Associative spectra of graph algebras

Erkko Lehtonen

Centro de Matemática e Aplicações, Faculdade de Ciências e Tecnologia
Universidade Nova de Lisboa

joint work with Tamás Waldhauser (University of Szeged)

Panglobal Algebra and Logic Seminar

17 November 2020

How to quantify (non)-associativity?

Associative law $x_1(x_2x_3) \approx (x_1x_2)x_3$

How to quantify the degree of (non)-associativity of a binary operation or the corresponding groupoid?

- distance from an associative operation
- number of triples satisfying the associative law
- ...

How to quantify (non)-associativity?

Associative law $x_1(x_2x_3) \approx (x_1x_2)x_3$

How to quantify the degree of (non)-associativity of a binary operation or the corresponding groupoid?

- distance from an associative operation
- number of triples satisfying the associative law
- ...
- number of “generalized associative laws” satisfied by the operation

Bracketings

bracketings

$$
\begin{aligned}
B_1 &= \{\, x_1 \,\} \\
B_2 &= \{\, (x_1 x_2) \,\} \\
B_3 &= \{\, (x_1(x_2 x_3)),\ ((x_1 x_2)x_3) \,\} \\
B_4 &= \{\, (x_1(x_2(x_3 x_4))),\ (x_1((x_2 x_3)x_4)),\ ((x_1 x_2)(x_3 x_4)), \\
&\qquad ((x_1(x_2 x_3))x_4),\ (((x_1 x_2)x_3)x_4) \,\} \\
&\ \vdots \\
B_n &= \{\, \text{valid bracketings of } x_1 x_2 \dots x_n \,\}
\end{aligned}
$$

bracketing identity: $t \approx t'$, where $t, t' \in B_n$ for some $n \in \mathbb{N}_+$

Example: $(x_1(x_2 x_3)) \approx ((x_1 x_2)x_3)$

Associative spectrum

$\mathbf{A} = (A; \circ)$ – a groupoid

fine associative spectrum of $\mathbf{A}$:

$$(\sigma_n(\mathbf{A}))_{n \in \mathbb{N}}, \quad \text{where} \quad \sigma_n(\mathbf{A}) = \{\, (s, t) \in B_n \times B_n \mid \mathbf{A} \models s \approx t \,\}$$

associative spectrum of $\mathbf{A}$:

$$(s_n(\mathbf{A}))_{n \in \mathbb{N}}, \quad \text{where} \quad s_n(\mathbf{A}) := |B_n / \sigma_n(\mathbf{A})|$$

$s_n(\mathbf{A})$ counts the number of distinct term operations induced by bracketings of n variables on $\mathbf{A}$.

$1 \leq s_n(\mathbf{A}) \leq C_{n-1}$

Extremes: associative and **antiassociative** operations

B. Csákány, T. Waldhauser, Associative spectra of binary operations, *Mult.-Valued Log.* **5** (2000) 175–200.

Associative spectrum

- **associative spectrum**

  B. Csákány, T. Waldhauser, Associative spectra of binary operations, *Mult.-Valued Log.* **5** (2000) 175–200.

  S. Liebscher, T. Waldhauser, On associative spectra of operations, *Acta Sci. Math.* (*Szeged*) **75** (2009) 433–456.

- **subassociativity type**

  M. S. Braitt, D. Silberger, Subassociative groupoids, *Quasigroups Related Systems* **14** (2006) 11-26.

- **number of $*$-equivalence classes of parenthesizations of** $x_0 * x_1 * \cdots * x_n$

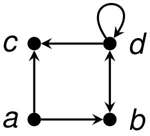
  N. Hein, J. Huang, Modular Catalan numbers, *European J. Combin.* **61** (2017) 197–218.

  J. Huang, M. Mickey, J. Xu, The nonassociativity of the double minus operation, *J. Integer Seq.* **20** (2017) Art. 17.10.3, 11 pp.

Graph algebras

graph

$$G = (V, E)$$



graph algebra

$$\mathbf{A}(G) = (V \cup \{\infty\}; \circ, \infty)$$

$$x \circ y = \begin{cases} x, & \text{if } (x, y) \in E \\ \infty, & \text{otherwise} \end{cases}$$

$\circ$	a	b	c	d	∞
a	∞	a	a	∞	∞
b	∞	∞	∞	b	∞
c	∞	∞	∞	∞	∞
d	∞	d	d	d	∞
∞	∞	∞	∞	∞	∞

C. R. Shallon, Non-finitely based binary algebras derived from lattices, Ph.D. thesis, University of California, Los Angeles, 1979.

Quiz

Which graphs are **idempotent**, i.e., satisfying $xx \approx x$?

Which graphs are **commutative**, i.e., satisfying $xy \approx yx$?

Which graphs are **associative**, i.e., satisfying $x(yz) \approx (xy)z$?

Quiz

Which graphs are **idempotent**, i.e., satisfying $xx \approx x$?

Every vertex has a loop.

Which graphs are **commutative**, i.e., satisfying $xy \approx yx$?

Which graphs are **associative**, i.e., satisfying $x(yz) \approx (xy)z$?

Quiz

Which graphs are **idempotent**, i.e., satisfying $xx \approx x$?

Every vertex has a loop.

Which graphs are **commutative**, i.e., satisfying $xy \approx yx$?

The only edges are loops.

Which graphs are **associative**, i.e., satisfying $x(yz) \approx (xy)z$?

Quiz

Which graphs are **idempotent**, i.e., satisfying $xx \approx x$?

Every vertex has a loop.

Which graphs are **commutative**, i.e., satisfying $xy \approx yx$?

The only edges are loops.

Which graphs are **associative**, i.e., satisfying $x(yz) \approx (xy)z$?

Hmmm… Some tools might be helpful.

Graphs associated with terms

t – a term in the language of graph algebras

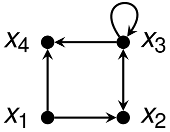
t is **trivial** if it contains ∞

For a nontrivial term t, let $G(t) = (V(t), E(t))$, where $V(t) = \mathrm{var}(t)$ and

- $E(t) = \emptyset$ if t is a variable,
- $E(t) = E(t_1) \cup E(t_2) \cup \{(L(t_1), L(t_2))\}$ if $t = (t_1 t_2)$.

Example:

$$t = \big(\big(x_1\big(x_2((x_3x_4)(x_3x_2))\big)\big)x_4\big) \qquad G(t) =$$



Graphs associated with terms

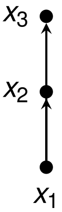
Theorem (Pöschel, Wessel)

Let s and t be nontrivial terms with $\mathrm{var}(s) = \mathrm{var}(t)$, and let $G = (V, E)$ be a graph. Then the following are equivalent:

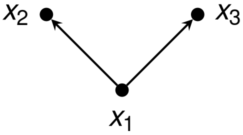
1. $\mathbf{A}(G) \models s \approx t$.
2. $\mathrm{Hom}(G(s), G) = \mathrm{Hom}(G(t), G) =: H$ *and* $h(L(s)) = h(L(t))$ *for every* $h \in H$.

R. Pöschel, W. Wessel, Classes of graphs definable by graph algebra identities or quasi-identities, *Comment. Math. Univ. Carolin.* **28** (1987) 581–592.

Associative graphs

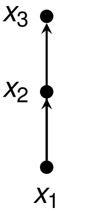


$G(\ x_1(x_2x_3)\ )$

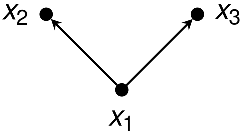


$G(\ (x_1x_2)x_3\ )$

Associative graphs



$G(\ x_1(x_2x_3)\ )$



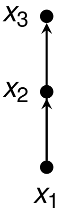
$G(\ (x_1x_2)x_3\ )$

Proposition (Poomsa-ard, 2000)

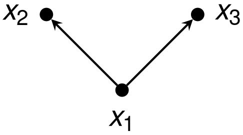
For any digraph $G = (V, E)$, the following are equivalent.

1. *G is associative.*

Associative graphs



$G(\ x_1(x_2x_3)\ )$



$G(\ (x_1x_2)x_3\ )$

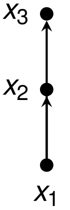
Proposition (Poomsa-ard, 2000)

For any digraph $G = (V, E)$, the following are equivalent.

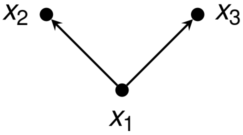
1. *G is associative.*

3. *The edge relation is transitive*

Associative graphs



$G(\ x_1(x_2x_3)\ )$



$G(\ (x_1x_2)x_3\ )$

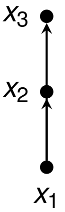
Proposition (Poomsa-ard, 2000)

For any digraph $G = (V, E)$, the following are equivalent.

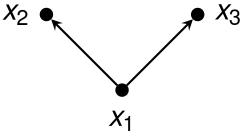
1. *G is associative.*

3. *The edge relation is transitive and the subgraph induced on the out-neighbourhood of any vertex is a complete graph (with loops).*

Associative graphs



$G(\ x_1(x_2x_3)\ )$



$G(\ (x_1x_2)x_3\ )$

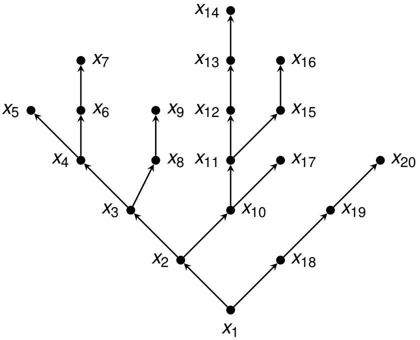
Proposition (Poomsa-ard, 2000)

For any digraph $G = (V, E)$, the following are equivalent.

1. *G is associative.*
2. *For any edge $(u, v) \in E$ and any vertex $w \in V$, $(u, w) \in E$ if and only if $(v, w) \in E$.*
3. *The edge relation is transitive and the subgraph induced on the out-neighbourhood of any vertex is a complete graph (with loops).*

Graphs associated with bracketings

The graph associated with any bracketing is a **DFS tree**.



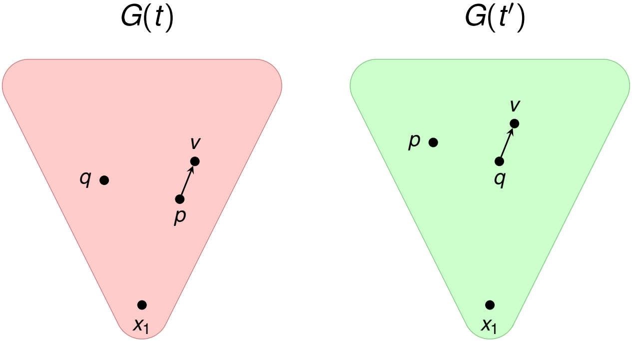
$G(t)$

$$t = ((x_1((x_2((x_3((x_4x_5)(x_6x_7)))(x_8x_9)))((x_{10}((x_{11}(x_{12}(x_{13}x_{14})))(x_{15}x_{16}))x_{17})))(x_{18}(x_{19}x_{20})))$$

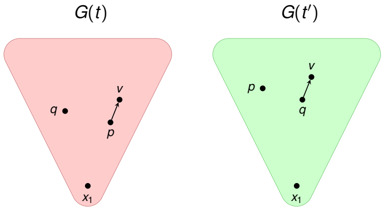
Satisfaction of bracketing identities

When does a graph algebra $\mathbf{A}(G)$ satisfy a bracketing identity $t \approx t'$?

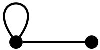
$$t, t' \in B_n,\ t \neq t'$$

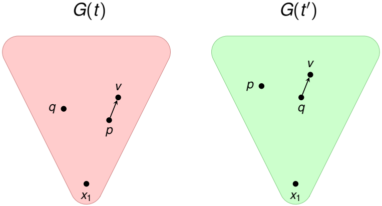


Undirected graphs

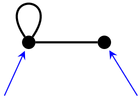


Undirected graphs





Undirected graphs

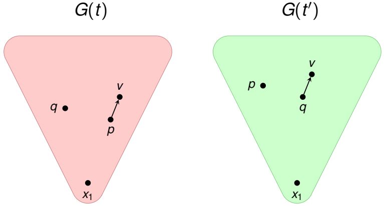


$\varphi(x)$

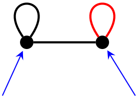
$x \in \mathrm{var}(t) \setminus \{v, q\}$

$\varphi(v)$

$\varphi(q)$



Undirected graphs

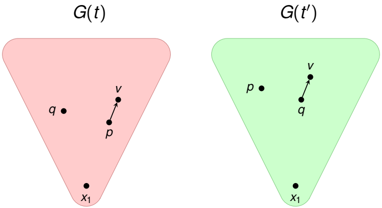


$\varphi(x)$

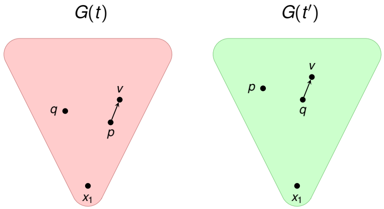
$x \in \mathrm{var}(t) \setminus \{v, q\}$

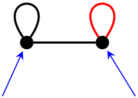
$\varphi(v)$

$\varphi(q)$



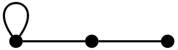
Undirected graphs



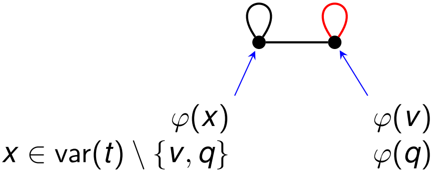


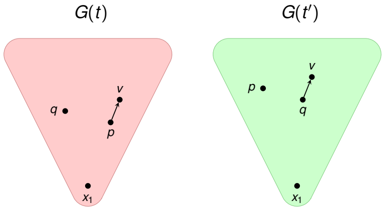
$\varphi(x)$ $\varphi(v)$

$x \in \mathrm{var}(t) \setminus \{v, q\}$ $\varphi(q)$

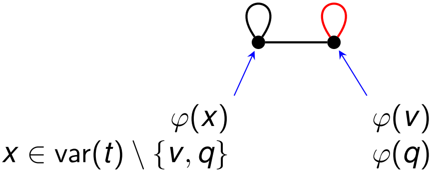


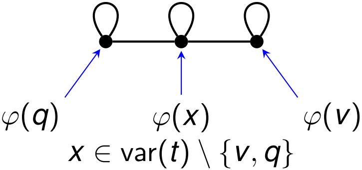
Undirected graphs

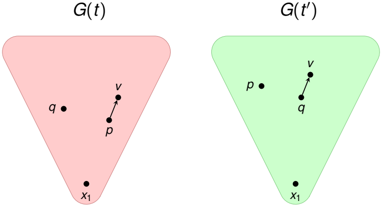




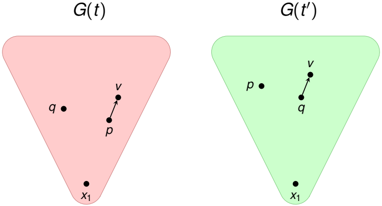
Undirected graphs

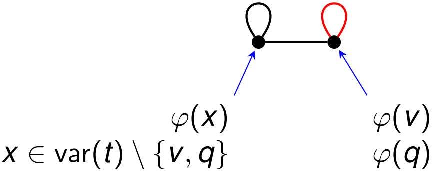


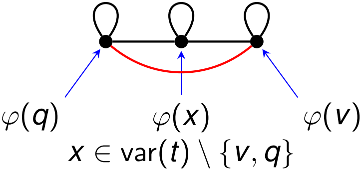




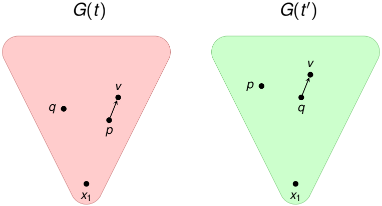
Undirected graphs

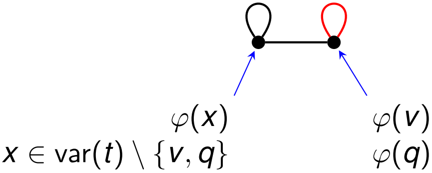


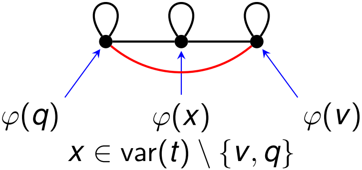




Undirected graphs

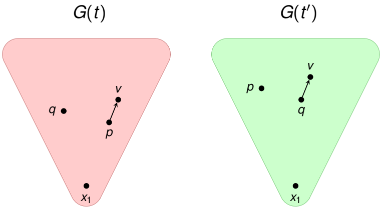






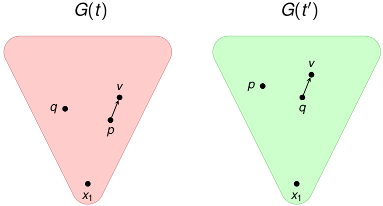
A connected component with a loop is a complete graph (with loops).

Undirected graphs



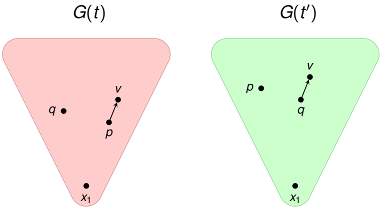
Undirected graphs

$$d_{G(t)}(x_i) \equiv d_{G(t')}(x_i) \pmod 2$$



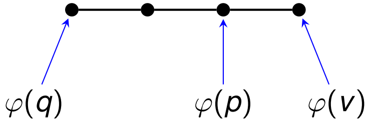
Undirected graphs

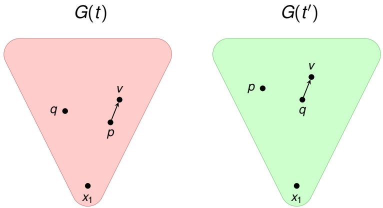
$$d_{G(t)}(x_i) \equiv d_{G(t')}(x_i) \pmod 2$$



Undirected graphs

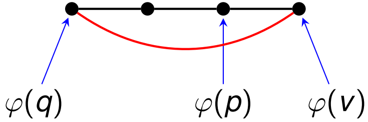
$$d_{G(t)}(x_i) \equiv d_{G(t')}(x_i) \pmod 2$$

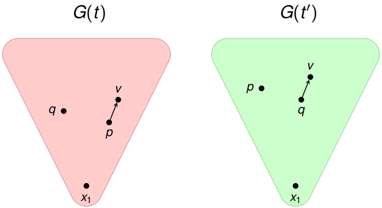




Undirected graphs

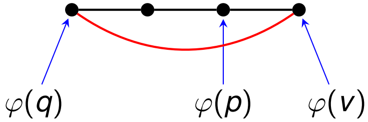
$$d_{G(t)}(x_i) \equiv d_{G(t')}(x_i) \pmod{2}$$

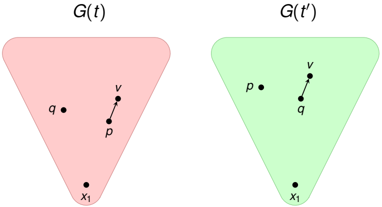




Undirected graphs

$$d_{G(t)}(x_i) \equiv d_{G(t')}(x_i) \pmod 2$$





A connected component without loops is complete bipartite.

Undirected graphs

Theorem

Let G be an undirected graph.

1. *If every connected component of G is either trivial or a complete graph (with loops), then $\mathbf{A}(G)$ satisfies every bracketing identity.* *In this case, $s_n(\mathbf{A}(G)) = 1$ for all $n \in \mathbb{N}_+$.*
2. *If every connected component of G is either trivial, a complete graph, or a complete bipartite graph, and the last case occurs at least once, then $\mathbf{A}(G)$ satisfies a bracketing identity $t \approx t'$ if and only if $d_{G(t)}(x_i) \equiv d_{G(t')}(x_i) \pmod{2}$ for all $x_i \in \mathrm{var}(t)$.* *In this case, $s_n(\mathbf{A}(G)) = 2^{n-2}$ for all $n \geq 2$.*
3. *Otherwise, G satisfies no nontrivial bracketing identity.* *In this case, $s_n(\mathbf{A}(G)) = C_{n-1}$ for all $n \in \mathbb{N}_+$.*

Antiassociative graphs

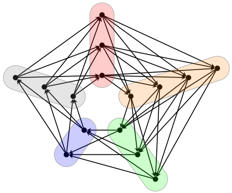
Vertices u and v are **strongly connected** if $u \to \cdots \to v$ and $v \to \cdots \to u$.

The relation of being strongly connected is an equivalence relation, and its equivalence classes are called **strongly connected components**.

A one-vertex graph with no loop is the **trivial** strongly connected graph.

A walk is **pleasant** if it only contains vertices from trivial strongly connected components.

An m**-whirl** is a strong homomorphic preimage of a directed m-cycle.
A **whirl** is an m-whirl for some m.

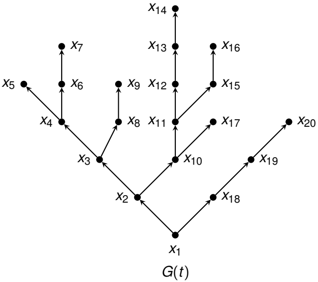


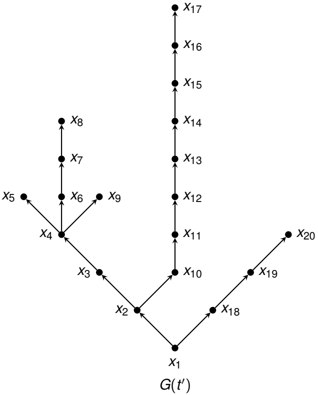
Theorem

Let G be a digraph. Then G is ***not*** *antiassociative if and only if the following conditions hold.*

1. *Every nontrivial strongly connected component of G is a whirl.*
2. *There is no path from a nontrivial strongly connected component of G to another.*
3. *There is a finite upper bound on the length of the pleasant paths of G.*
4. *There is a finite upper bound on the numbers m such that G contains an m-whirl.*

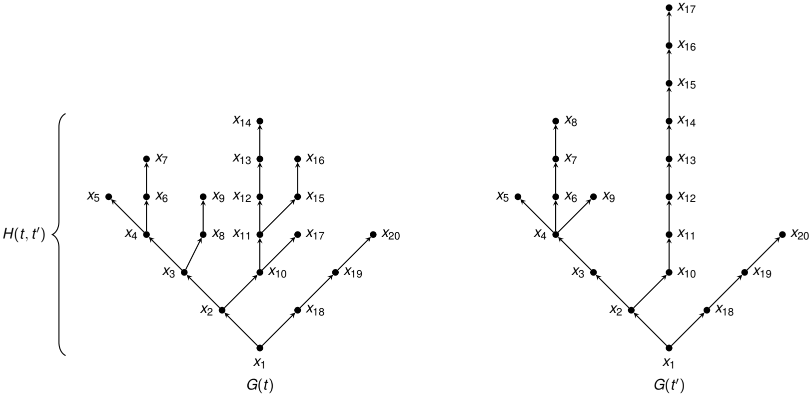
Parameters on pairs of DFS trees





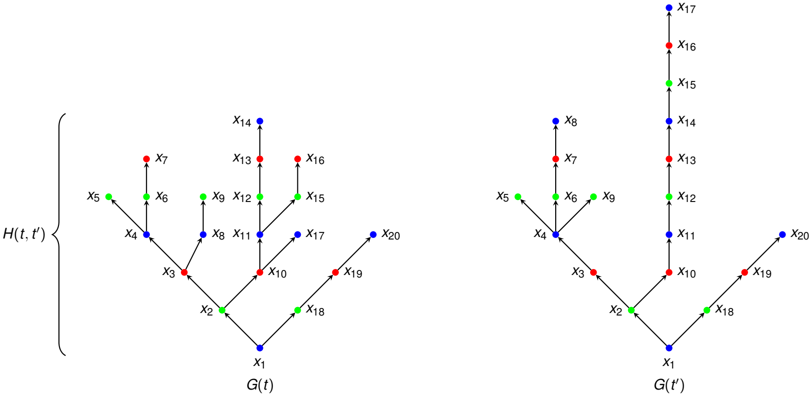
$$H_{t,t'} = 6 \qquad M_{t,t'} = 3 \qquad L_{t,t'} = 2$$

Parameters on pairs of DFS trees



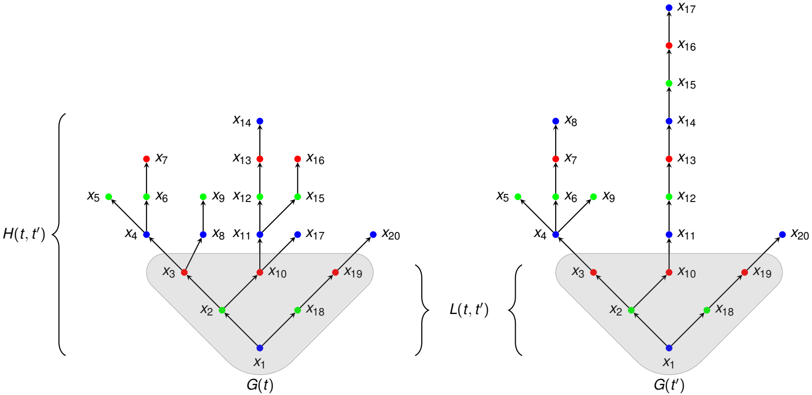
$$H_{t,t'} = 6 \qquad M_{t,t'} = 3 \qquad L_{t,t'} = 2$$

Parameters on pairs of DFS trees



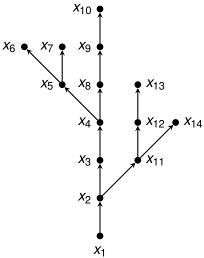
$$H_{t,t'} = 6 \qquad M_{t,t'} = 3 \qquad L_{t,t'} = 2$$

Parameters on pairs of DFS trees

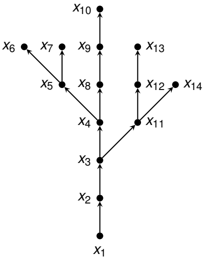


$$H_{t,t'} = 6 \qquad M_{t,t'} = 3 \qquad L_{t,t'} = 2$$

Parameters on pairs of DFS trees



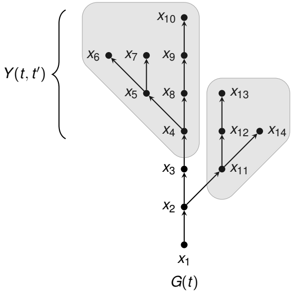
$G(t)$



$G(t')$

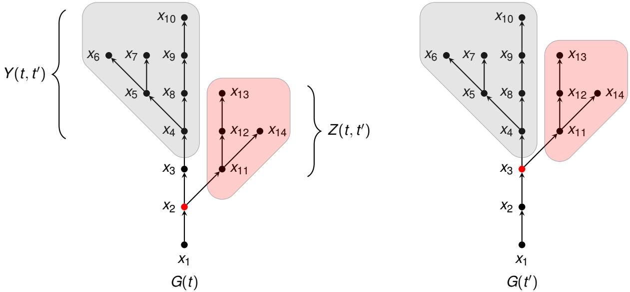
$$Y_{t,t'} = 3 \qquad Z_{t,t'} = 2$$

Parameters on pairs of DFS trees



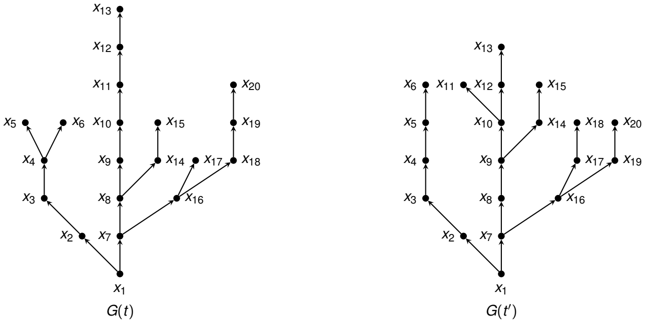
$$Y_{t,t'} = 3 \qquad Z_{t,t'} = 2$$

Parameters on pairs of DFS trees



$$Y_{t,t'} = 3 \qquad Z_{t,t'} = 2$$

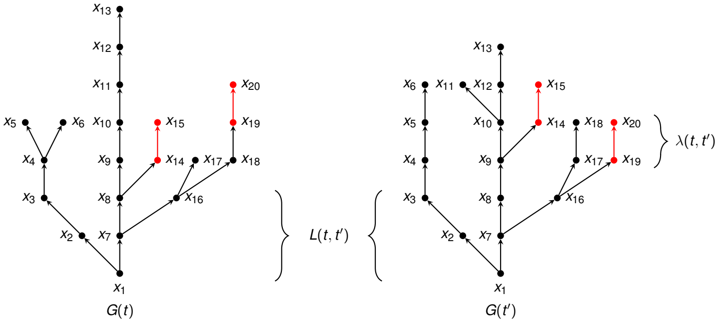
Parameters on pairs of DFS trees



$$\omega_{t,t'} = (6, 4, 4, 3, \dots)$$

$$\lambda_{t,t'} = 1.$$

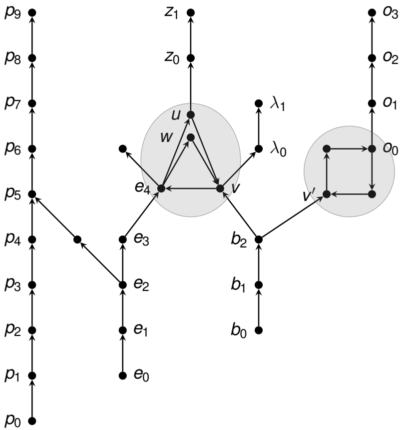
Parameters on pairs of DFS trees



$$\omega_{t,t'} = (6, 4, 4, 3, \dots)$$

$$\lambda_{t,t'} = 1.$$

Parameters on graphs



$$M_G = 12 \qquad Z_G = 1$$
$$P_G = 9 \qquad B_G = 2$$
$$E_G = 4 \qquad \lambda_G = 1$$
$$O_G = 3$$

$$\omega_G(\ell, r)$$

$\ell \setminus r$	1	2	3	4	5	6	7	8	$\cdots$
1	3	–	–	–	–	–	–	–	$\cdots$
2	6	4	–	–	–	–	–	–	$\cdots$
3	7	7	5	–	–	–	–	–	$\cdots$
4	8	8	8	6	–	–	–	–	$\cdots$
5	8	8	8	7	7	–	–	–	$\cdots$
6	8	8	8	8	8	8	–	–	$\cdots$
7	9	9	9	9	9	9	9	–	$\cdots$
8	10	10	10	10	10	10	10	10	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Directed graphs

Theorem

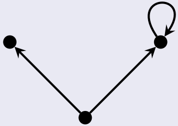
Let G be a digraph, and let $t, t' \in B_n$ with $t \neq t'$. Then $\mathbf{A}(G)$ satisfies the identity $t \approx t'$ if and only if the following conditions hold:

1. *Every nontrivial strongly connected component of G is a whirl.*
2. *There is no path from a nontrivial strongly connected component of G to another.*
3. $M_G \mid M_{t,t'}$.
4. $P_G < H_{t,t'}$.
5. $E_G \leq L_{t,t'} + 1$.
6. $O_G \leq Y_{t,t'} + 1$.
7. $Z_G < Z_{t,t'}$.
8. $B_G < L_{t,t'}$.
9. $\omega_G(L_{t,t'} + 1, r) < \omega_{t,t'}(r)$ *for all* $r \in \{1, \ldots, L_{t,t'} + 1\}$.
10. *If* $E_G = L_{t,t'} + 1$, *then* $\lambda_G < \lambda_{t,t'}$.

Theorem

For any digraph G we have the following mutually exclusive cases.

1. *The associative spectrum of* $\mathbf{A}(G)$ *is constant* 1*, i.e.,* $\mathbf{A}(G)$ *is associative.*
2. *The associative spectrum of* $\mathbf{A}(G)$ *is constant* 2*. This holds if and only if each weakly connected component of G is either associative or a directed bipartite graph with at least one edge, and the latter occurs at least once.*
3. *Otherwise the associative spectrum of* $\mathbf{A}(G)$ *is bounded below by the spectrum of* $\mathbf{A}(H)$ *where H is shown below, i.e.,* $s_n(\mathbf{A}(G)) \geq s_n(\mathbf{A}(H)) = \Theta(\alpha^n)$*, where* $\alpha \approx 1.755$*.*



Literature

E. LEHTONEN, T. WALDHAUSER,
Associative spectra of graph algebras I. Foundations, undirected graphs, antiassociative graphs,
arXiv:2011.07621.

E. LEHTONEN, T. WALDHAUSER,
Associative spectra of graph algebras II. Satisfaction of bracketing identities, spectrum dichotomy,
arXiv:2011.08522.

The End

Thank you for your attention!