Dual Ramsey properties for classes of algebras

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Finite Ramsey Phenomena

Finite Ramsey Theorem (sets).
For all $a, b \in \mathbb{N}$ and $k \geqslant 2$ there is a $c \in \mathbb{N}$ such that $c \longrightarrow (b)^a_k$.

Here, $c \longrightarrow (b)^a_k$ means:

for every c-element set C and every coloring

$$\chi : \binom{C}{a} \to k$$

there is a b-element set $B \subseteq C$ such that $|\chi(\binom{B}{a})| = 1$.

Frank P. Ramsey
1903 – 1930
Image courtesy of Wikipedia

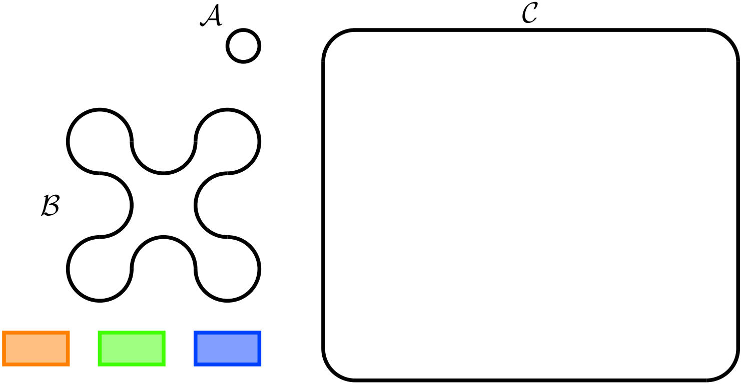
Finite Ramsey Phenomena

Finite Ramsey Theorem (chains). For all finite chains a and $b \geqslant a$ and all $k \geqslant 2$ there is a finite chain c such that $c \longrightarrow (b)^a_k$.

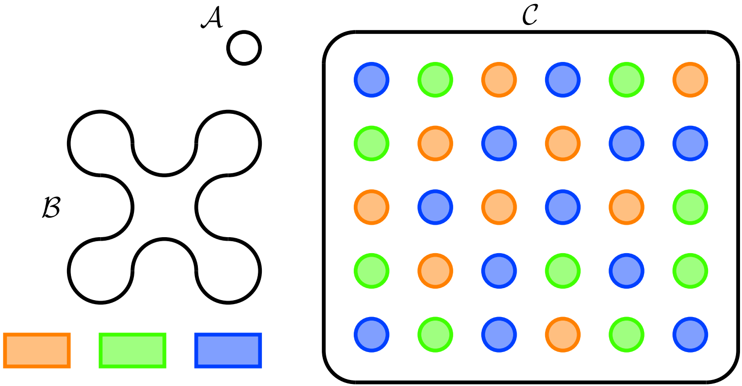
Definition. A class **K** of finite structures has the Ramsey property if:

> for all $\mathcal{A}, \mathcal{B} \in \mathbf{K}$ such that $\mathcal{A} \hookrightarrow \mathcal{B}$ and all $k \geqslant 2$ there is a $\mathcal{C} \in \mathbf{K}$ such that $\mathcal{C} \longrightarrow (\mathcal{B})^{\mathcal{A}}_k$.

$$\mathcal{C} \longrightarrow (\mathcal{B})^{\mathcal{A}}_{k}$$

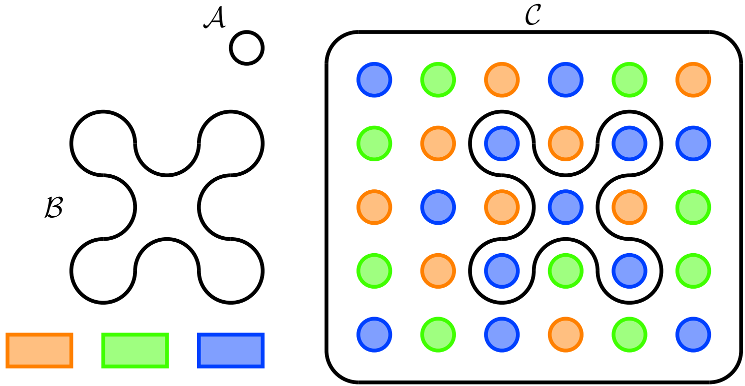


$\mathcal{C} \longrightarrow (\mathcal{B})^{\mathcal{A}}_k$



for every coloring $\chi : \binom{\mathcal{C}}{\mathcal{A}} \to k$

$\mathcal{C} \longrightarrow (\mathcal{B})^{\mathcal{A}}_{k}$



there is a $\tilde{\mathcal{B}} \in \binom{\mathcal{C}}{\mathcal{B}}$ such that $\left|\chi\left(\binom{\tilde{\mathcal{B}}}{\mathcal{A}}\right)\right| = 1$.

Finite Ramsey Phenomena

Alas, most combinatorially interesting classes of structures do NOT have the Ramsey property.

Finite Ramsey Phenomena

Alas, most combinatorially interesting classes of structures do NOT have the Ramsey property.

Two approaches to rectify the injustice:

1. early 1970's $\rightarrow$ add more structure to get the Ramsey property ($\rightsquigarrow$ *precompact Ramsey expansions*, Nguyen Van Thé 2013);
2. late 1990's, Fouché $\rightarrow$ relax the Ramsey property (*Ramsey degrees*).

Finite Ramsey Phenomena

Adding structure.

Combinatorially interesting classes equipped with an appropriate linear order **enjoy** the Ramsey property:

1. Finite graphs with a linear order: $(V, E, <)$
2. Finite posets with a linear *extension*: $(A, \sqsubseteq, <_e)$
3. Finite equiv rels with a *convex* lin order: $(A, \varrho, <_c)$

Nešetřil-Rödl Theorem. The class of linearly ordered finite relational structures of the same type defined by a finite set of *forbidden substructures* (Forb($\mathcal{F}$) where structs in $\mathcal{F}$ are *amalgamation irreducible*) has the Ramsey property.

Relaxing the property.

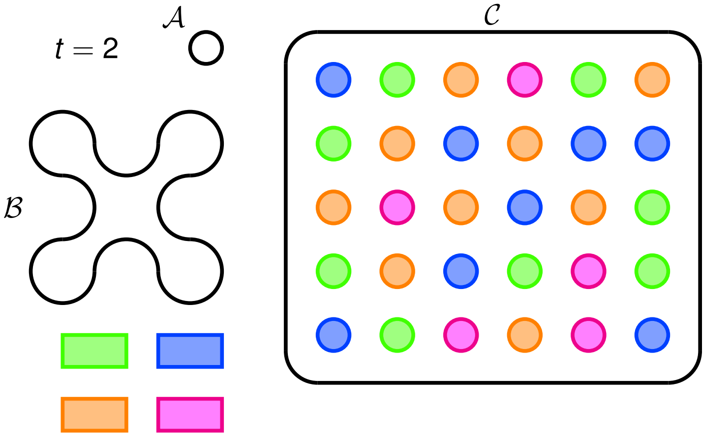
W. Fouché in a series of papers 1997–1999:
(small) Ramsey degrees

A class **K** of finite structures has *small Ramsey degrees* if for every $\mathcal{A} \in \mathbf{K}$ there is a positive integer $t = t(\mathcal{A})$ such that

$$\mathcal{C} \longrightarrow (\mathcal{B})^{\mathcal{A}}_{k,t} \text{ in } \mathbf{K}.$$

Finite Ramsey Phenomena

Relaxing the property.



Finite Ramsey Phenomena

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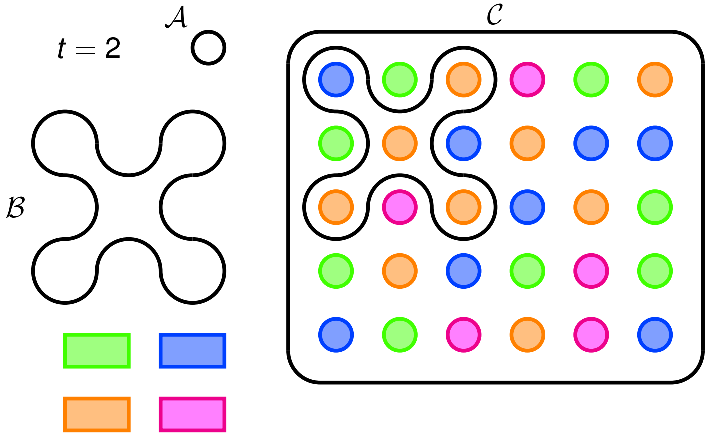

Finite Ramsey Phenomena

2016 Andy Zucker: the two approaches are equivalent!

- An amalgamation class of finite structures has finite Ramsey degrees iff it has a precompact Ramsey expansion.

Finite Ramsey Phenomena

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- An amalgamation class of finite structures has finite Ramsey degrees iff it has a precompact Ramsey expansion.

Hypothesis. Every amalgamation class of finite relational structures has a precompact Ramsey expansion
(or, equivalently, small Ramsey degrees).

Infinite Ramsey Phenomena

Infinite Ramsey Theorem. For every finite chain n and every $k \geqslant 2$ we have that $\omega \longrightarrow (\omega)^n_k$.

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Devlin 1979: For every $n \geqslant 1$ there is an integer $t = t(n)$ such that $\mathbb{Q} \longrightarrow (\mathbb{Q})^n_{k,t}$.

- The least such t is referred to as the big Ramsey degree of n in $\mathbb{Q}$ and denoted by $T(n, \mathbb{Q})$.

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Generalization to structs … Kechris-Pestov-Todorčević 2005

Infinite Ramsey Phenomena

Examples.

1. [Ramsey 1930] $T(n, \omega) = 1$
2. [Galvin, Devlin 1968/79] $T(n, \mathbb{Q}) < \infty$
3. [Sauer 2006] $T(G, \mathcal{R}) < \infty$ ($\mathcal{R}$ is the Rado graph)
4. [Dobrinen 2017] $T(G, \mathcal{H}_n) < \infty$ ($\mathcal{H}_n$ is the Henson graph)
5. [Hubička 2021] $T(A, \mathcal{P}) < \infty$ ($\mathcal{P}$ is the random poset)

Zucker's Theorem (2020). Let **K** be the class of all finite relational systems of the same finite binary type defined by a finite set of *forbidden substructures* (Forb($\mathcal{F}$)), and let $\mathcal{L}$ be the Fraïssé limit of **K.** Then $T(\mathcal{A}, \mathcal{L}) < \infty$ for all $\mathcal{A} \in$ **K.**

Finite Dual Ramsey Theorem

Finite Dual Ramsey Theorem (Graham, Rothschild 1971). For all finite chains a, b and $k \geqslant 2$ there is a finite chain c such that $c \xleftarrow{rs} (b)^a_k$.

- $c \xleftarrow{rs} (b)^a_k$:

  for every coloring $\chi : \mathrm{RSurj}(c, a) \to k$ there is a $w \in \mathrm{RSurj}(c, b)$ such that $|\chi(\mathrm{RSurj}(b, a) \circ w)| = 1$.
- $\mathrm{RSurj}(c, a)$ ... rigid surjections from c to a (c and a are finite chains)
- $f : c \to a$ is a rigid surjection if the image of every initial segment of c is an initial segment of a.

Finite Ramsey VS Finite Dual Ramsey

Finite Ramsey	**Finite Dual Ramsey**
For all finite chains a, b and $k \geqslant 2$ there is a finite chain c such that: $c \longrightarrow (b)^a_k$	For all finite chains a, b and $k \geqslant 2$ there is a finite chain c such that: $c \overset{rs}{\longleftarrow} (b)^a_k$
$c \longrightarrow (b)^a_k$:	$c \overset{rs}{\longleftarrow} (b)^a_k$:
$\forall\chi : \mathrm{Emb}(a, c) \to k$	$\forall\chi : \mathrm{RSurj}(c, a) \to k$
$\exists w \in \mathrm{Emb}(b, c)$	$\exists w \in \mathrm{RSurj}(c, b)$
such that	such that
$\lvert\chi(w \circ \mathrm{Emb}(a, b))\rvert = 1$	$\lvert\chi(\mathrm{RSurj}(b, a) \circ w)\rvert = 1$

Infinite Dual Ramsey Theorem

Theorem (Carlson, Simpson 1984). For every finite chain n and every $k \geqslant 2$ we have that $\omega \xleftarrow{\flat} (\omega)^n_k$.

- $\omega \xleftarrow{\flat} (\omega)^n_k$: for every Borel coloring $\chi : \mathrm{RSurj}(\omega, n) \to k$ there is a $w \in \mathrm{RSurj}(\omega, \omega)$ such that $|\chi(\mathrm{RSurj}(\omega, n) \circ w)| = 1$.

Infinite Ramsey	**Infinite Dual Ramsey**
For each finite chain n and $k \geqslant 2$: $\omega \longrightarrow (\omega)^n_k$	For each finite chain n and $k \geqslant 2$: $\omega \xleftarrow{\flat} (\omega)^n_k$

Summary

	Relational structures	Algebras
"Direct" Ramsey Phenomena	MANY EXAMPLES	few examples (semilat's; unary alg's)
Dual Ramsey Phenomena	few examples (mostly chains)	?

- Rel structs $\rightarrow$ rich source of Ramsey Phenomena

Summary

	Relational structures	Algebras
"Direct" Ramsey Phenomena	MANY EXAMPLES	few examples (semilat's; unary alg's)
Dual Ramsey Phenomena	few examples (mostly chains)	MANY EXAMPLES

- Rel structs $\rightarrow$ rich source of Ramsey Phenomena
- **In this talk:**
  Algebras $\rightarrow$ rich source of Dual Ramsey Phenomena

Finite Dual Ramsey for finite algebras in a variety

$\Omega \dots$ arbitrary algebraic language = functions + constants

$\mathbf{V} \dots$ arbitrary nontrivial variety of Ω-algebras

Theorem. The class of all finite algebras in $\mathbf{V}$ has small dual Ramsey degrees (wrt epimorphisms).

Finite Dual Ramsey for finite algebras in a variety

Corollary. The following classes of algebras have small dual Ramsey degrees (wrt epimorphisms):

1. finite groups in an arbitrary nontrivial variety of groups; (in particular, all finite groups, all finite abelian groups. . .);

   (NB. We do not know whether the class of all finite groups has a precompact Ramsey expansion!)

2. finitely dimensional vector spaces over a finite field;

3. finite lattices in an arbitrary nontrivial variety of lattices; (in particular, all finite lattices, finite distributive lattices, finite modular lattices. . .).

Infinite Dual Ramsey for varieties of algebras

$\Omega \ldots$ arbitrary countable algebraic language

$\mathbf{V} \ldots$ arbitrary nontrivial variety of Ω-algebras

Theorem. If $\mathcal{A} \in \mathbf{V}$ is a finite algebra then $\mathcal{A}$ has finite big dual Ramsey degree (wrt Borel coloring of epimorphisms) in the free algebra on ω generators. More precisely, if $\mathcal{A}$ has n elements then

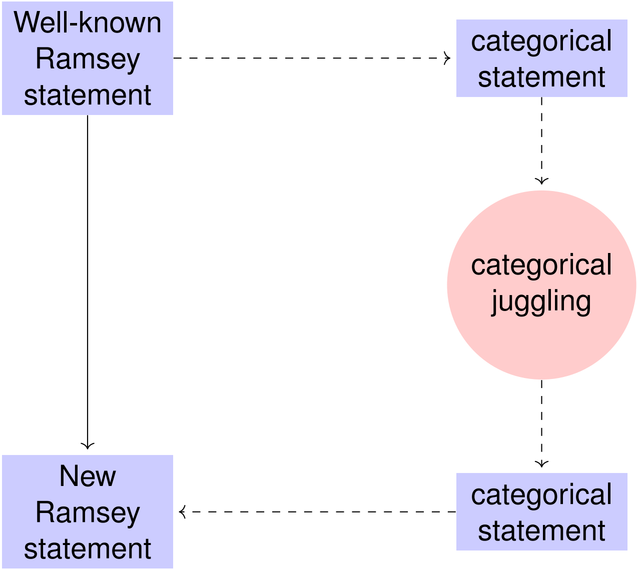
$$T^{\flat\partial}_{\mathbf{V}_{epi}}(\mathcal{A}, \mathcal{F}_{\mathbf{V}}(\omega)) \leqslant n \cdot n!.$$

Infinite Dual Ramsey for varieties of algebras

Corollary. With respect to Borel colorings of epimorphisms:

1. every finite group has finite big dual Ramsey degree in the free group on ω generators;
2. every finitely dimensional vector space over a finite field F has finite big dual Ramsey degree in F^ω;
3. every finite (distrib/modular/...) lattice has has finite big dual Ramsey degree in the free (distrib/modular/...) lattice on ω generators.

Proof strategy



Structural Ramsey Theory and Category Theory

Ramsey property (for morphisms) in a locally small category $\mathbf{C}$:

- $C \longrightarrow (B)^A_{k,t}$:

  for every coloring $\chi : \mathrm{hom}(A, C) \to k$ there is a $w \in \mathrm{hom}(B, C)$ such that $|\chi(w \cdot \mathrm{hom}(A, B))| \leqslant t$.

- **C** has the Ramsey property:

  for every $k \geqslant 2$ and all $A, B \in \mathrm{Ob}(\mathbf{C})$ there is a $C \in \mathrm{Ob}(\mathbf{C})$ such that $C \longrightarrow (B)^A_{k,1}$.

- **C** has the dual Ramsey property if $\mathbf{C}^{op}$ has the Ramsey property.

Structural Ramsey Theory and Category Theory

Small Ramsey degrees in a locally small category **C**:

- $C \longrightarrow (B)^{A}_{k,t}$:

  for every coloring $\chi : \mathrm{hom}(A, C) \to k$ there is a $w \in \mathrm{hom}(B, C)$ such that $|\chi(w \cdot \mathrm{hom}(A, B))| \leqslant t$.

- $t_{\mathbf{C}}(A) = \min\{s : C \longrightarrow (B)^{A}_{k,s}\}$, or ∞ if no such s exists

- $t^{\partial}_{\mathbf{C}}(A) = t_{\mathbf{C}^{op}}(A)$.

Structural Ramsey Theory and Category Theory

Big Ramsey degrees for Borel colorings in a locally small category **C** enriched over **Top**:

- $C \overset{\flat}{\longrightarrow} (B)^{A}_{k,t}$:

  for every Borel coloring $\chi : \mathrm{hom}(A, C) \to k$ there is a $w \in \mathrm{hom}(B, C)$ such that $|\chi(w \cdot \mathrm{hom}(A, B))| \leqslant t$.

- $T^{\flat}_{\mathbf{C}}(A, F) = \min\{s : (\forall k \geqslant 2) F \longrightarrow (F)^{A}_{k,s}\}$, or ∞ if no such s exists

- $T^{\flat\partial}_{\mathbf{C}}(A, F) = T^{\flat}_{\mathbf{C}^{op}}(A, F)$

Monads

Let $\mathbf{C}$ be a category and $T : \mathbf{C} \to \mathbf{C}$ an endofunctor.

A monad is a triple (T, μ, η) where:

- $\mu : TT \to T$ is a multiplication (associative):

$$\begin{tikzcd} TTT(A) \arrow[r, "\mu_{T(A)}"] \arrow[d, "T(\mu_A)"'] & TT(A) \arrow[d, "\mu_A"] \\ TT(A) \arrow[r, "\mu_A"'] & T(A) \end{tikzcd} \quad \text{for all } A \in \mathrm{Ob}(\mathbf{C}),$$

- and $\eta : \mathrm{ID} \to T$ is a unit for μ:

$$\begin{tikzcd} T(A) \arrow[r, "\eta_{T(A)}"] \arrow[dr, "\mathrm{id}"'] & TT(A) \arrow[d, "\mu_A"] & T(A) \arrow[l, "T(\eta_A)"'] \arrow[dl, "\mathrm{id}"] \\ & T(A) & \end{tikzcd} \quad \text{for all } A \in \mathrm{Ob}(\mathbf{C}).$$

Monads

Example.

- The term algebra monad (over a fixed alg language Ω): (T, μ, η) where

$$\begin{aligned} T &: \mathbf{Set} \to \mathbf{Set} : X \mapsto \mathrm{Term}_\Omega(X) \\ \eta_X &: X \to T(X) : x \mapsto x \\ \mu_X &: TT(X) \to T(X) : t_0(\langle t_1 \rangle, \ldots, \langle t_n \rangle) \mapsto t_0(t_1, \ldots, t_n) \end{aligned}$$

Algebras for an endofunctor

Let $T : \mathbf{C} \to \mathbf{C}$ be an endofunctor.

A *T*-algebra is a pair (A, α) where $A \in \mathrm{Ob}(\mathbf{C})$ and $\alpha : T(A) \to A$.

A homomorphism between T-algebras (A, α) and (B, β) is a $\mathbf{C}$-morphism $f : A \to B$ such that

$$\begin{tikzcd} T(A) \arrow[r, "T(f)"] \arrow[d, "\alpha"'] & T(B) \arrow[d, "\beta"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Algebras for a monad

Let (T, μ, η) be a monad where $T : \mathbf{C} \to \mathbf{C}$.

An Eilenberg-Moore algebra is a T-algebra (A, α) such that

$$\begin{tikzcd} TT(A) \arrow[r, "T(\alpha)"] \arrow[d, "\mu_A"'] & T(A) \arrow[d, "\alpha"] \\ T(A) \arrow[r, "\alpha"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "\mathrm{id}_A"'] & T(A) \arrow[d, "\alpha"] \\ & A \end{tikzcd}$$

The Eilenberg-Moore category $\mathbf{EM}(T, \mu, \eta)$:

- objects are Eilenberg-Moore algebras;
- morphisms are homomorphisms between T-algebras.

Algebras for “half a monad”

Let $T : \mathbf{C} \to \mathbf{C}$ be an endofunctor and $\mu : TT \to T$ a multiplication.

A weak Eilenberg-Moore algebra is a T-algebra (A, α) such that

$$\begin{tikzcd} TT(A) \arrow[r, "T(\alpha)"] \arrow[d, "\mu_A"'] & T(A) \arrow[d, "\alpha"] \\ T(A) \arrow[r, "\alpha"'] & A \end{tikzcd}$$

The weak Eilenberg-Moore category $\mathbf{EM}^{w}(T, \mu)$:

- objects are weak Eilenberg-Moore algebras;
- morphisms are homomorphisms between T-algebras.

Algebras for “half a monad”

Theorem. Let $T : \mathbf{C} \to \mathbf{C}$ be an endofunctor and (T, μ, η) a monad. If **C** has the dual Ramsey property then $\mathbf{EM}(T, \mu, \eta)$ has the dual Ramsey property.

Proof. Obvious because right adjoints preserve the dual Ramsey porperty.

Algebras for "half a monad"

Theorem. Let $T : \mathbf{C} \to \mathbf{C}$ be an endofunctor and (T, μ, η) a monad. If $\mathbf{C}$ has the dual Ramsey property then $\mathbf{EM}(T, \mu, \eta)$ has the dual Ramsey property.

Proof. Obvious because right adjoints preserve the dual Ramsey porperty.

Theorem. Let $T : \mathbf{C} \to \mathbf{C}$ be an endofunctor and $\mu : TT \to T$ a multiplication. If $\mathbf{C}$ has the dual Ramsey property then every full subcategory of $\mathbf{EM}^w(T, \mu)$ which contains all the free T-algebras (i.e. algebras of the form $(T(C), \mu_C)$) has the dual Ramsey property.

Proof. Easy but technical – use pre-adjunctions.

Finite Dual Ramsey for finite algebras in a variety

Ω ... arbitrary algebraic language = functions + constants

V ... arbitrary nontrivial variety of Ω-algebras

Theorem. $\mathbf{V}^{fin}$ has small dual Ramsey degrees (wrt epimorphisms).

Idea of proof:

1. FDRT: $\mathbf{Wch}_{rs}^{fin}$ has dual Ramsey property
2. upgrade the free algebra monad from **Set** to $\mathbf{Wch}_{rs}$;
3. form the weak Eilenberg-Moore category for this monad (these are now well-ordered algs + rigid epis);

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Theorem. $\mathbf{V}^{fin}$ has small dual Ramsey degrees (wrt epimorphisms).

Idea of proof:

4. using a pre-adjunction $\mathbf{Wch}_{rs} \rightleftarrows \mathbf{EM}^{w}(\hat{T}, \hat{\mu})$ prove that for finite $\hat{\mathcal{A}}, \hat{\mathcal{B}}$: $\hat{\mathcal{F}}_{\mathbf{V}}(\omega) \longleftarrow (\hat{\mathcal{B}})^{\hat{\mathcal{A}}}_{k}$;
5. use a compactness argument to prove that $\hat{\mathcal{C}} \longleftarrow (\hat{\mathcal{B}})^{\hat{\mathcal{A}}}_{k}$ for some finite $\hat{\mathcal{C}}$;
6. finally, use additive property of small (dual) Ramsey degrees to get rid of linear orders.

Infinite Dual Ramsey for varieties of algebras

Ω ... arbitrary countable algebraic language

V ... arbitrary nontrivial variety of Ω-algebras

Theorem. If $\mathcal{A} \in \mathbf{V}$ is a finite algebra with n elements then

$$T^{\flat\partial}_{\mathbf{V}_{epi}}(\mathcal{A}, \mathcal{F}_{\mathbf{V}}(\omega)) \leqslant n \cdot n!.$$

Idea of proof:

1. NB: categories are enriched over **Top** in the obvious way
2. ∞DRT: $T^{\flat\partial}_{\mathbf{Wch}_{rs}}(n, \omega) = 1$.
3. Use weak E-M construction to piggyback on that

Infinite Dual Ramsey for varieties of algebras

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$$T^{\flat\partial}_{\mathbf{V}_{epi}}(\mathcal{A}, \mathcal{F}_{\mathbf{V}}(\omega)) \leqslant n \cdot n!.$$

Idea of proof:

4. Result: in the category of ordered algs + rigid epis: $T^{\flat\partial}(\hat{\mathcal{A}}, \hat{\mathcal{F}}_{\mathbf{V}}(\omega)) \leqslant n$ for every $\hat{\mathcal{A}}$ with n elements.
5. Then use additive property of big (dual) Ramsey degrees to get rid of orders.

Open problems

Open problem 1. Does the class of finite groups have small Ramsey degrees (equivalently, precompact Ramsey expansion)?

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- Strengthen some of Sokić's results about the Ramsey classes of unary algebras. Are there more colorful classes of algebras that can fit into this setting?
- Comonads have been in effective use in the functional programming community for at least two decades. Coalgebras for a comonad are abstract models of special computational processes. What does it mean that a class of computational processes has the Ramsey property?