

    - $F(g \circ f) = Fg \circ Ff$
    - $F\ \iota d_A = \iota d_{F(A)}$

Functors and Coalgebras



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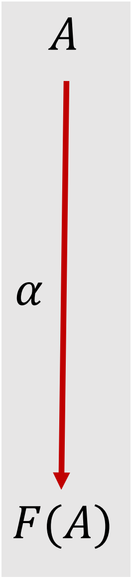
F-coalgebras

Homomorphism



F-coalgebras

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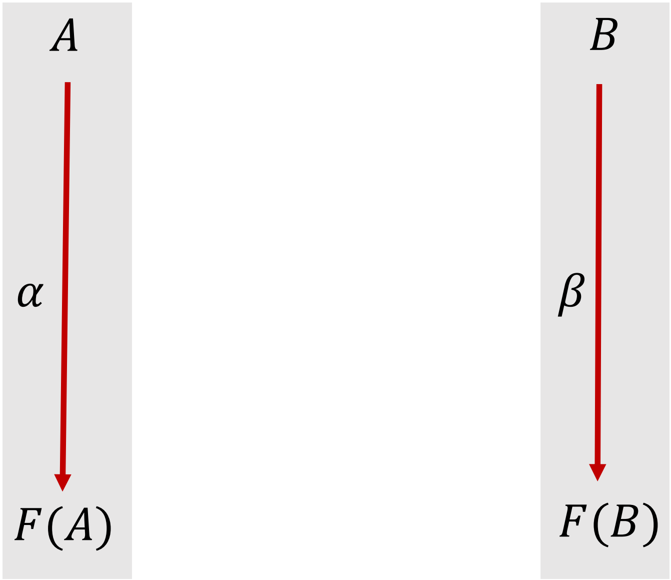
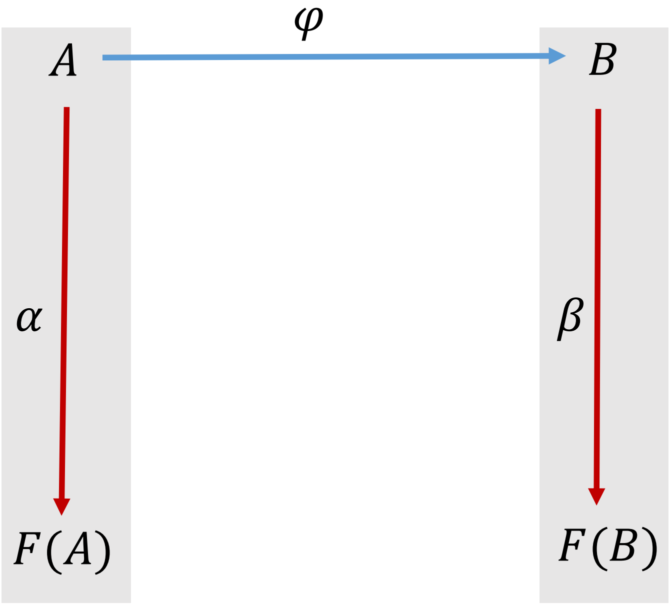


F-coalgebras and homomorphisms

Homomorphism



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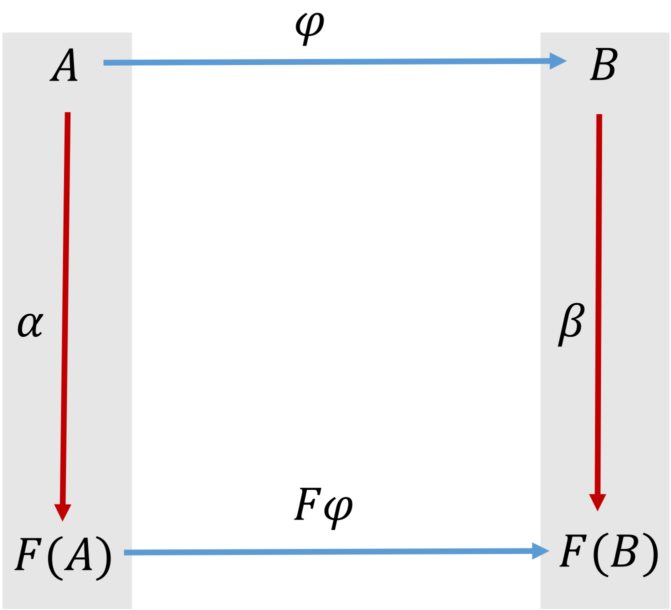

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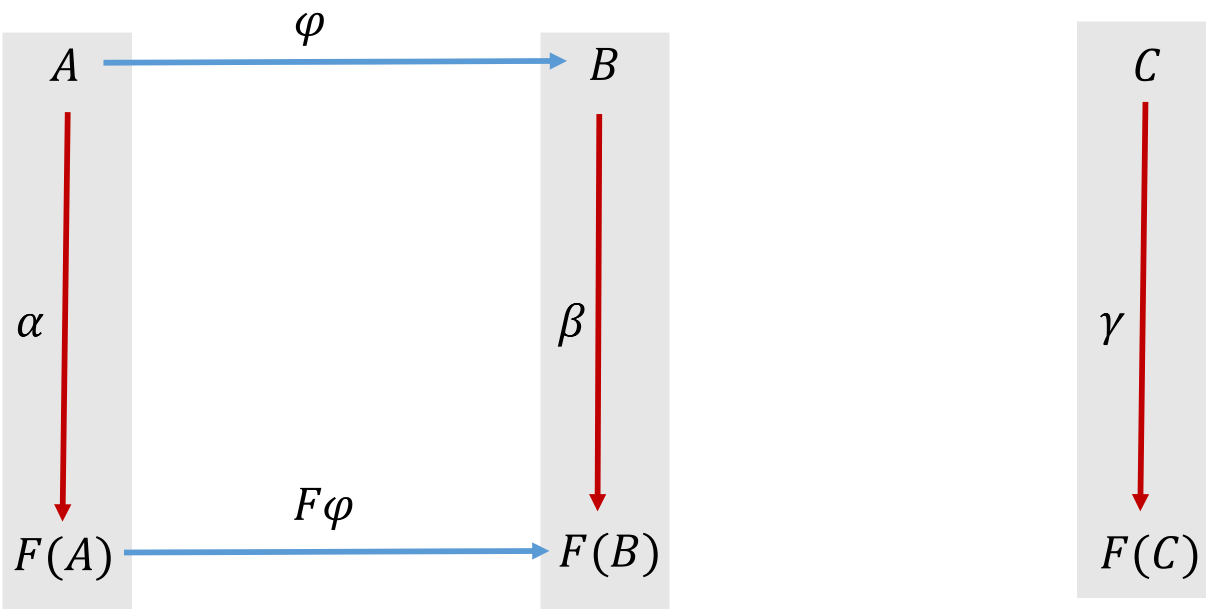

Homomorphisms compose

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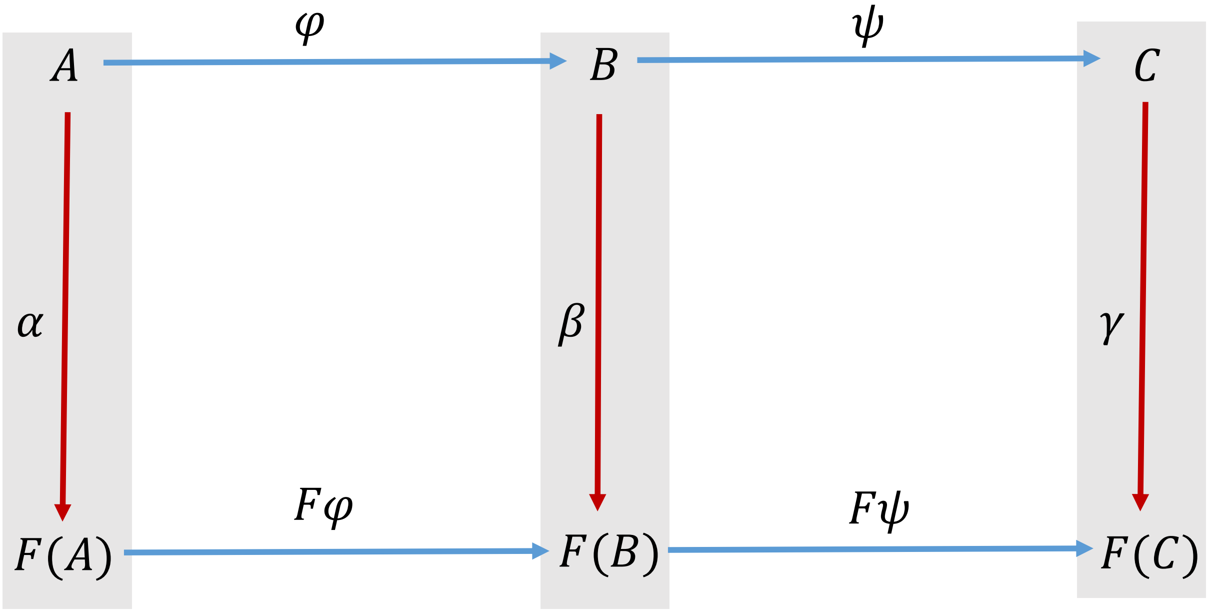

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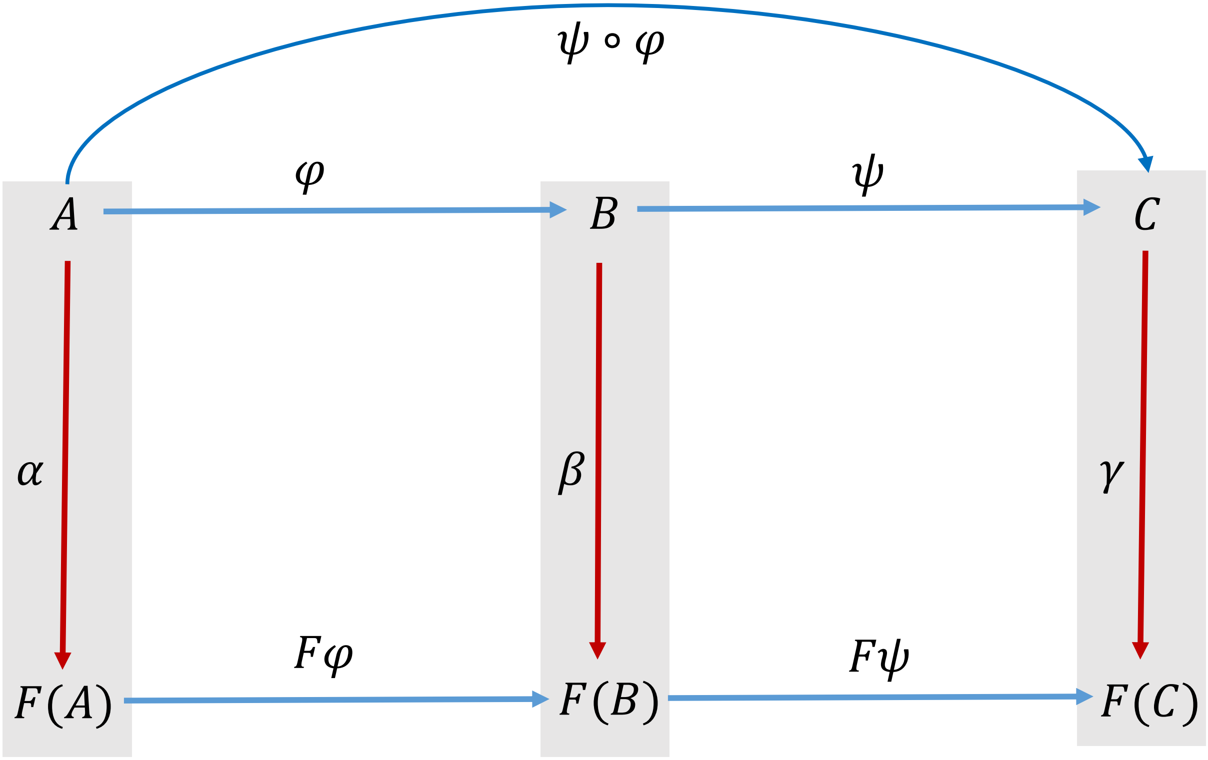

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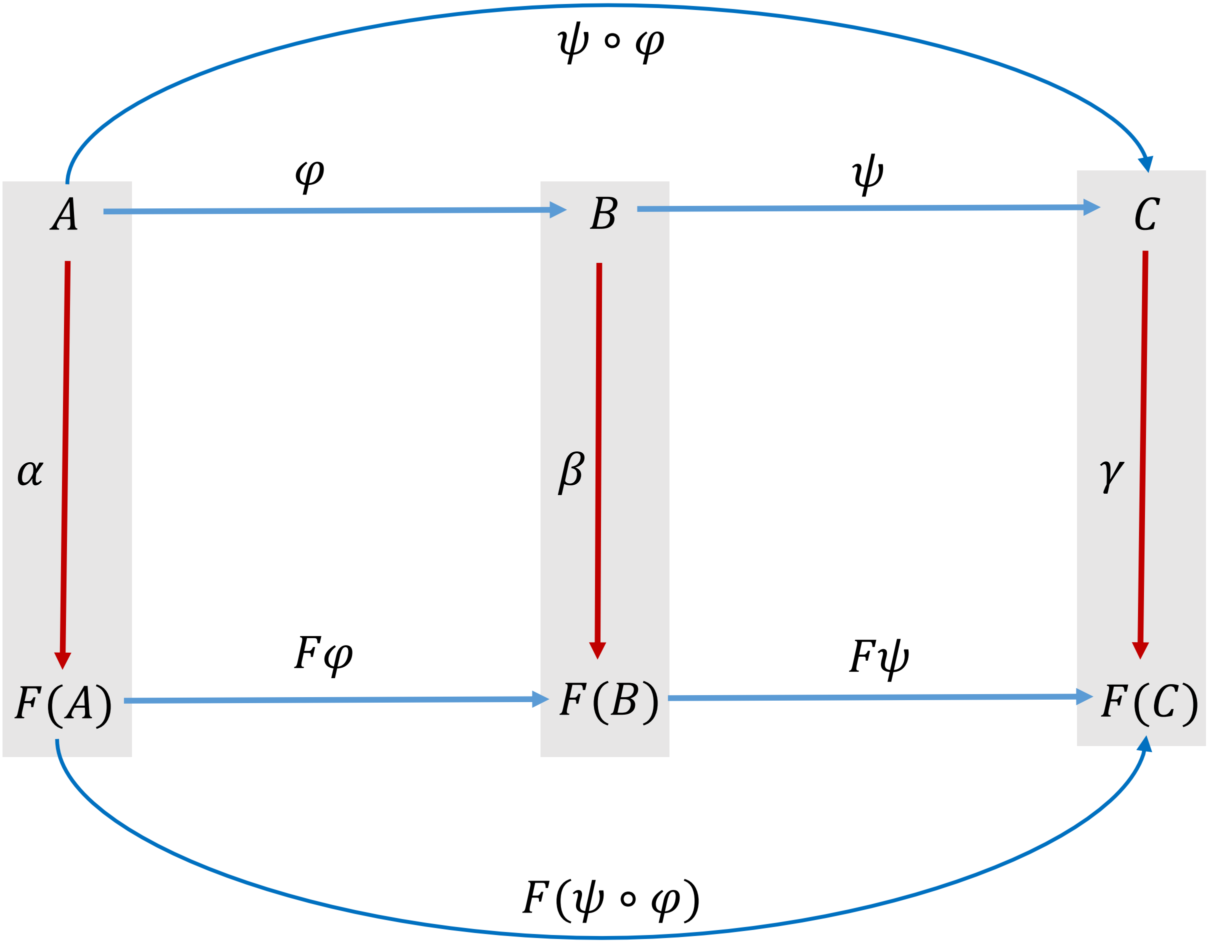
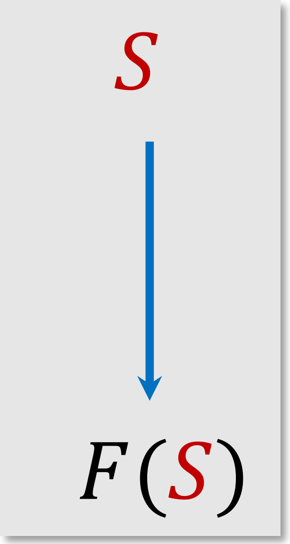


Homomorphisms compose

$F(\psi \circ \varphi) = F\psi \circ F\varphi$

Set_F



- F-coalgebras form a category Set_F
  - Homomorphism theorems
  - Substructures, quotients, congruences, …
  - Co-equations, Co-Birkhoff
  - Modal logic
  - …
- Properties of F determine structure of Set_F
  - *weak pullback preservation*

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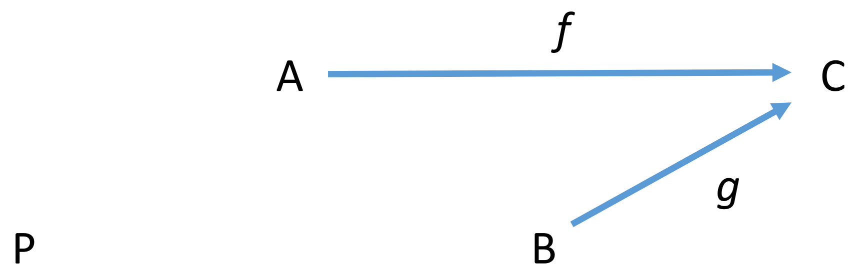
- (Weak) pullback



$P = \{(a, b) \in A \times B \mid f(a) = g(b)\}$

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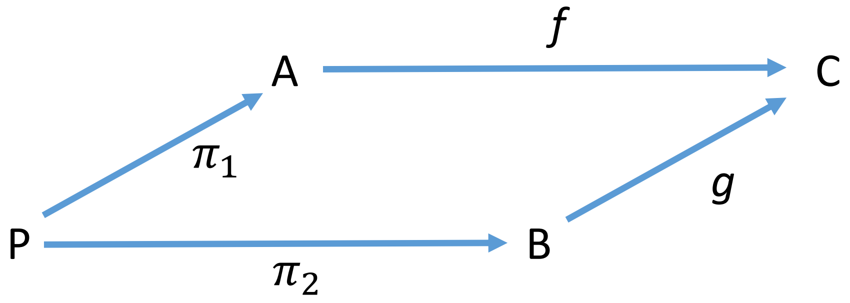

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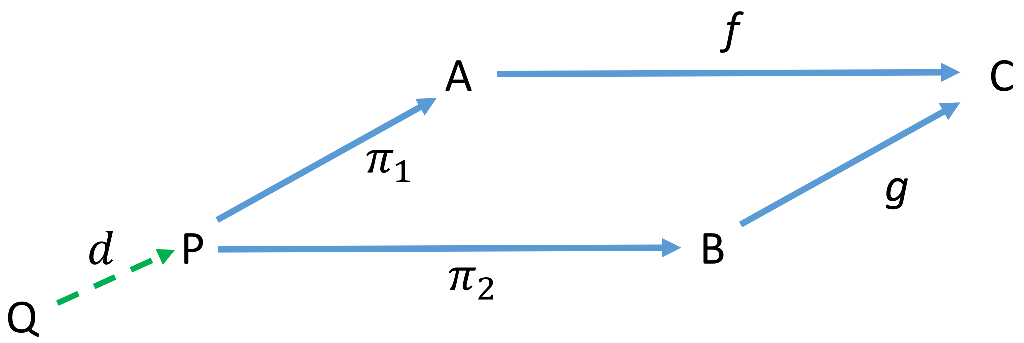

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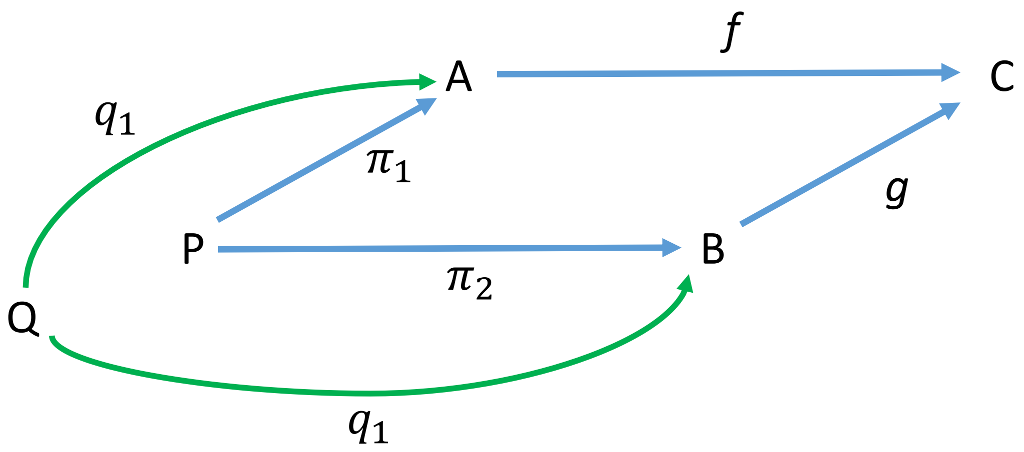

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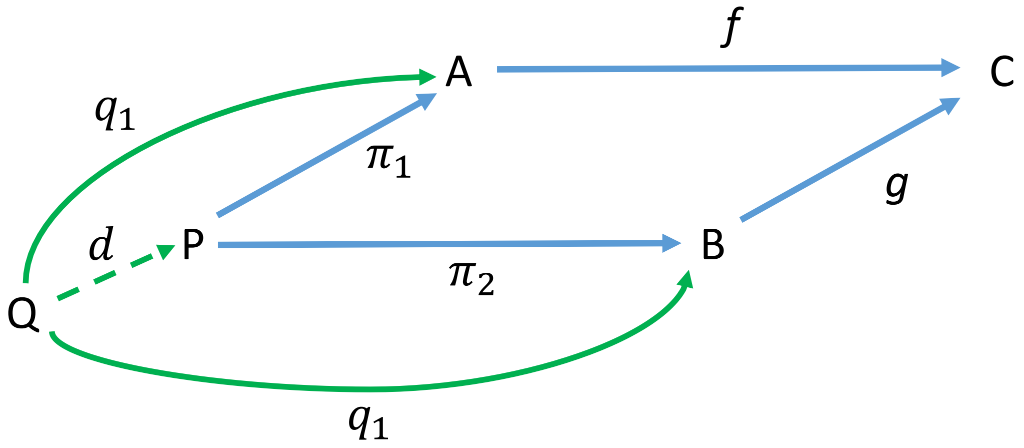

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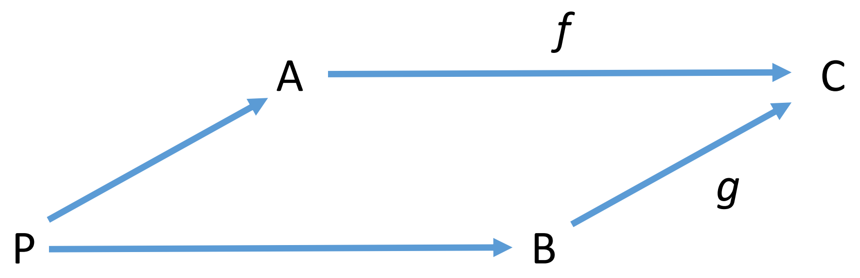

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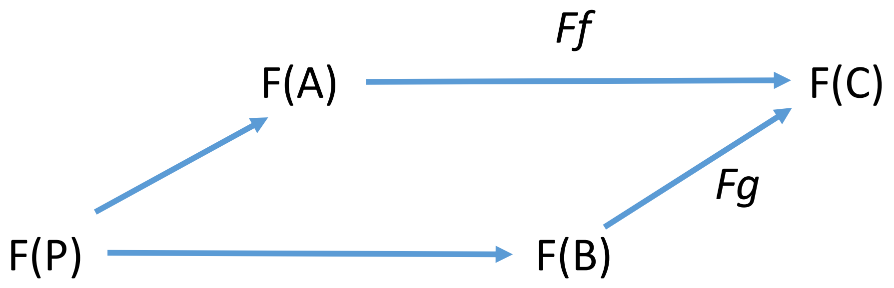
- apply F



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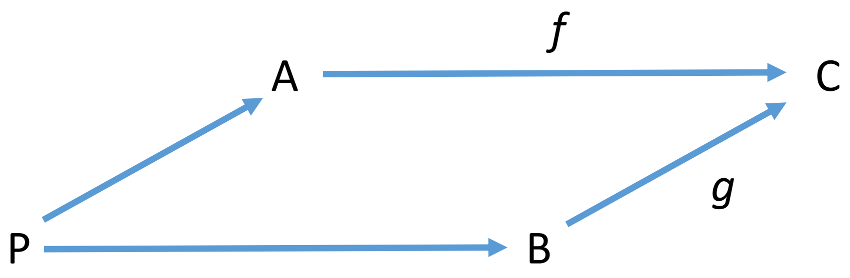

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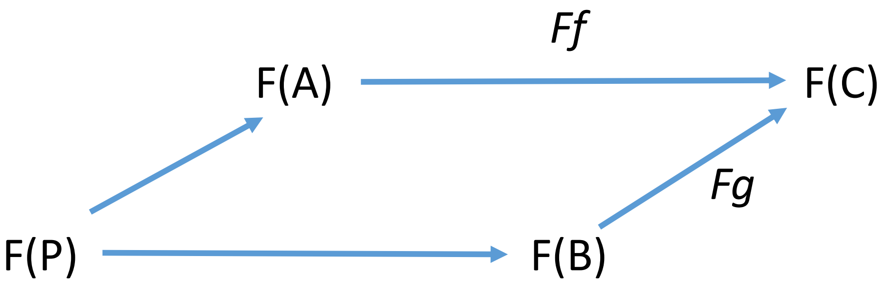
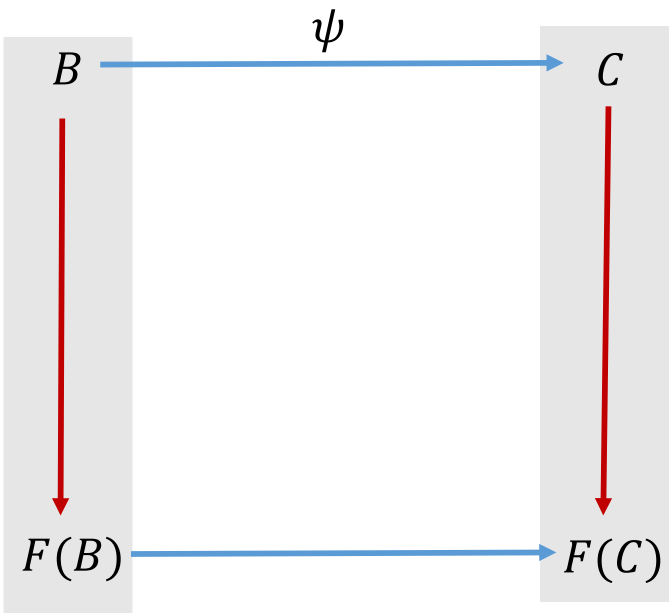


Is this a weak pullback diagram?

Observational equivalence



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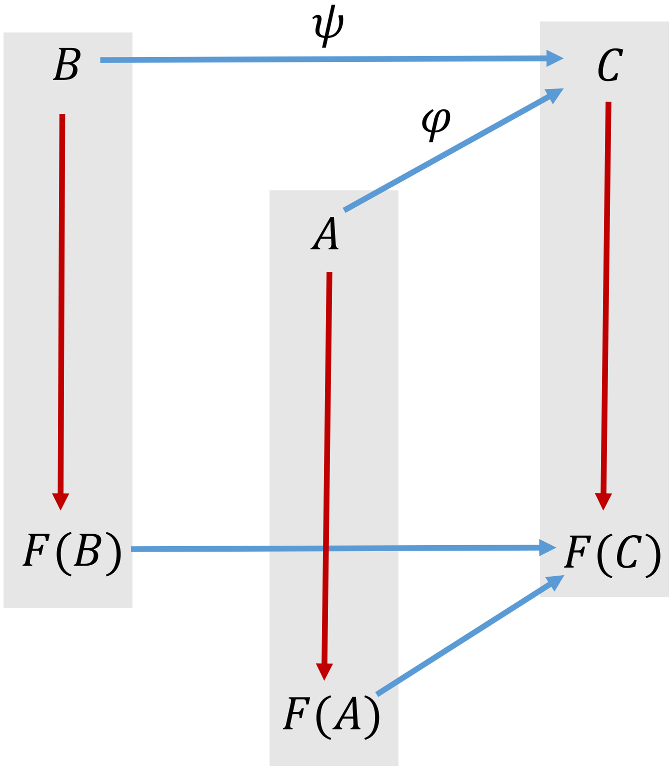

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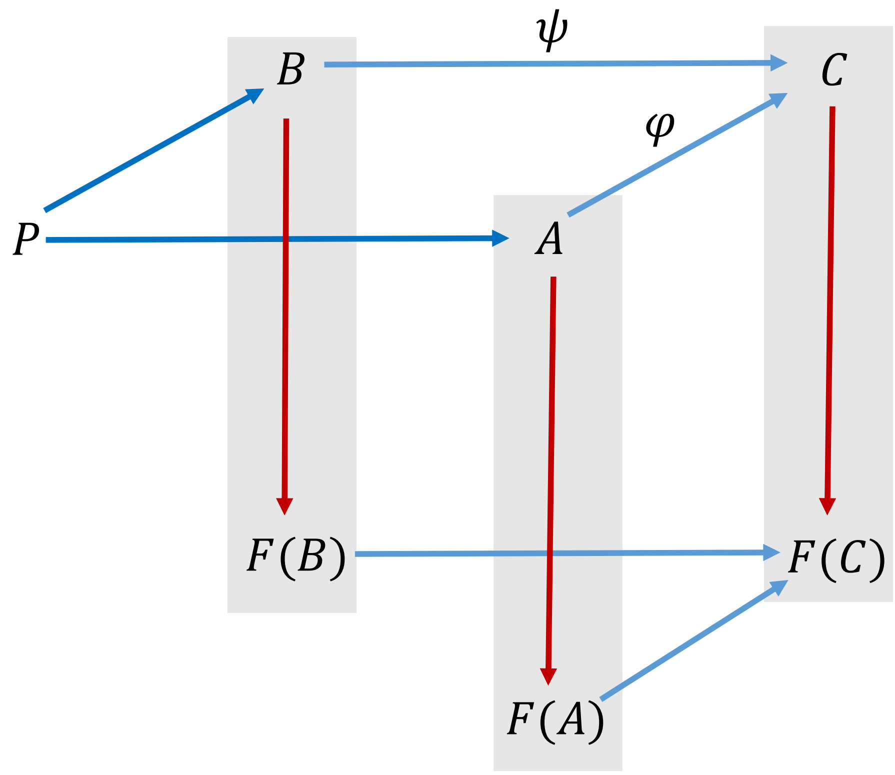

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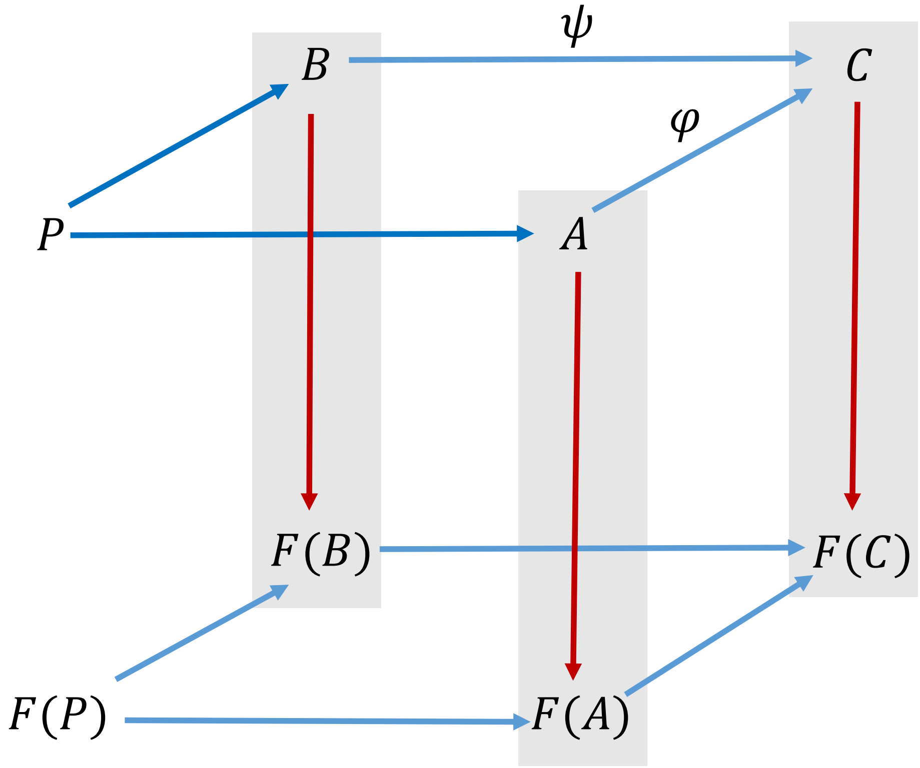

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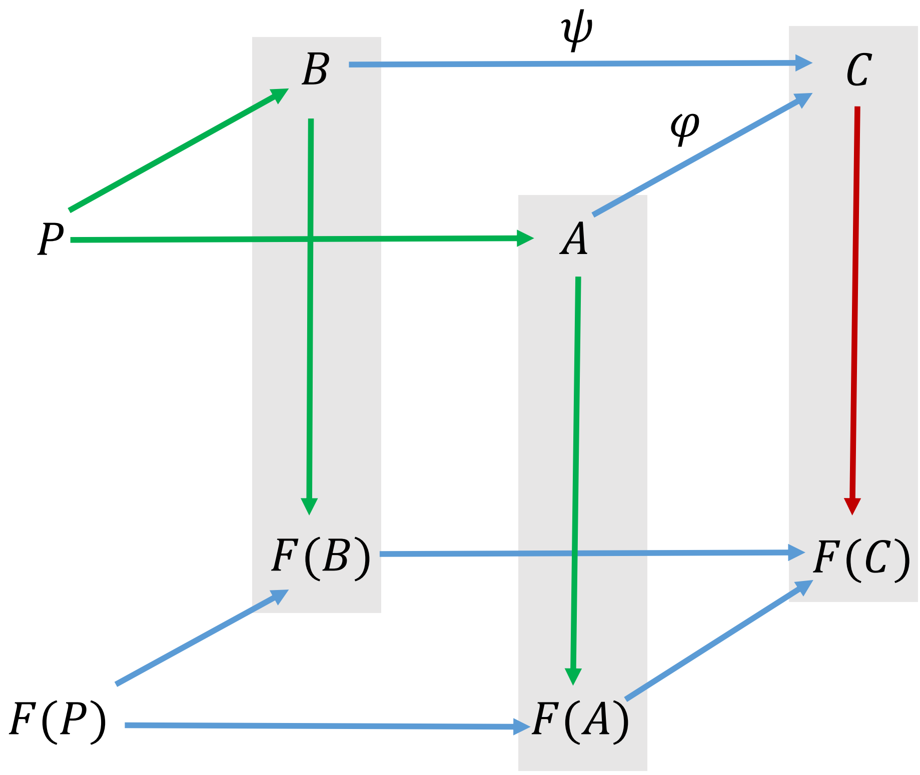
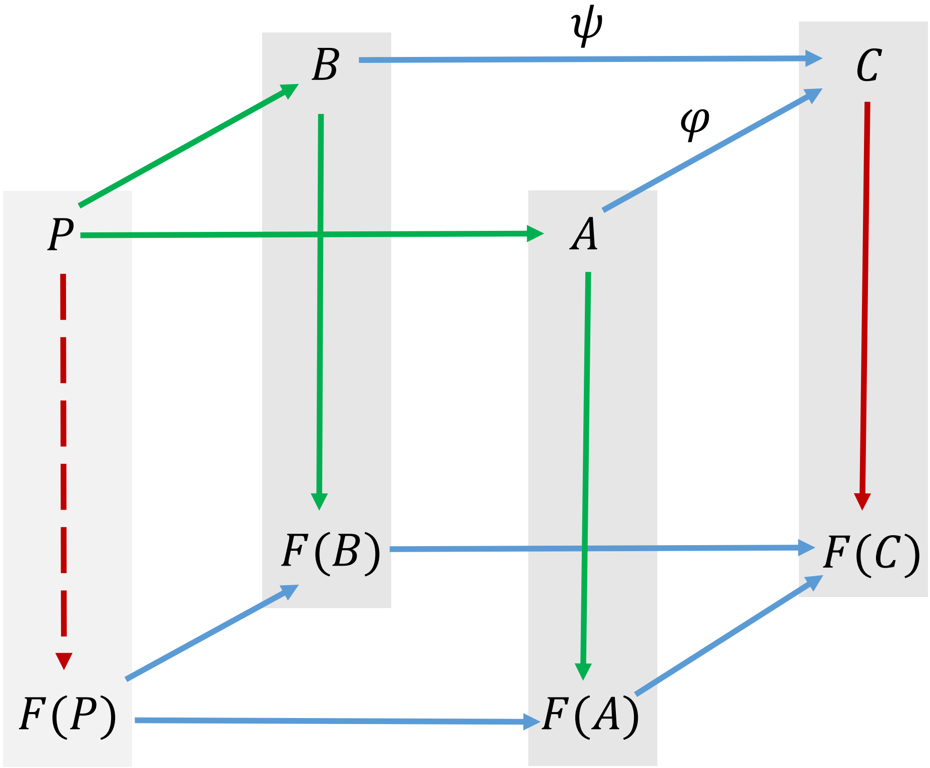


Bisimilarity

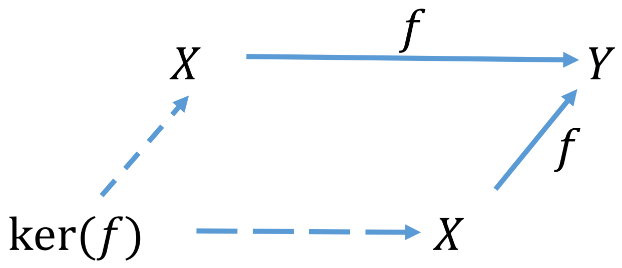
Observational equivalence …



… is bisimilarity

Weak pullback preservation

- F weakly preserves *pullbacks*, iff F preserves
  - *kernel pairs*



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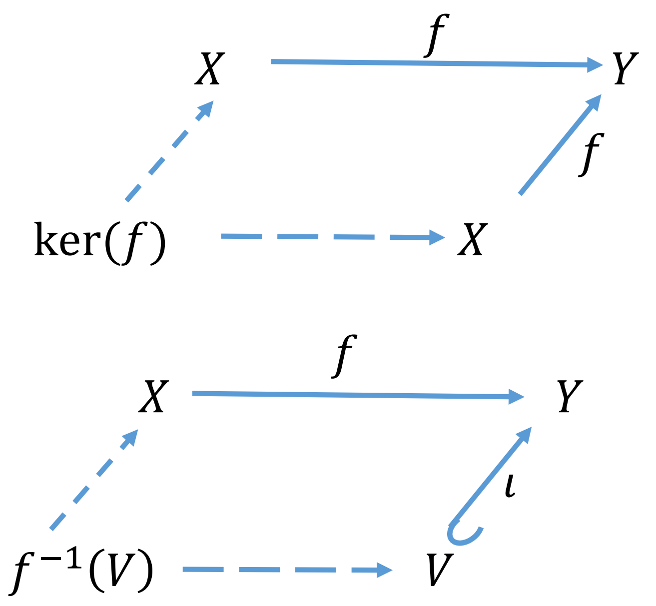

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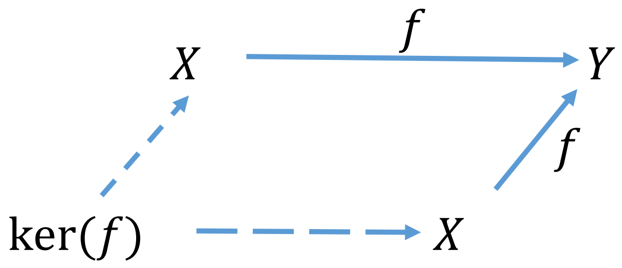

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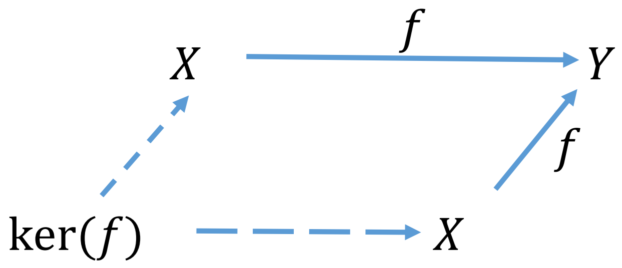
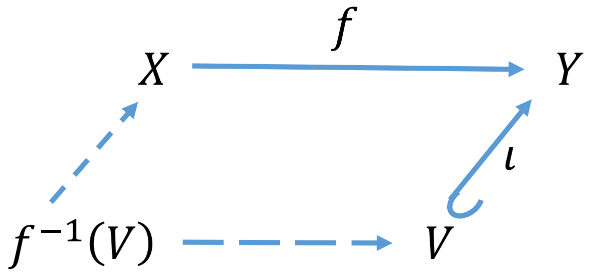


- F weakly preserves *preimages*
  - iff $HS(\mathfrak{K}) = SH(\mathfrak{K})$, for any class $\mathfrak{K}$
    (provided $|F(1)| > 1$)



Functors weakly preserving special pullbacks

- all pullbacks
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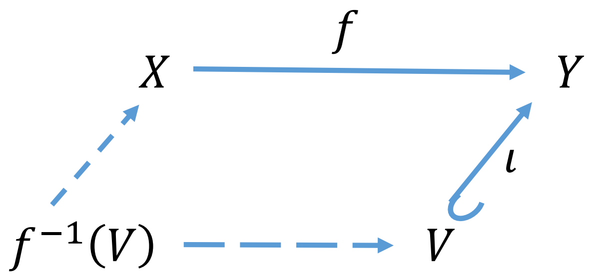

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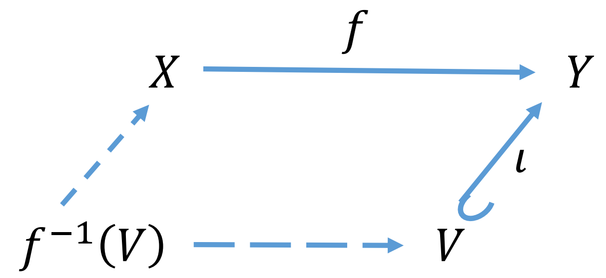
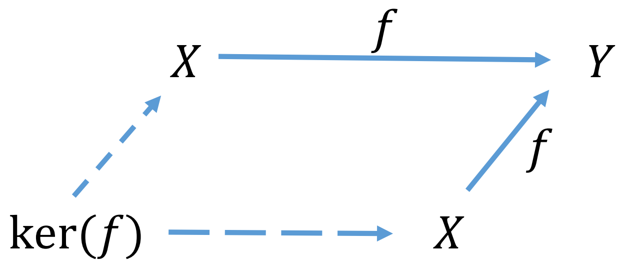


- kernel pairs
  - $F(X) = \ \dots\ 2^{2^X},\ X^2 - X + 1, Odd(X), \dots$



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Parameterizing functors by algebras

- Let $F_{\mathcal{A}}$ depend on some algebra $\mathcal{A}$

- Choose $\mathcal{A}$ so that $F_{\mathcal{A}}$ has desirable properties, e.g. (weakly) preserves
    - *kernel pairs*
    - *preimages*
    - *pullbacks*

L-fuzzy functor

1. Generalize $\mathbb{P}(X) = 2^X$

For $U \subseteq X$ put $\chi_U(x) = \begin{cases} 1, & x \in U \\ 0, & x \notin U \end{cases}$

- Replace $2 = \{0,1\}$ by a complete lattice L
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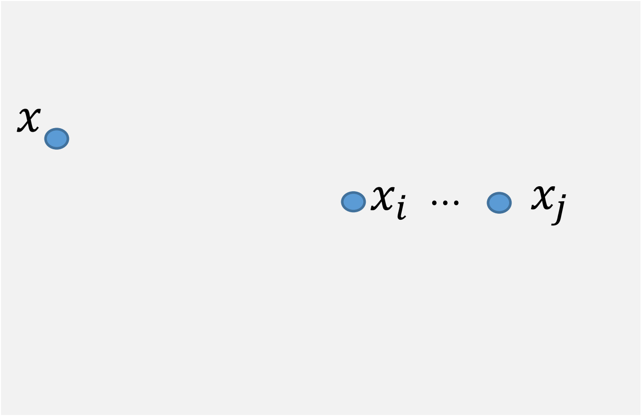

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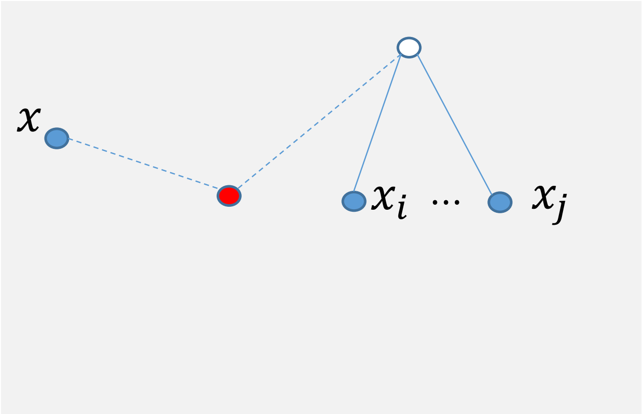

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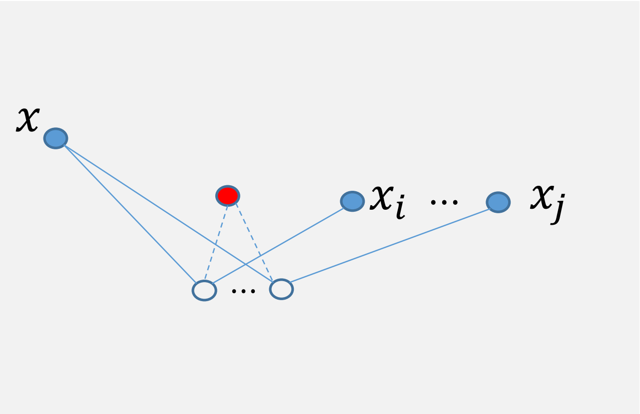

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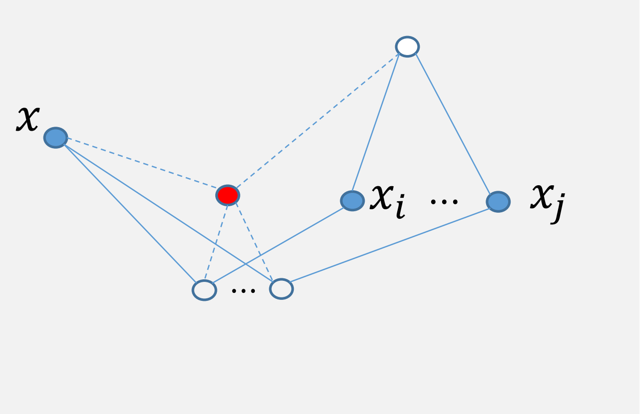

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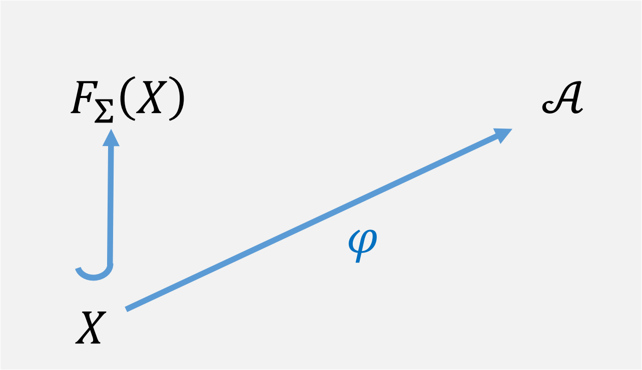

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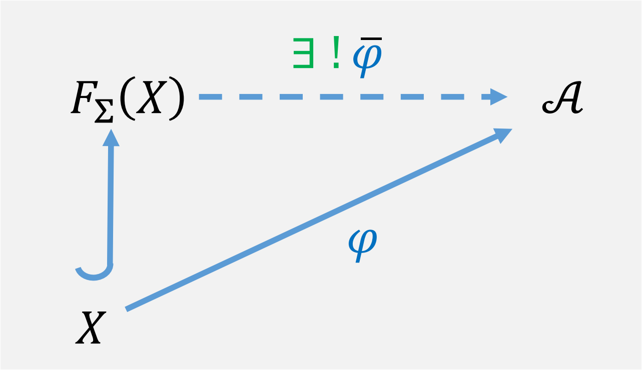

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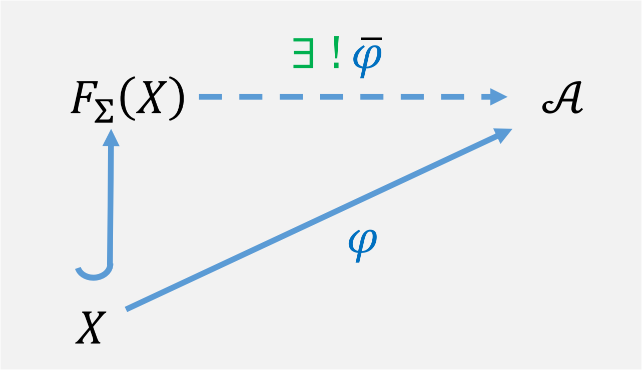

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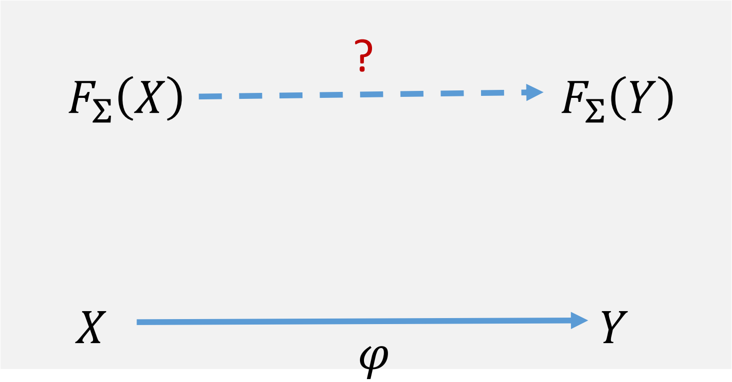

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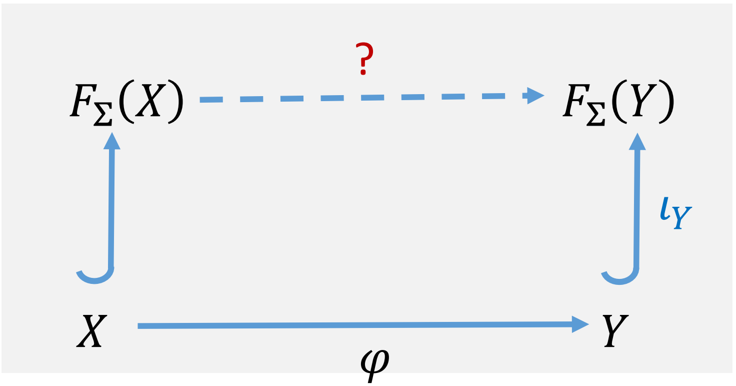

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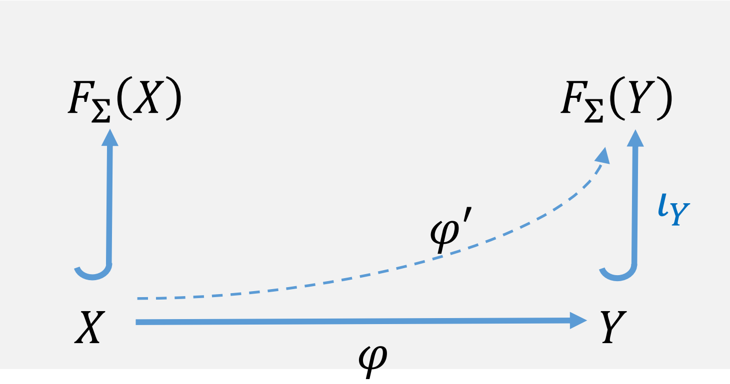

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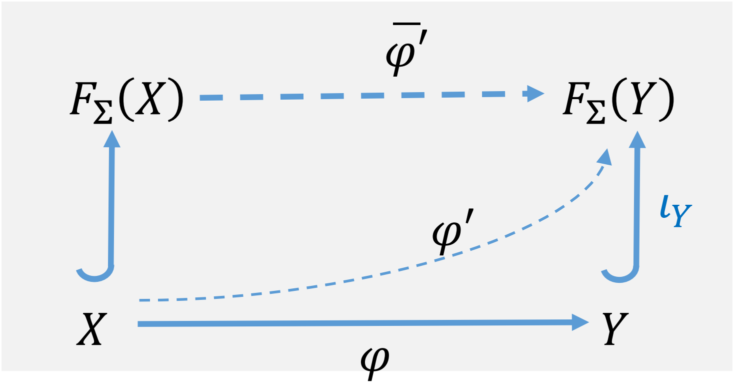

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Derivative of Σ:

$$\Sigma' \coloneqq \{p(x, \vec{z}) \approx p(y, \vec{z}) \mid p(x, \vec{v}) \text{ weakly independent of } x\}$$

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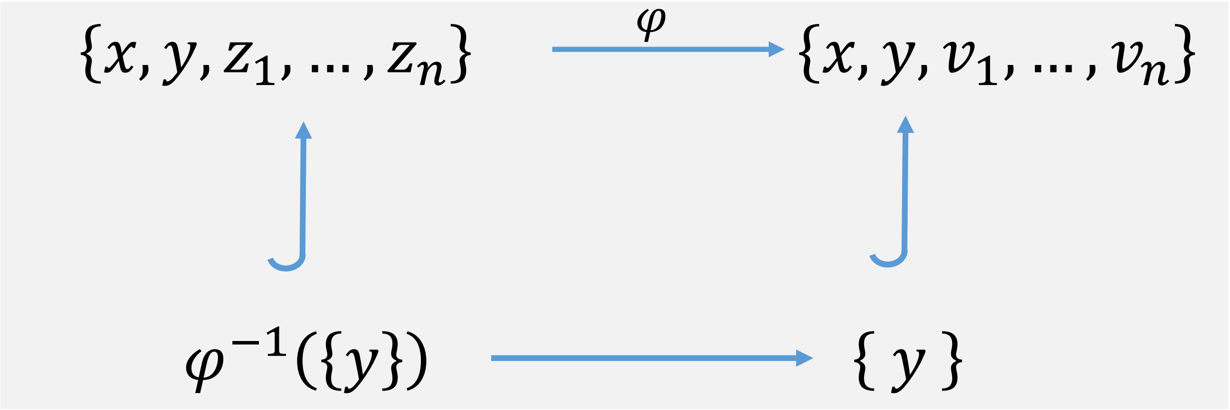

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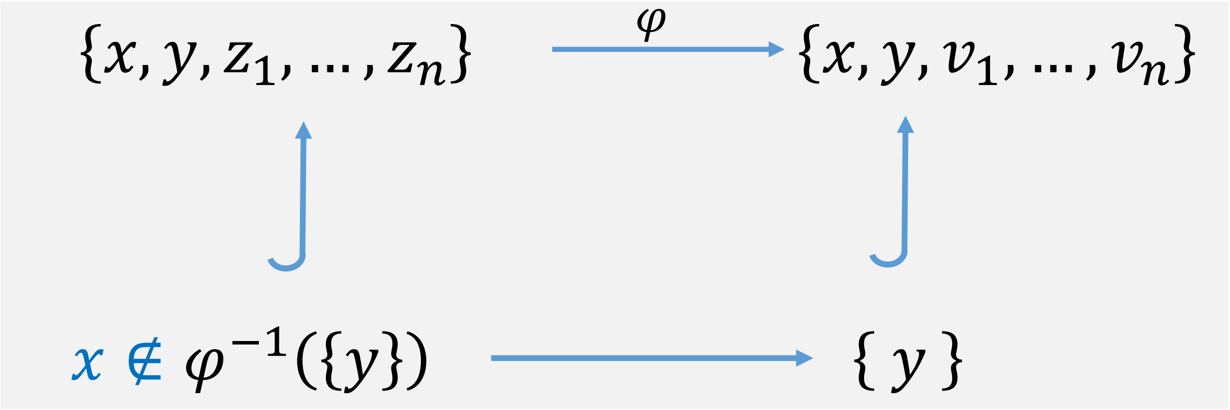

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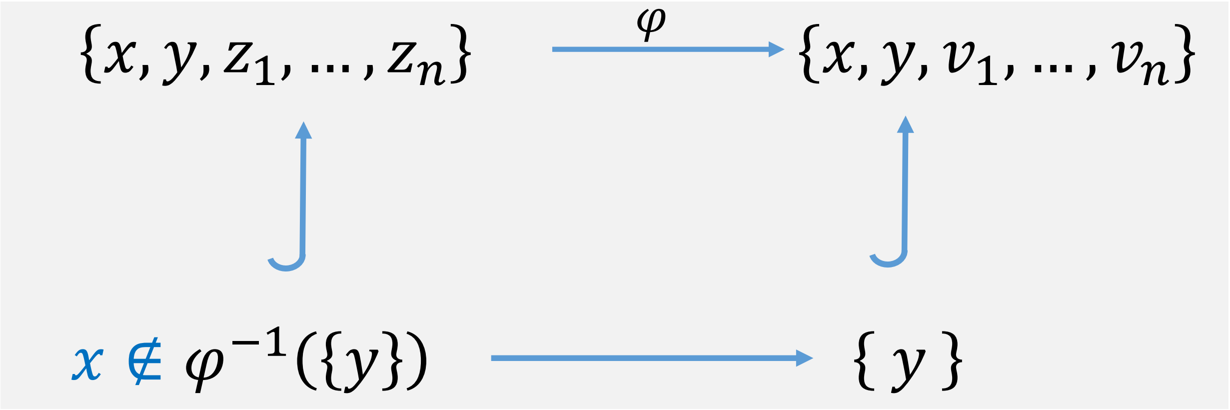

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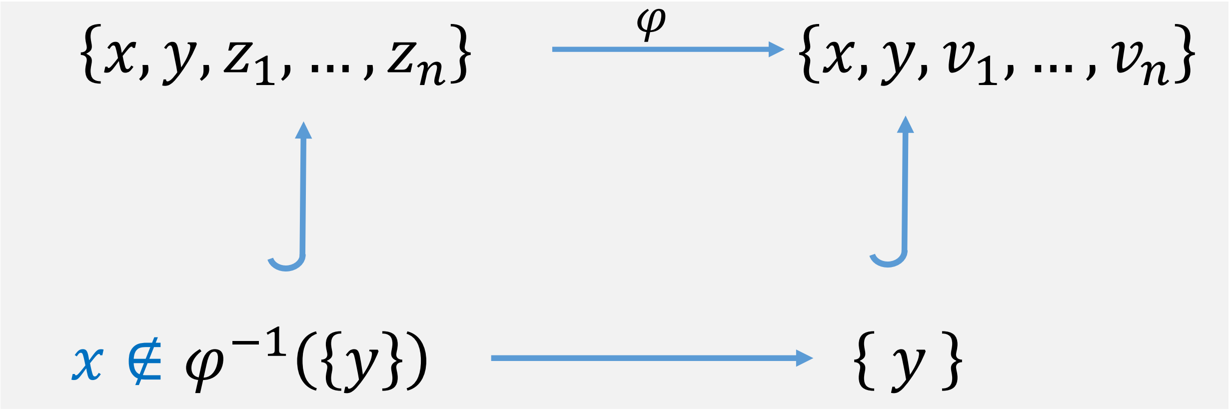

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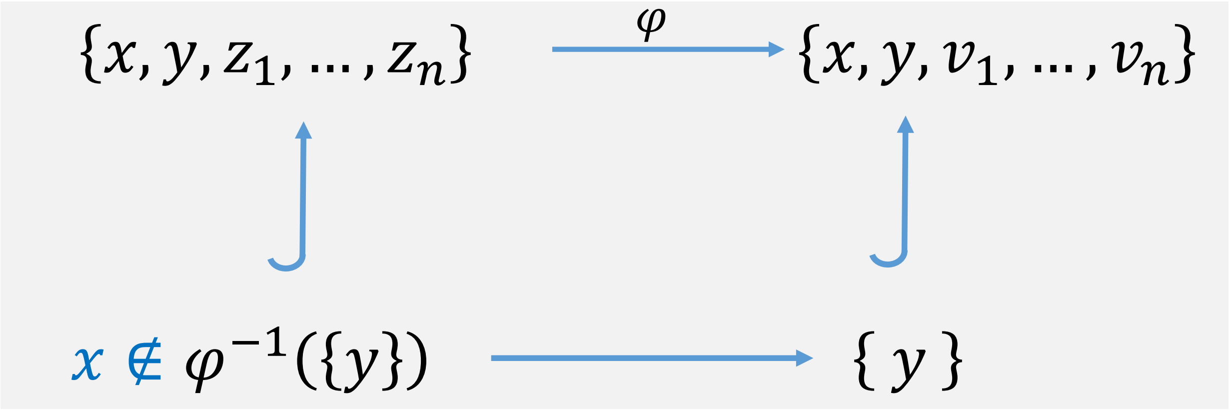

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- <u>Thm</u>: $F_\Sigma(-)$ preserves preimages iff $\Sigma \vdash \Sigma'$

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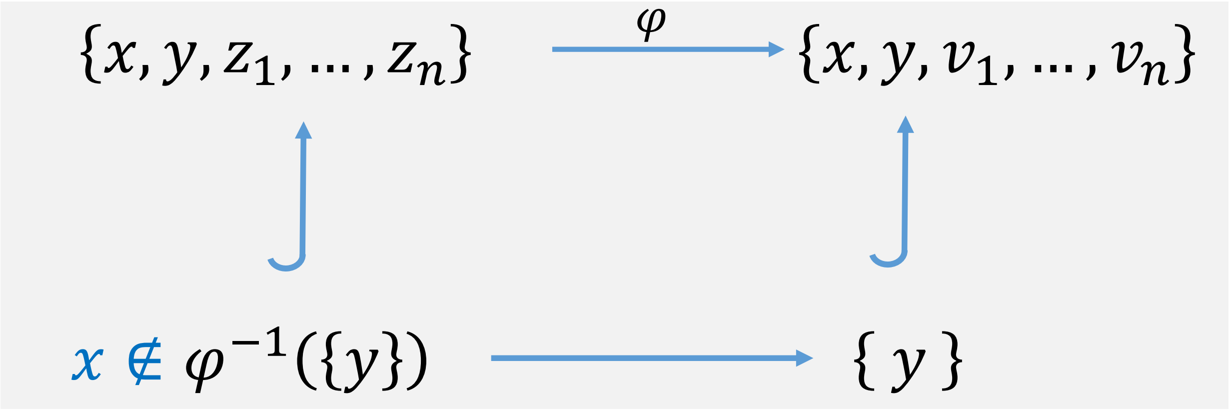

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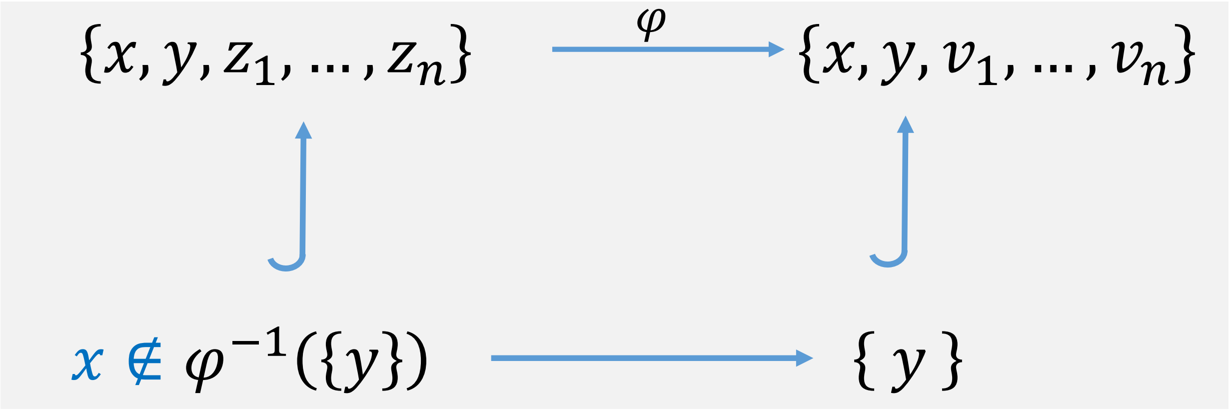
$$\Rightarrow p(x, z_1, \dots, z_n) \in \bar{\varphi}^{-1}(F_\Sigma(\{y\}))$$



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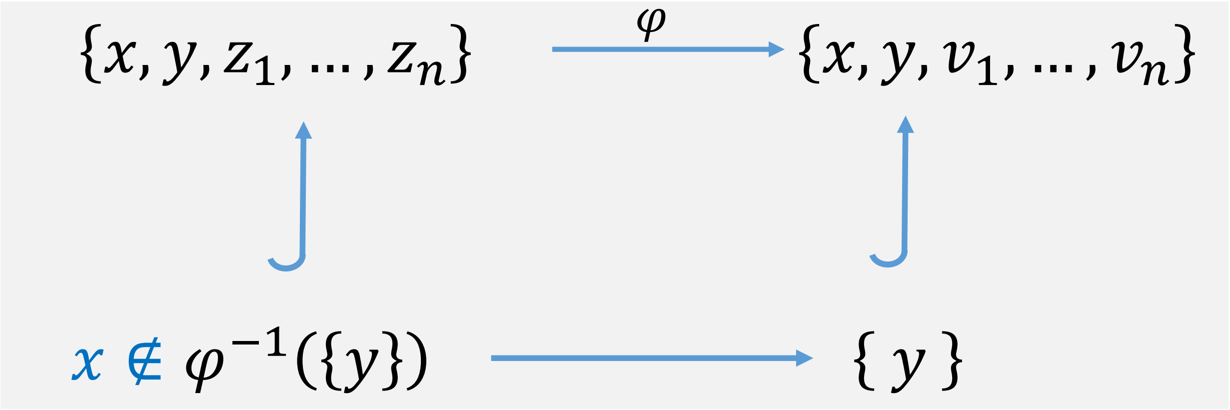

$$\begin{aligned} \Rightarrow p(x, z_1, \dots, z_n) &\in \bar{\varphi}^{-1}(F_\Sigma(\{y\})) \\ &= F_\Sigma(\varphi^{-1}\{y\}) \end{aligned}$$

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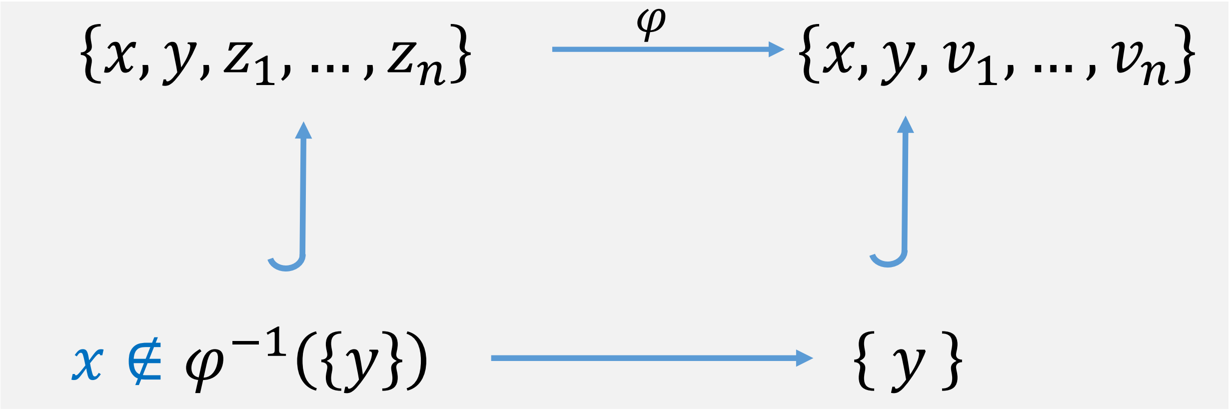

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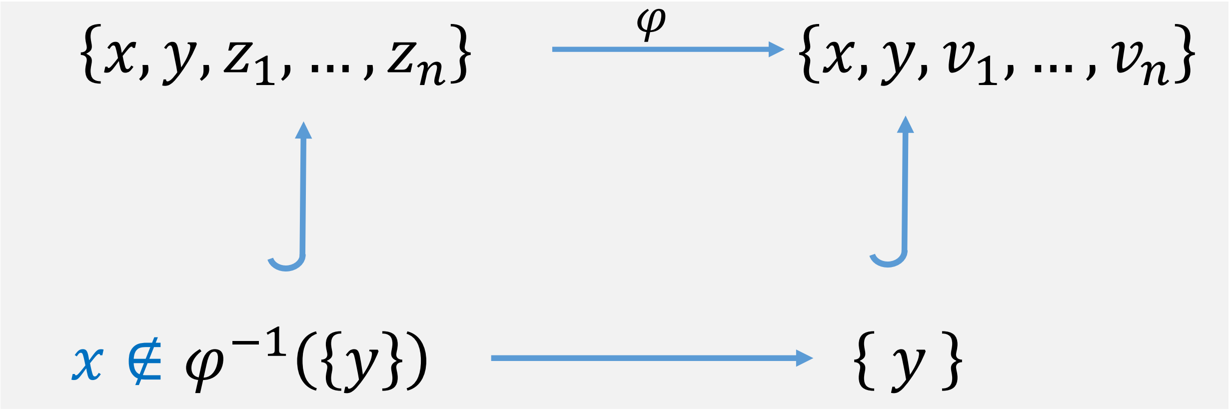

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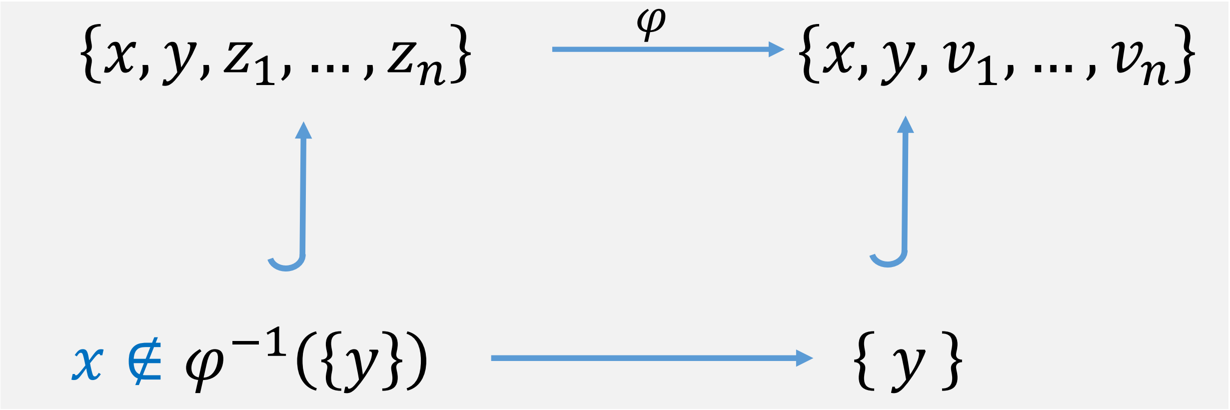

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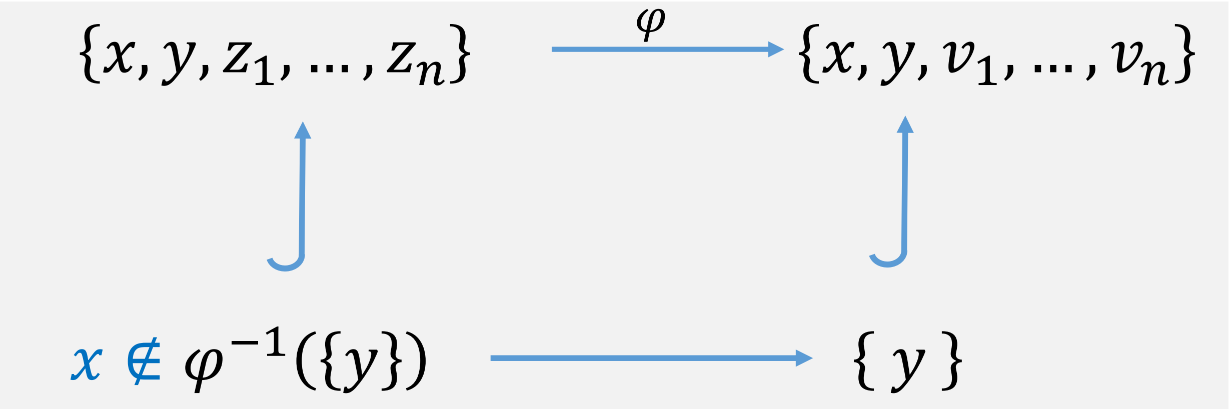

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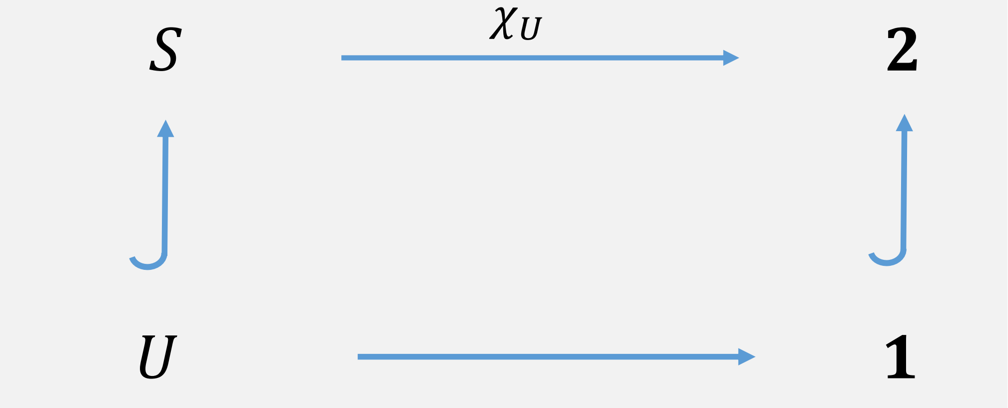

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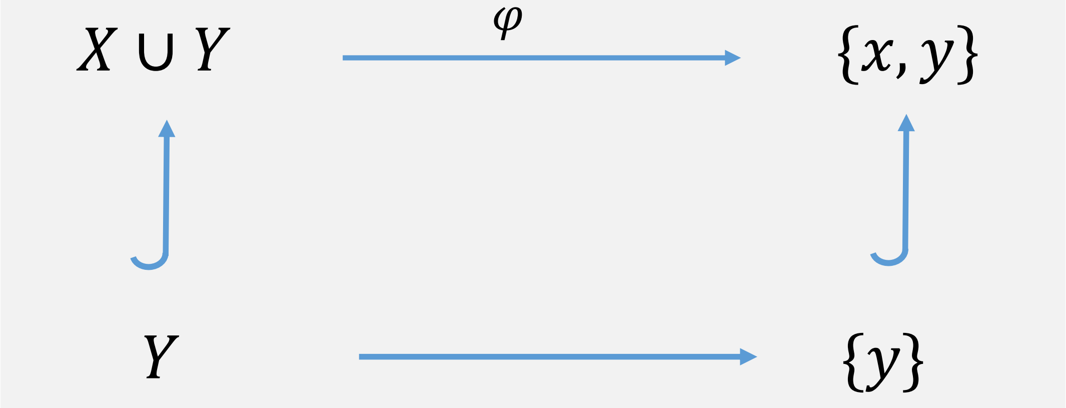

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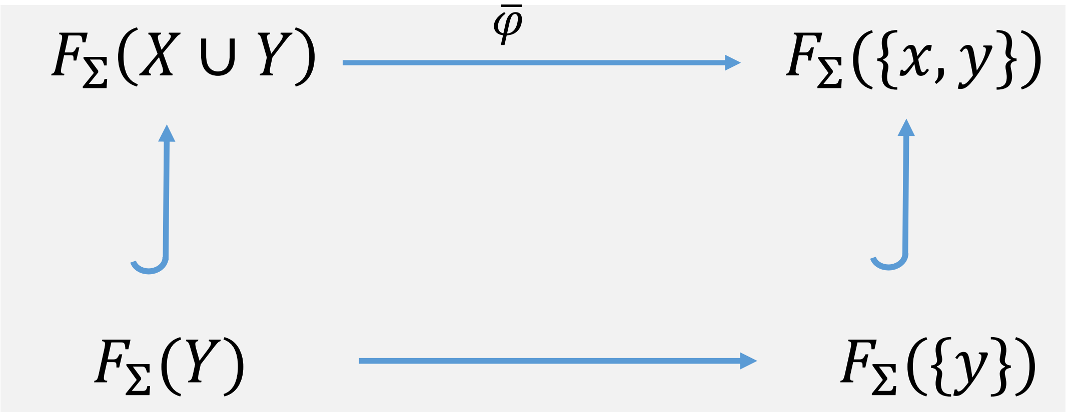

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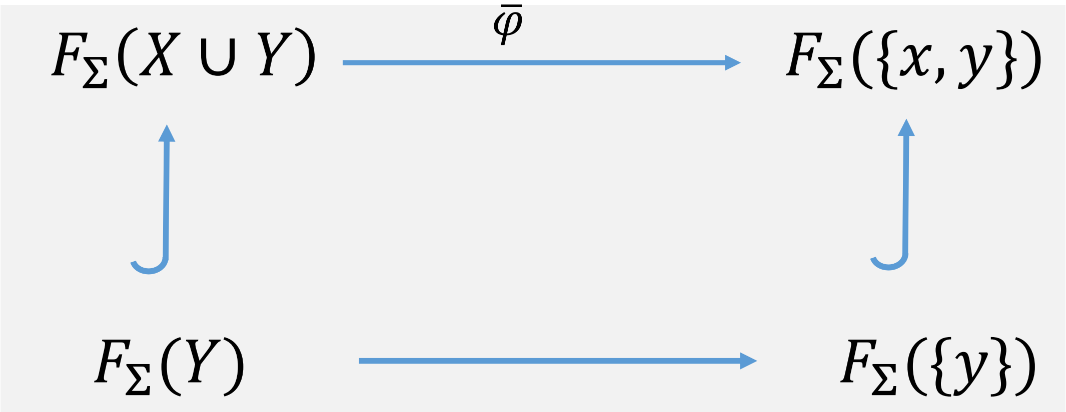

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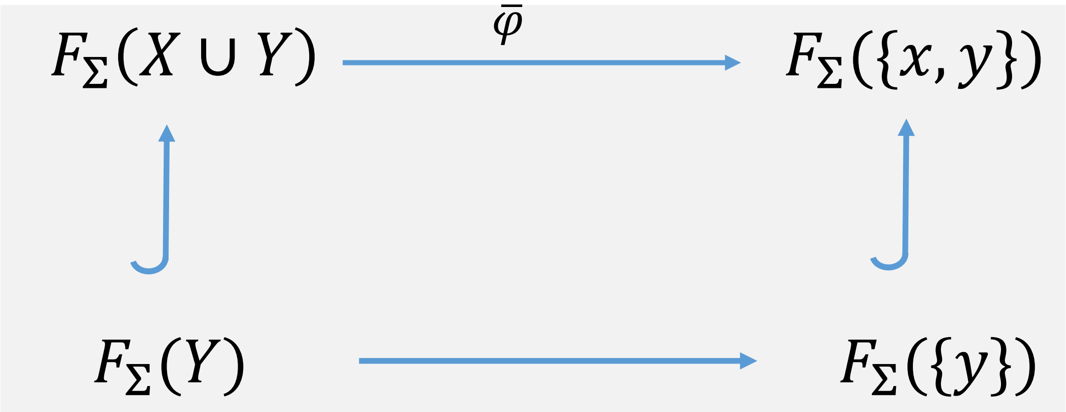
$$p \quad = \quad p(x_1, \dots, x_n, y_1, \dots, y_m)$$



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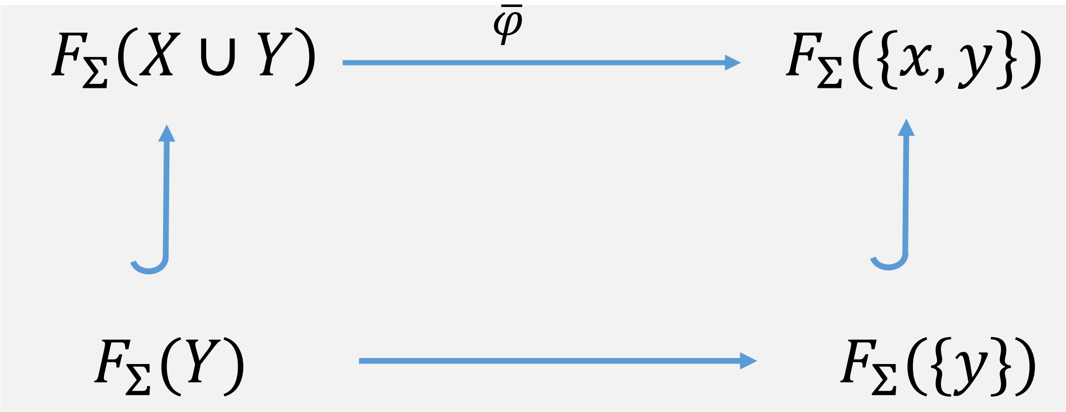

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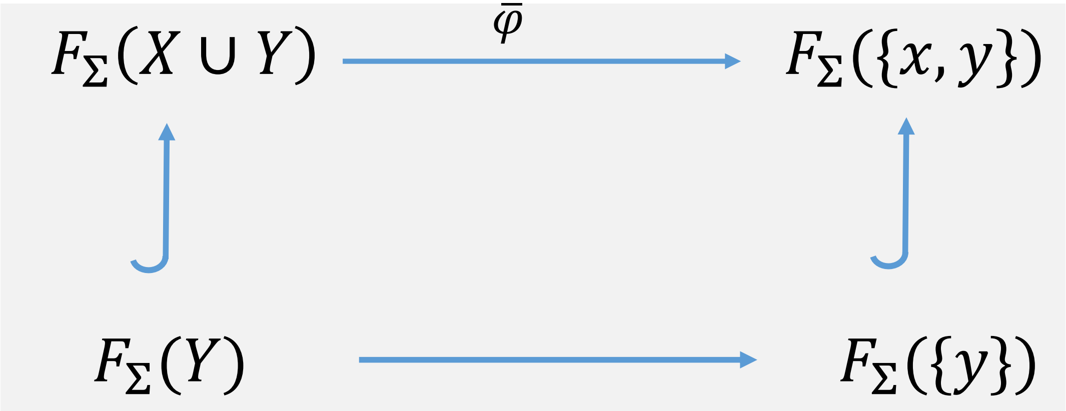
$$\bar{\varphi} p = \bar{\varphi} p(x_1, \dots, x_n, y_1, \dots, y_m) = p(x, \dots, x, y, \dots, y)$$



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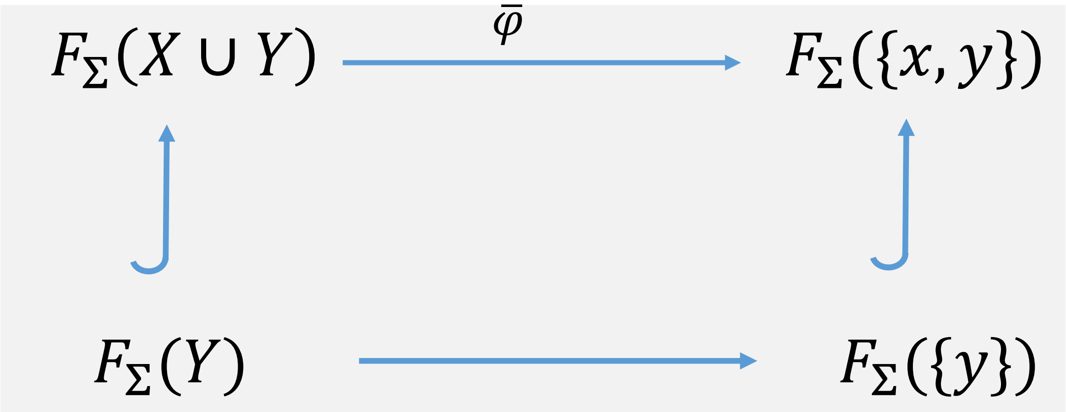

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$$\approx$$
$$\approx$$
$$\approx$$
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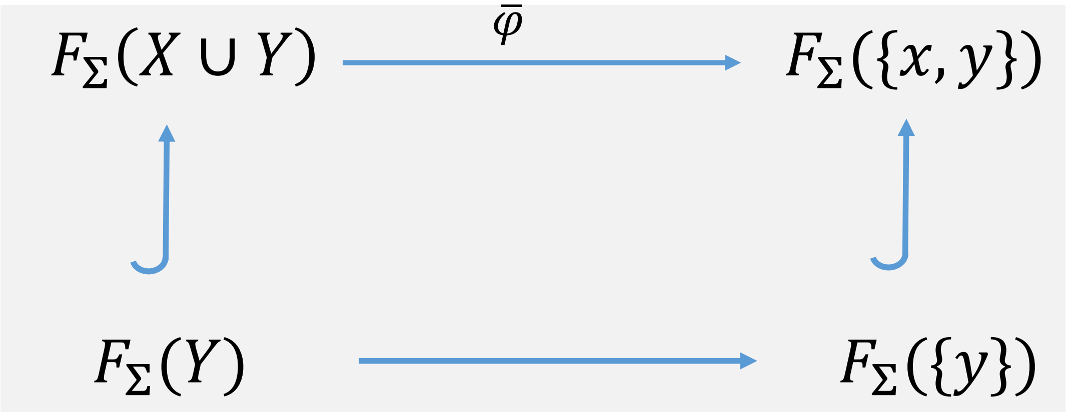

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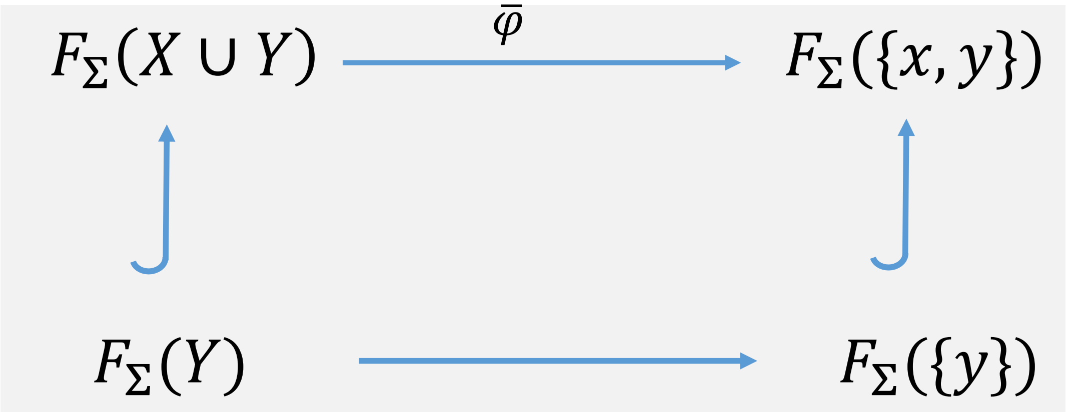

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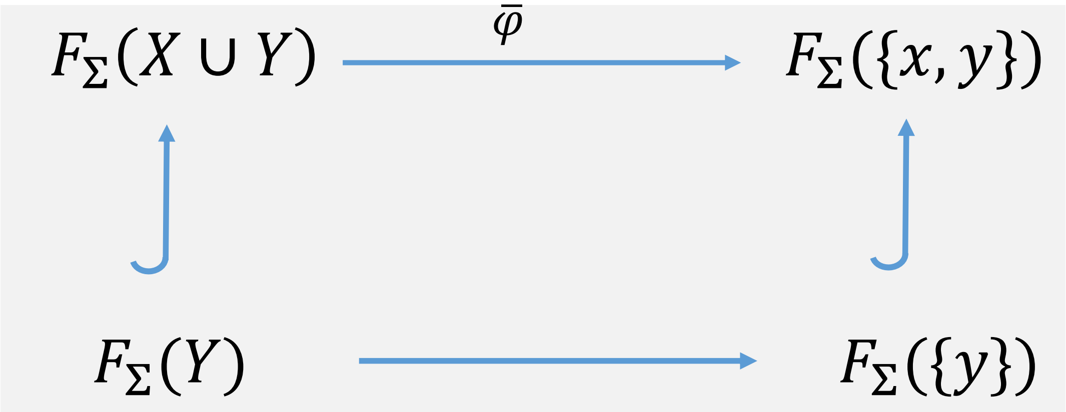

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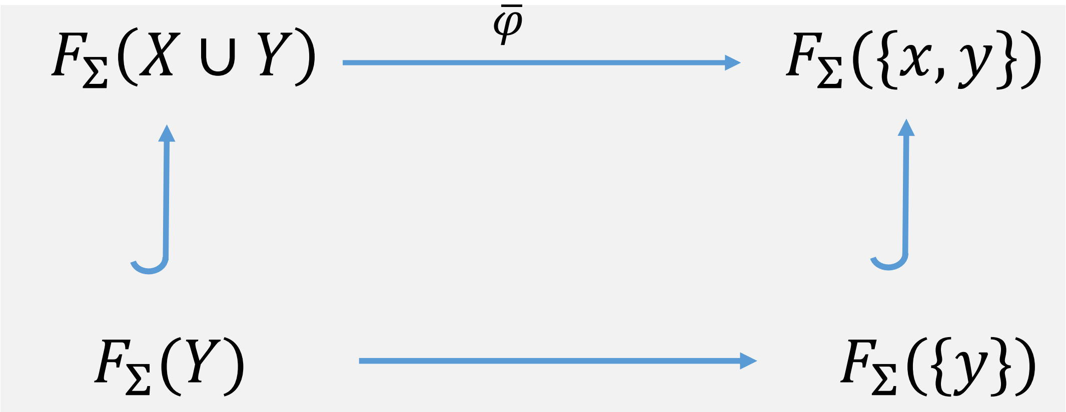

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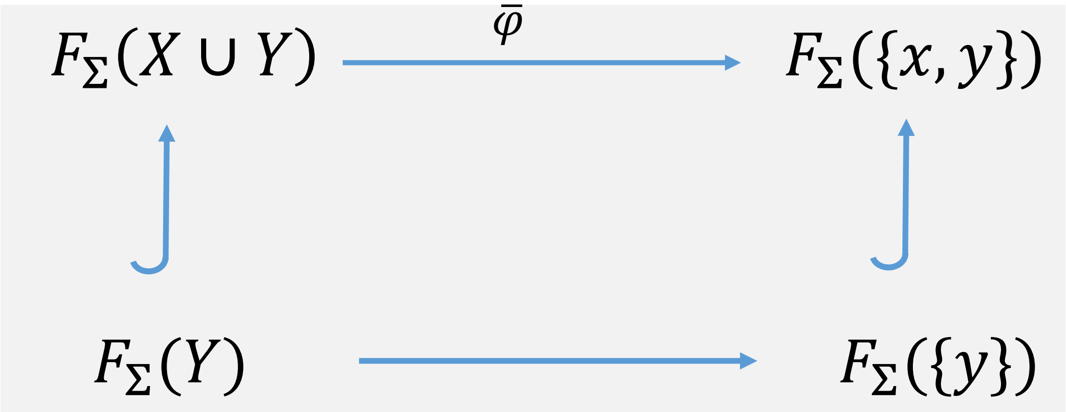

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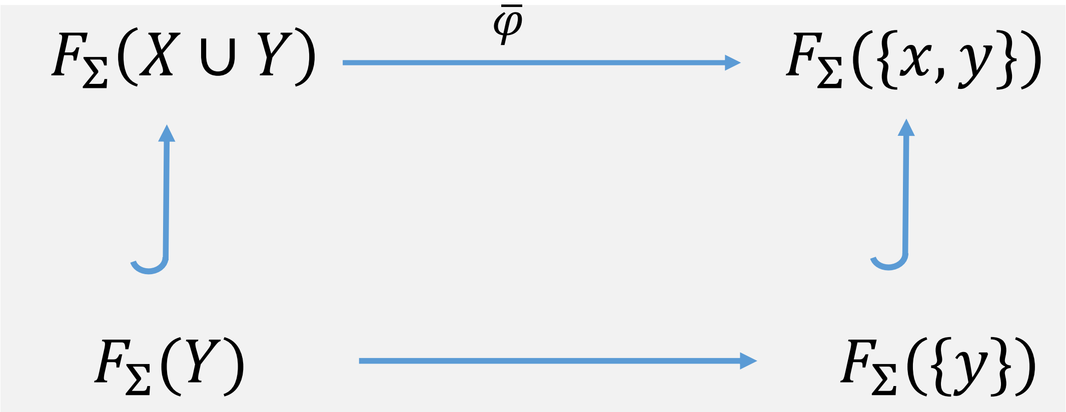

$$\begin{aligned} p(x_1, x_2, \dots, x_n, y_1, \dots, y_m) &\approx p(z_1, x_2, \dots, x_n, y_1, \dots, y_m) \\ &\approx p(z_1, z_2, \dots, x_n, y_1, \dots, y_m) \\ &\approx \cdots \\ &\approx p(z_1, z_2, \dots, z_n, y_1, \dots, y_m) \approx p(y, y, \dots, y, y_1, \dots, y_m) \in F_\Sigma(Y) \checkmark \end{aligned}$$

Intro

State based systems

Functors and Coalgebras

Functor properties

Weak Pullback Preservation

Functors parameterized by algebras

Free-algebra functor

Preimage preservation

Weak kernel preservation

Conclusion

- and Breaking News

Mal‘cev term

- Variety $\mathcal{V}(\Sigma)$ = all algebras satisfying Σ

Mal‘cev term

А. И. Мальцев
1909-1967

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- *Mal‘cev variety* $:\Leftrightarrow \exists m.$

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Groups:
$$m(x,y,z) = x * y^{-1} * z$$
Quasigroups:
$$m(x,y,z) = (x/(y\backslash y)) * (y\backslash z)$$
Rings:
$$m(x,y,z) = x - y + z$$
…

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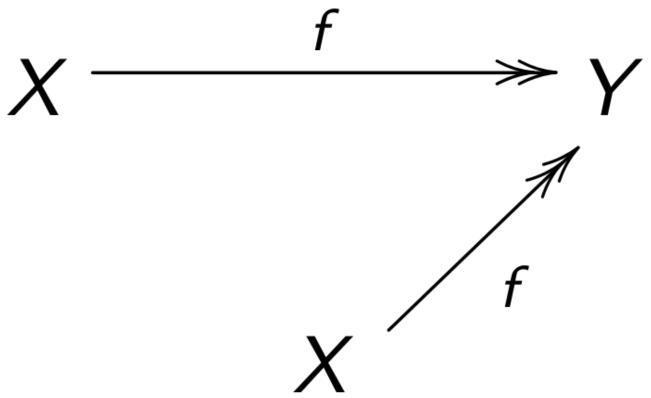
… the above, and also …

- *implication* algebras
- all *congruence regular* algebras

…

When does $F_\Sigma(-)$ preserve kernel pairs

- Thm: If $\mathcal{V}$ is Mal'cev, then $F_\Sigma(-)$ weakly preserves kernel pairs



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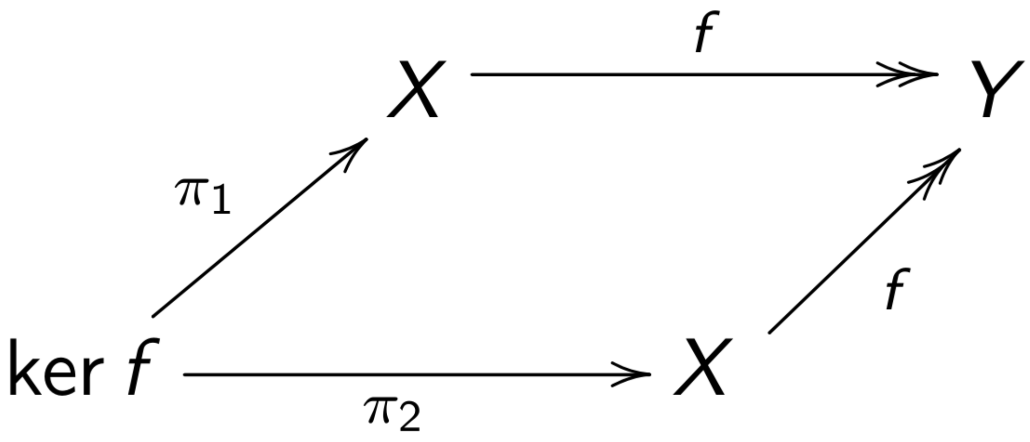
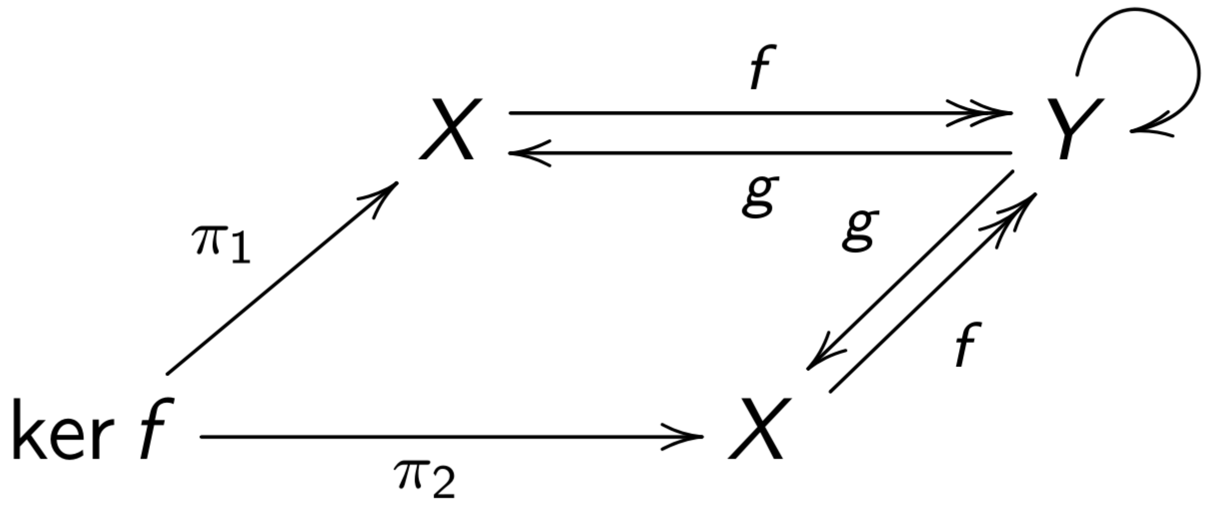


When does $F_\Sigma(-)$ preserve kernel pairs

- <u>Thm</u>: If $\mathcal{V}$ is Mal'cev, then $F_\Sigma(-)$ weakly preserves kernel pairs



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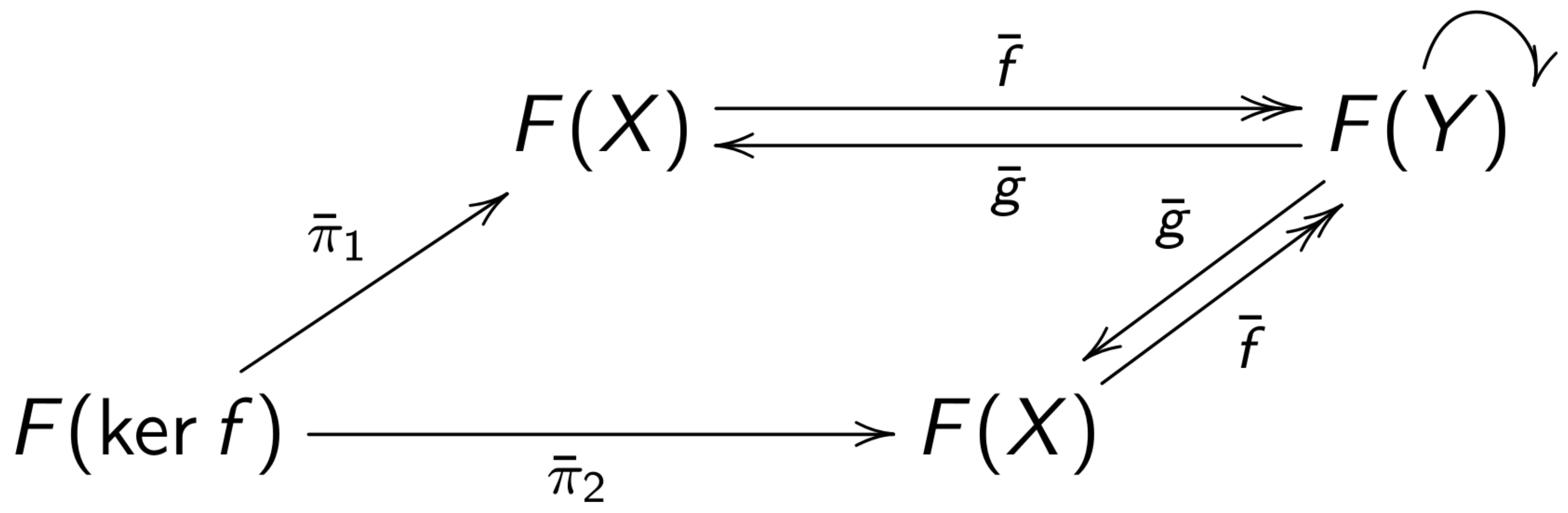

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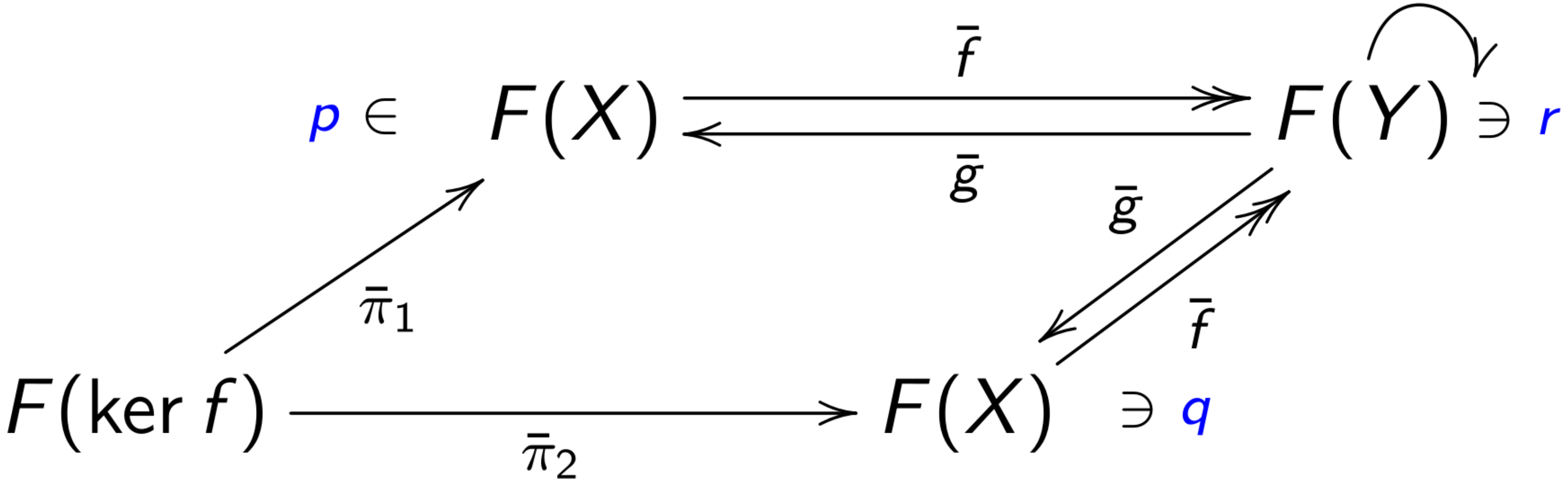

When does $F_\Sigma(-)$ preserve kernel pairs

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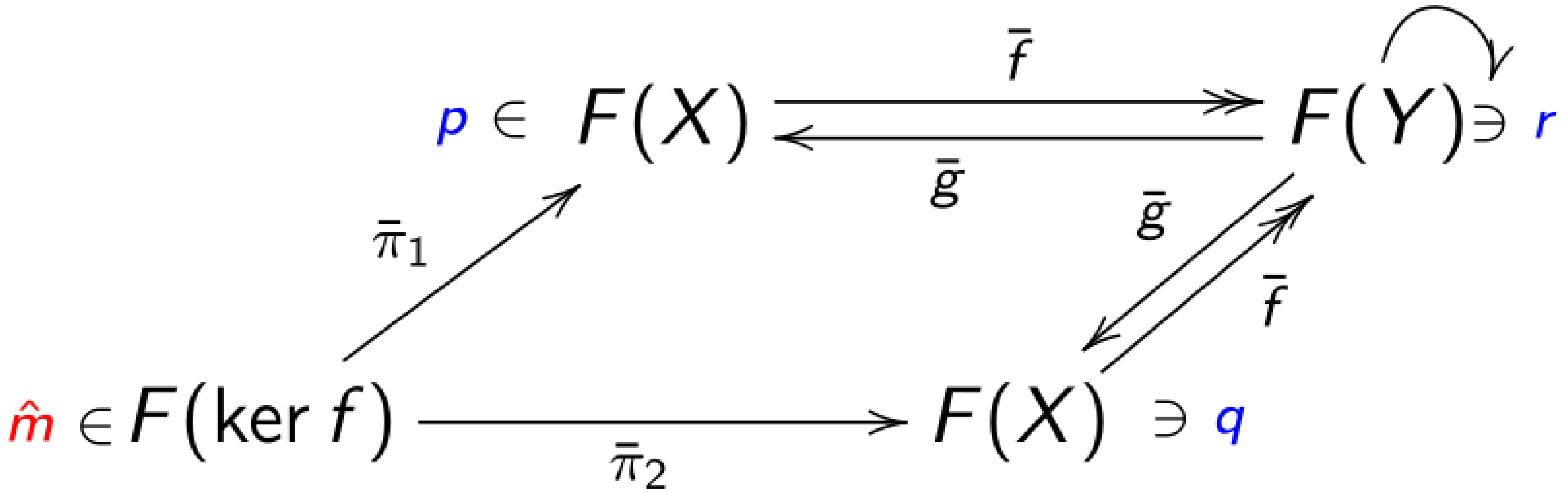
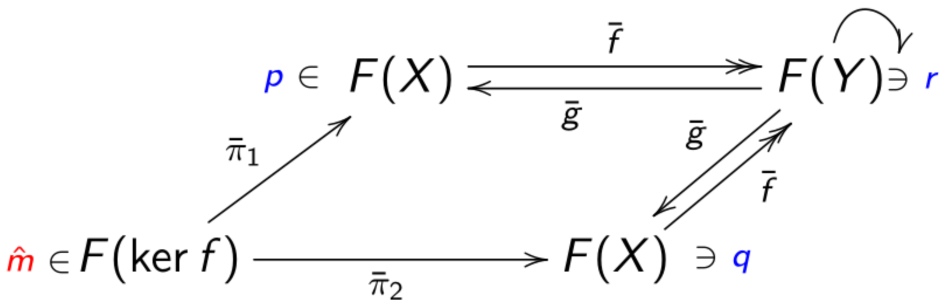


Mal‘cev ⇒ weak kernel preservation

- <u>Thm</u>: If $\mathcal{V}$ is Mal‘cev, then $F_\Sigma(-)$ weakly preserves kernel pairs

$\hat{m} := m(\hat{p}, \hat{r}, \hat{q})$ where
$\hat{p} := p(\langle x_1, gfx_1\rangle, \ldots, \langle x_n, gfx_n\rangle)$
$\hat{r} = r(\langle gy_1, gy_1\rangle, \ldots, \langle gy_k, gy_k\rangle)$,
$\hat{q} := q(\langle gfx'_1, x'_1\rangle, \ldots, \langle gfx'_m, x'_m\rangle)$



$$\bar{\pi}_1(\hat{m}) =$$
$$=$$
$$=$$
$$=$$
$$=$$
$$= \qquad = p$$

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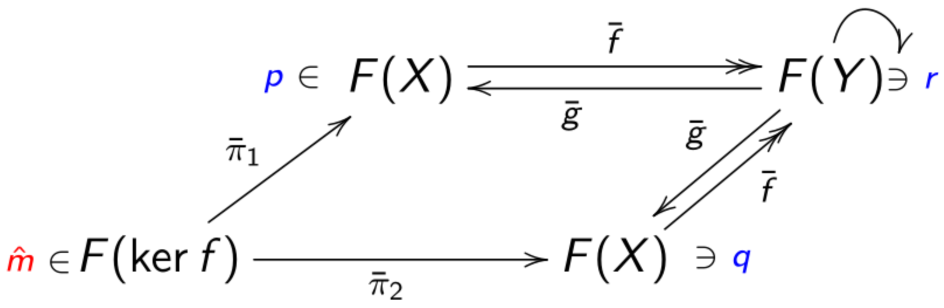

$$\bar{\pi}_1(\hat{m}) = \bar{\pi}_1(m(\hat{p}, \hat{r}, \hat{q}))$$
$=$
$=$
$=$
$=$
$= \qquad = p$

Mal'cev ⇒ weak kernel preservation

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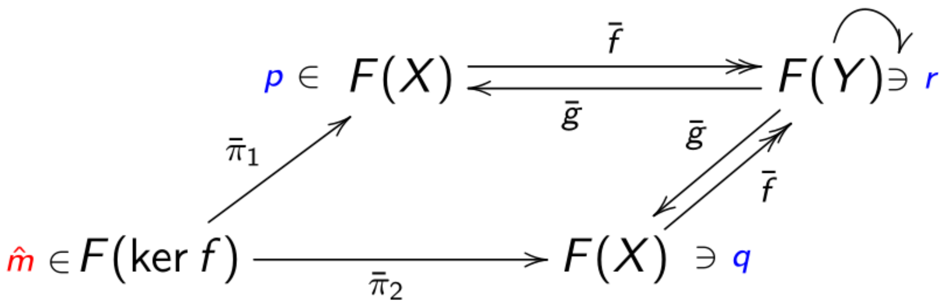

$$
\begin{aligned}
\bar{\pi}_1(\hat{m}) &= \bar{\pi}_1(m(\hat{p}, \hat{r}, \hat{q})) \\
&= m(\bar{\pi}_1\hat{p}, \bar{\pi}_1\hat{r}, \bar{\pi}_1\hat{q}) \\
&= \\
&= \\
&= \\
&= \qquad\qquad = p
\end{aligned}
$$

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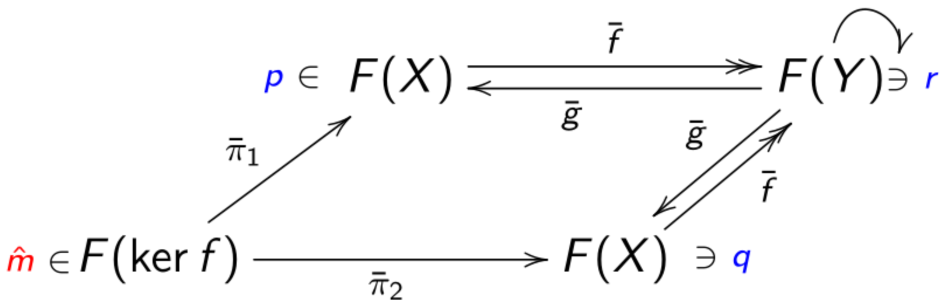

$$
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\bar{\pi}_1(\hat{m}) &= \bar{\pi}_1(m(\hat{p}, \hat{r}, \hat{q})) \\
&= m(\bar{\pi}_1\hat{p}, \bar{\pi}_1\hat{r}, \bar{\pi}_1\hat{q}) \\
&= m(p(x_1 \ldots, x_n), r(gy_1, \ldots, gy_k), q(gfx'_m, \ldots, gfx'_m)) \\
&= \\
&= \\
&= \qquad\qquad\qquad = p
\end{aligned}
$$

Mal‘cev ⇒ weak kernel preservation

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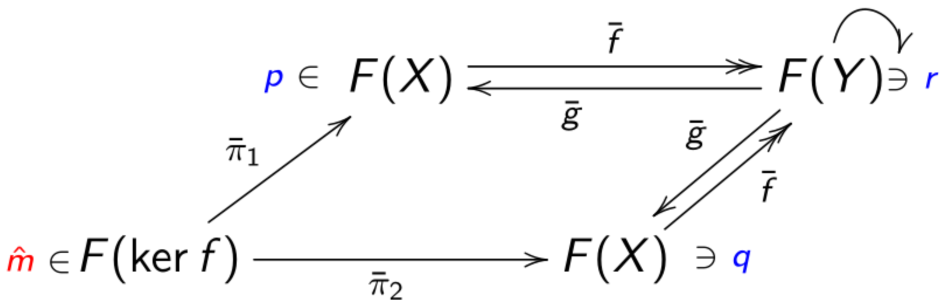

$$\begin{aligned}
\bar{\pi}_1(\hat{m}) &= \bar{\pi}_1(m(\hat{p}, \hat{r}, \hat{q})) \\
&= m(\bar{\pi}_1\hat{p}, \bar{\pi}_1\hat{r}, \bar{\pi}_1\hat{q}) \\
&= m(p(x_1 \ldots, x_n), r(gy_1, \ldots, gy_k), q(gfx'_m, \ldots, gfx'_m)) \\
&= m(p, \bar{g}r(y_1, \ldots, y_k), \bar{g}q(fx'_m, \ldots, fx'_m)) \\
&= \\
&= \qquad\qquad = p
\end{aligned}$$

Mal‘cev ⇒ weak kernel preservation

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$\hat{m} := m(\hat{p}, \hat{r}, \hat{q})$ where
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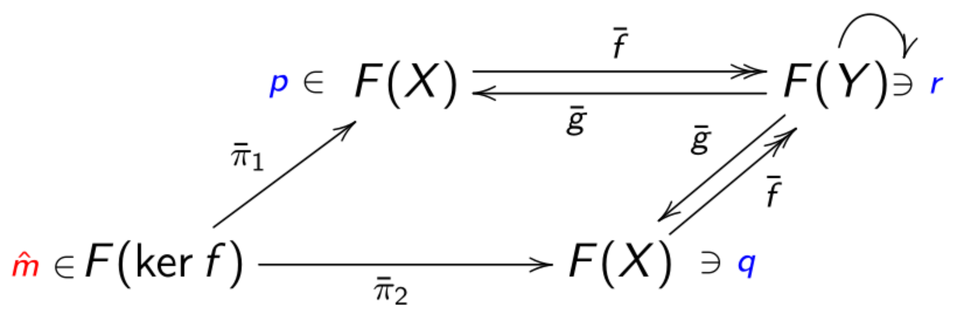
$\hat{r} = r(\langle gy_1, gy_1\rangle, \ldots, \langle gy_k, gy_k\rangle)$,
$\hat{q} := q(\langle gfx_1', x_1'\rangle, \ldots, \langle gfx_m', x_m'\rangle)$

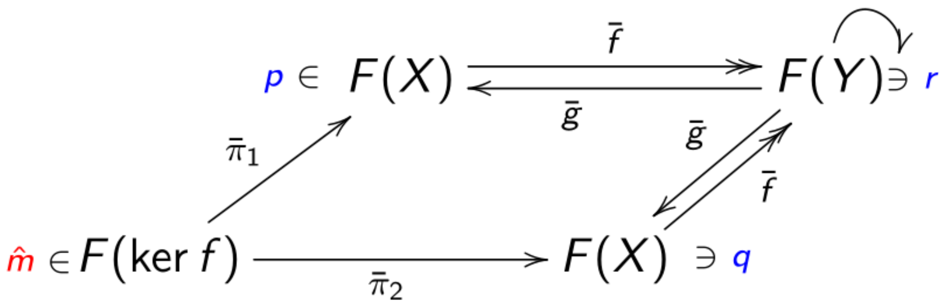


$$
\begin{aligned}
\bar\pi_1(\hat{m}) &= \bar\pi_1(m(\hat{p}, \hat{r}, \hat{q})) \\
&= m(\bar\pi_1\hat{p}, \bar\pi_1\hat{r}, \bar\pi_1\hat{q}) \\
&= m(p(x_1 \ldots, x_n), r(gy_1, \ldots, gy_k), q(gfx_m', \ldots, gfx_m')) \\
&= m(p, \bar{g}r(y_1, \ldots, y_k), \bar{g}q(fx_m', \ldots, fx_m')) \\
&= m(p, \bar{g}r, \bar{g}\bar{f}q(x_m', \ldots, x_m')) \\
&= \qquad\qquad = p
\end{aligned}
$$

Mal‘cev ⇒ weak kernel preservation

- <u>Thm</u>: If $\mathcal{V}$ is Mal‘cev, then $F_\Sigma(-)$ weakly preserves kernel pairs

$$
\begin{aligned}
\hat{m} &:= m(\hat{p}, \hat{r}, \hat{q}) \text{ where} \\
\hat{p} &:= p(\langle x_1, gfx_1\rangle, \ldots, \langle x_n, gfx_n\rangle) \\
\hat{r} &= r(\langle gy_1, gy_1\rangle, \ldots, \langle gy_k, gy_k\rangle), \\
\hat{q} &:= q(\langle gfx'_1, x'_1\rangle, \ldots, \langle gfx'_m, x'_m\rangle)
\end{aligned}
$$



$$
\begin{aligned}
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&= m(\bar{\pi}_1\hat{p}, \bar{\pi}_1\hat{r}, \bar{\pi}_1\hat{q}) \\
&= m(p(x_1 \ldots, x_n), r(gy_1, \ldots, gy_k), q(gfx'_m, \ldots, gfx'_m)) \\
&= m(p, \bar{g}r(y_1, \ldots, y_k), \bar{g}q(fx'_m, \ldots, fx'_m)) \\
&= m(p, \bar{g}r, \bar{g}\bar{f}q(x'_m, \ldots, x'_m)) \\
&= m(p, \bar{g}r, \bar{g}r) = p \qquad \checkmark
\end{aligned}
$$

n-permutable + wkp $\Leftrightarrow$ Mal‘cev

- Lemma: If F_Σ weakly preserves kernel pairs then for any terms p, q

$$p(x,x,y) = q(x,y,y) \Leftrightarrow \quad \exists\, s.$$

$$p(x,y,z) = s(x,y,z,z)$$
$$q(x,y,z) = s(x,x,y,z)$$

n-permutable + wkp $\Leftrightarrow$ Mal'cev

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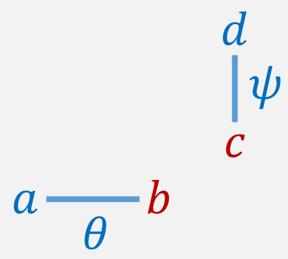

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- <u>Lemma</u>: If F_Σ weakly preserves kernel pairs then for any terms p, q

$$p(x,x,y) = q(x,y,y) \Leftrightarrow \quad \exists\, s.$$

$$\begin{aligned} p(x,y,z) &= s(x,y,z,z) \\ q(x,y,z) &= s(x,x,y,z) \end{aligned}$$

d

ψ

c

$a \overset{}{\underset{\theta}{\text{———}}} b \quad \Longrightarrow \quad p(a,b,c) \qquad q(b,c,d) \qquad p(b,b,c) = q(b,c,c)$

n-permutable + wkp ⇔ Mal'cev

- Lemma: If F_Σ weakly preserves kernel pairs then for any terms p, q

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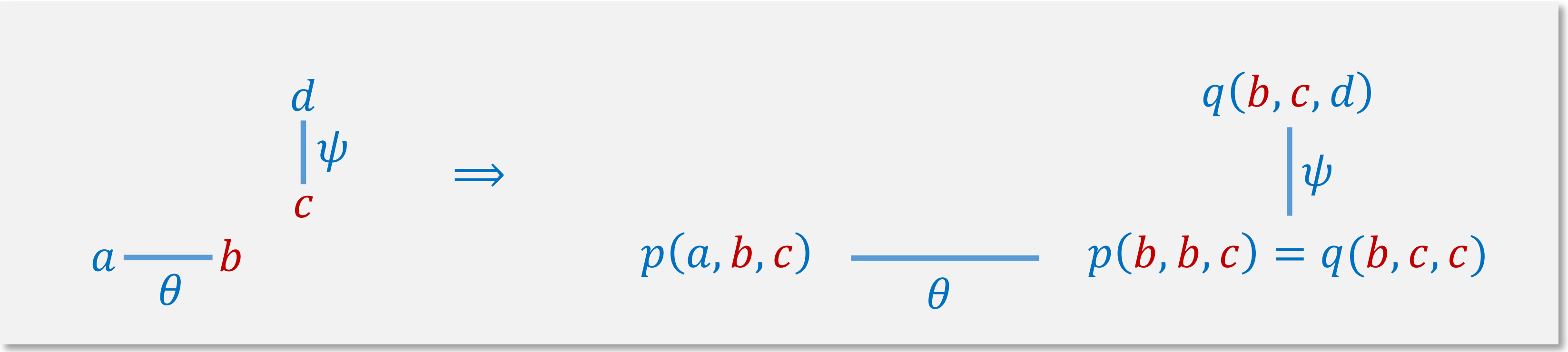

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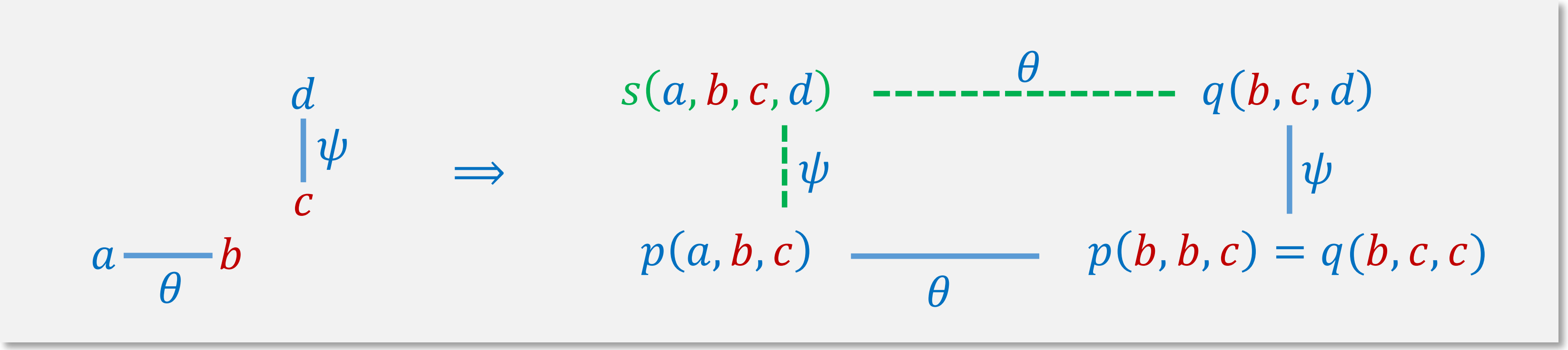

n-permutable + wkp ⇔ Mal‘cev

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$$q(x,y,z) = s(x,x,y,z)$$

- Thm: Mal‘cev ⇔ $n-permutable$ + F_Σ weakly preserves kernel pairs

$x = m(x,y,y)$ $m(x,x,y) = y$	⇔	*WKP*	+	$x = m_1(x,y,y),$ $m_i(x,x,y) = m_{i+1}(x,y,y)$ $m_k(x,x,y) = y$

n-permutable + wkp $\Leftrightarrow$ Mal'cev

- Lemma: If F_Σ weakly preserves kernel pairs then for any terms p, q

$$p(x,x,y) = q(x,y,y) \Leftrightarrow \quad \exists\, s.$$

$$p(x,y,z) = s(x,y,z,z)$$
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- Thm: Mal'cev $\Leftrightarrow$ n $-permutable$ + F_Σ weakly preserves kernel pairs

Proof $\Leftarrow$:

$$x = m_1(x,y,y),$$
$$\dots$$
$$m_i\,(x,x,y) \quad = m_{i+1}(x,y,y)$$
$$\dots$$
$$m_k(x,x,y) = y$$

n-permutable + wkp $\Leftrightarrow$ Mal'cev

- Lemma: If F_Σ weakly preserves kernel pairs then for any terms p, q

$$p(x,x,y) = q(x,y,y) \Leftrightarrow \quad \exists\, s.$$

$$p(x,y,z) = s(x,y,z,z)$$
$$q(x,y,z) = s(x,x,y,z)$$

- Thm: Mal'cev $\Leftrightarrow$ n *−permutable* + F_Σ weakly preserves kernel pairs

Proof $\Leftarrow$:

$$x = m_1(x,y,y),$$
$$\dots$$
$$m_{i-1}(x,x,y) = m_i(x,y,y)$$
$$m_i(x,x,y) = m_{i+1}(x,y,y)$$
$$m_{i+1}(x,x,y) = m_{i+2}(x,y,y)$$
$$\dots$$
$$m_k(x,x,y) = y$$

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- Thm: Mal'cev ⇔ n −*permutable* + F_Σ weakly preserves kernel pairs

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$$x = m_1(x,y,y),$$
$$\cdots$$
$$m_{i-1}(x,x,y) = m_i(x,y,y)$$
$$m_i(x,x,y) = m_{i+1}(x,y,y)$$
$$m_{i+1}(x,x,y) = m_{i+2}(x,y,y)$$
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n-permutable + wkp $\Leftrightarrow$ Mal‘cev

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$$\cdots$$
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$$m_i(x,x,y) = m_{i+1}(x,y,y)$$
$$m_{i+1}(x,x,y) = m_{i+2}(x,y,y)$$
$$\cdots$$
$$m_k(x,x,y) = y$$

$$m_i(x,y,z) = s(x,y,z,z)$$
$$m_{i+1}(x,y,z) = s(x,x,y,z)$$

n-permutable + wkp ⇔ Mal‘cev

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$$\dots$$
$$m_{i-1}(x,x,y) = m_i(x,y,y)$$
$$m_i(x,x,y) = m_{i+1}(x,y,y)$$
$$m_{i+1}(x,x,y) = m_{i+2}(x,y,y)$$
$$\dots$$
$$m_k(x,x,y) = y$$

$$m_i(x,y,z) = s(x,y,z,z)$$
$$m_{i+1}(x,y,z) = s(x,x,y,z)$$

$$m(x,y,z) := s(x,y,y,z)$$

n-permutable + wkp ⇔ Mal'cev

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- Thm: Mal'cev ⇔ n *−permutable* + F_Σ weakly preserves kernel pairs

Proof ⇐ :

$$x = m_1(x,y,y),$$
$$\dots$$
$$m_{i-1}(x,x,y) = m_i(x,y,y)$$
$$m_i(x,x,y) = m_{i+1}(x,y,y)$$
$$m_{i+1}(x,x,y) = m_{i+2}(x,y,y)$$
$$\dots$$
$$m_k(x,x,y) = y$$

$$m_i(x,y,z) = s(x,y,z,z)$$
$$m_{i+1}(x,y,z) = s(x,x,y,z)$$

$$m(x,y,z) := s(x,y,y,z)$$

$$m(x,y,y) = s(x,y,y,y) = m_i(x,y,y)$$

n-permutable + wkp ⇔ Mal'cev

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Proof ⇐ :

$$x = m_1(x,y,y),$$
$$\dots$$
$$m_{i-1}(x,x,y) = m_i(x,y,y) = m(x,y,y)$$
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$$m_{i+1}(x,x,y) = m_{i+2}(x,y,y)$$
$$\dots$$
$$m_k(x,x,y) = y$$

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$$m_{i+1}(x,y,z) = s(x,x,y,z)$$

$$m(x,y,z) := s(x,y,y,z)$$

n-permutable + wkp ⇔ Mal‘cev

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$$\begin{aligned} p(x,y,z) &= s(x,y,z,z) \\ q(x,y,z) &= s(x,x,y,z) \end{aligned}$$

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$$\dots$$
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$$\begin{aligned} m_i(x,y,z) &= s(x,y,z,z) \\ m_{i+1}(x,y,z) &= s(x,x,y,z) \end{aligned}$$

$$m(x,y,z) := s(x,y,y,z)$$

$$\begin{aligned} m(x,x,y) &= s(x,x,x,y) \\ &= m_{i+1}(x,x,y) \end{aligned}$$

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Proof $\Leftarrow$:

$$x = m_1(x,y,y),$$

$$\dots$$

$$m_{i-1}(x,x,y) = m_i(x,y,y) = m(x,y,y)$$

$$m_i\,(x,x,y) = m_{i+1}(x,y,y)$$

$$m(x,y,y) = m_{i+1}(x,x,y) = m_{i+2}(x,y,y)$$

$$\dots$$

$$m_k(x,x,y) = y$$

$$\begin{aligned} m_i\,(x,y,z) &= s(x,y,z,z) \\ m_{i+1}(x,y,z) &= s(x,x,y,z) \end{aligned}$$

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$$\begin{aligned} m(x,x,y) &= s(x,x,x,y) \\ &= m_{i+1}(x,x,y) \end{aligned}$$

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Proof ⇐ :

$$x = m_1(x,y,y),$$
$$\dots$$
$$m_{i-1}(x,x,y) = m_i(x,y,y) = m(x,y,y)$$
$$m_i(x,x,y) = m_{i+1}(x,y,y)$$
$$m(x,y,y) = m_{i+1}(x,x,y) = m_{i+2}(x,y,y)$$
$$\dots$$
$$m_k(x,x,y) = y$$

$$m_i(x,y,z) = s(x,y,z,z)$$
$$m_{i+1}(x,y,z) = s(x,x,y,z)$$

$$m(x,y,z) := s(x,y,y,z)$$

n-permutable + wkp ⇔ Mal'cev

- Lemma: If F_Σ weakly preserves kernel pairs then for any terms p, q

$$p(x,x,y) = q(x,y,y) \Leftrightarrow \quad \exists\, s.$$

$$p(x,y,z) = s(x,y,z,z)$$
$$q(x,y,z) = s(x,x,y,z)$$

- Thm: Mal'cev ⇔ n −*permutable* + F_Σ weakly preserves kernel pairs

Proof ⇐ :

$$x = m_1(x,y,y),$$
$$\ldots$$
$$m_{i-1}(x,x,y) \;= m(x,y,y)$$

$$m(x,x,y) = m_{i+2}(x,y,y)$$
$$\ldots$$
$$m_k(x,x,y) = y$$

$$m_i\;(x,y,z) \;= s(x,y,z,z)$$
$$m_{i+1}(x,y,z) = s(x,x,y,z)$$

$$m(x,y,z) \;:= s(x,y,y,z)$$

Conclusion

- F_{Σ} preserves
  - preimages $\Leftrightarrow$ weak independence implies independence
  - kernel pairs $\Leftarrow \mathcal{V}(\Sigma)$ is Mal‘cev
    $\Rightarrow$ (n-permutable $\Rightarrow$ Mal‘cev)

  Open: $\Leftrightarrow$???

Distributive and modular varieties

- If $F_{\mathcal{V}}(X)$ preserves kernel pairs then
  - $\mathcal{V}$ is congruence distributive iff
    $\exists k \in \mathbb{N}. \exists m. m$ is k–ary majority term, i.e.
    $$m(x, \dots, x, y) = \cdots = m(x, \dots, x, y, x, \dots, x) = \cdots = m(y, x, \dots, x) = x$$
  - $\mathcal{V}$ is congruence modular iff
    $\exists k \in \mathbb{N}. \exists m. \exists q.$
    $$m(x, \dots, x, y) = \cdots = m(x, \dots, x, y, x, \dots, x) = \cdots = m(x, y, x, \dots, x) = x$$
    $$m(y, x, \dots, x) = q(x, x, y)$$
    $$q(x, y, y) = x$$