

Syntactical characterization of co-extensive varieties of universal algebras

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Overview

- ▶ Characterisation Example
- ▶ Products of commutative semirings
- ▶ Coextensivity
- ▶ Characterisation of left coextensivity
- ▶ Diagonalising terms
- ▶ Characterisation of coextensivity

Famous Example

A variety \mathcal{C} is called **Mal'cev** if it satisfies the following equivalent conditions:

1. Any reflexive homomorphic relation in \mathcal{C} is a congruence
2. \mathcal{C} is congruence permutable: $R \circ S = S \circ R$
3. \mathcal{C} has a Mal'cev term $t(x, z, z) = x, t(x, x, z) = z$

Where a relation R on an algebra A is called **homomorphic** if it is a subalgebra of $A \times A$. A category \mathcal{C} is called **Mal'cev** if:

1. Any reflexive internal relation in \mathcal{C} is an equivalence.

When \mathcal{C} is regular it is possible to define composition of internal relations and so \mathcal{C} is Mal'cev if and only if:

2. \mathcal{C} is congruence permutable: $R \circ S = S \circ R$

Commutative semirings

A **commutative semiring** is a system $A = (A, 0, +, 1, \cdot)$ in which:

1. $(A, 0, +)$ and $(A, 1, \cdot)$ are commutative monoids;
2. $a \cdot 0 = 0$ and $a \cdot (b + c) = a \cdot b + a \cdot c$ (distributivity)

The category **CSemiRings** of commutative semirings forms a variety of universal algebras. Then the product $A \times B$ is, as usual, the product of the underlying sets $A \times B$ with component-wise operations:

$$(a, b) + (a', b') = (a + a', b + b')$$

$$(a, b) \cdot (a', b') = (a \cdot a', b \cdot b')$$

Note that $(1, 0) \cdot (0, 1) = 0$ and $(1, 0) + (0, 1) = 1$

Commutative semirings

To consider a commutative semiring S as a product is to give two such elements $e_1, e_2 \in S$ with:

$$e_1 e_2 = 0 \text{ and } e_1 + e_2 = 1$$

- ▶ $e_i = e_i \cdot 1 = e_i(e_1 + e_2) = e_i e_1 + e_i e_2 = e_i^2$
 - ▶ $e_i S = \{e_i s \mid s \in S\}$ is a commutative semiring with $1 = e_i$ and inheriting all other operations from S
1. $e_i(e_i s) = (e_i e_i) s = e_i s$ (multiplicative unit)
 2. $e_i s + e_i s' = e_i(s + s') \in e_i S$ (addition)
 3. $(e_i s)(e_i s') = (e_i e_i)(s s') = e_i(ss') \in e_i S$ (multiplication)

Commutative semirings

We define maps:

$$\phi : S \longrightarrow e_1 S \times e_2 S \quad \phi(s) = (e_1 s, e_2 s)$$

$$\psi : e_1 S \times e_2 S \longrightarrow S \quad \psi(s, t) = s + t$$

Then $S \simeq e_1 S \times e_2 S$:

$$\psi\phi(s) = e_1 s + e_2 s = (e_1 + e_2)s = 1 \cdot s = s$$

$$\phi\psi(e_1 s, e_2 t) = (e_1(e_1 s + e_2 t), e_2(e_1 s + e_2 t)) = (e_1 s, e_2 t)$$

In particular for $S = A \times B$ we have:

$$e_1 = (1, 0), e_2 = (0, 1)$$

$$(1, 0)S = \{(a, 0) \in A \times B\} \simeq A$$

$$(0, 1)S = \{(0, b) \in A \times B\} \simeq B$$

So every product is of the form $S \simeq e_1 S \times e_2 S$.

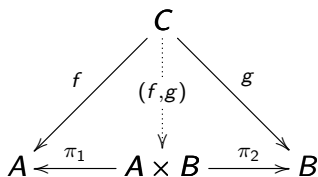
Coextensivity Overview

- ▶ Products
- ▶ Pushouts
- ▶ Initial objects
- ▶ Coextensivity of commutative semirings

Products

For objects A, B in a category \mathcal{C} , the product $A \times B$ is an object equipped with morphisms π_1, π_2 such that:

- ▶ for all C, f, g there exists a unique (f, g) such that $\pi_1(f, g) = f$ and $\pi_2(f, g) = g$

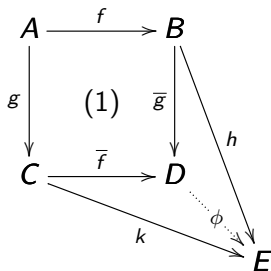


When \mathcal{C} is a variety, $A \times B$ is the usual product of two algebras:

- ▶ $\pi_1((a, b)) = a$
- ▶ $\pi_2((a, b)) = b$

Pushouts

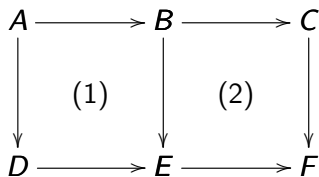
We say the square (1) is a pushout if:



- ▶ $\bar{g}f = \bar{f}g$
- ▶ for all h, k such that $hf = kg$ there exists a unique ϕ such that $\phi\bar{g} = h$ and $\phi\bar{f} = k$

Pasting Law

One useful result about pushouts is the pasting law. In a diagram:



If square (1) is a pushout then square (2) is a pushout if and only if the rectangle (1)+(2) is a pushout

Co-extensivity

We say a category \mathcal{C} is Co-extensive, if for any diagram:

$$\begin{array}{ccccc} X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \\ \downarrow f & & \downarrow h & & \downarrow g \\ A' & \xleftarrow{p'_1} & S & \xrightarrow{p'_2} & B' \end{array}$$

the bottom row of the following diagram is a product diagram if and only if both squares are pushouts.

Initial objects

We say that an object 0 is **initial** in some category \mathcal{C} , if for any object $A \in \mathcal{C}$ there exists a unique morphism $!_A : 0 \longrightarrow A$.

In a variety, the initial object is the free algebra generated on the empty set $F(\emptyset)$, whose elements are constant terms.

For example, in **CSemiRings**, we have constants $0, 1$ and binary operations $+, \cdot$, so the initial object 0 is $F(\emptyset) \simeq \mathbb{N}$

Co-extensivity Simplified

Assume the condition holds when both objects in the top row are initial. Then (1) and (3) are pushouts in the diagram:

$$\begin{array}{ccccc} 0 & \xleftarrow{\pi_1} & 0 \times 0 & \xrightarrow{\pi_2} & 0 \\ \downarrow !_X & & \downarrow !_X \times !_Y & & \downarrow !_Y \\ X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \\ \downarrow f & & \downarrow h & & \downarrow g \\ A' & \xleftarrow{p'_1} & S & \xrightarrow{p'_2} & B' \end{array}$$

(1) (3)

(2) (4)

So, by the pasting law, (2) and (4) are pushouts \iff (1)+(2) and (3)+(4) are pushouts \iff the bottom row is a product diagram. Therefore this is equivalent to co-extensivity.

Co-extensivity of commutative semirings

To find a homomorphism $h : 0 \times 0 \rightarrow S$ is exactly to find two elements $e_1 = h(1, 0)$, $e_2 = h(0, 1)$ with:

$$e_1 \cdot e_2 = h(1, 0) \cdot h(0, 1) = h(0, 0) = 0$$

$$e_1 + e_2 = h(1, 0) + h(0, 1) = h(1, 1) = 1$$

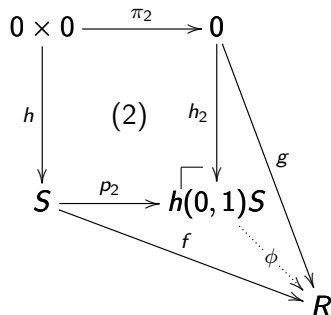
So we have $S \simeq h(1, 0)S \times h(0, 1)S$. It remains to show that the squares (1) and (2) are pushouts:

$$\begin{array}{ccccc} 0 & \xleftarrow{\pi_1} & 0 \times 0 & \xrightarrow{\pi_2} & 0 \\ \downarrow h_1 & & \downarrow h & & \downarrow h_2 \\ h(1, 0)S & \xleftarrow{p_1} & S & \xrightarrow{p_2} & h(0, 1)S \end{array}$$

(1) (2)

pushout diagram

Let $g\pi_2 = fh$ for some homomorphisms f, g .



Then the unique $\phi : h(0, 1)S \rightarrow R$ is given for all $s \in S$ by $\phi(h(0, 1)s) = f(h(0, 1)s) = f(s)$.

Therefore (2) and similarly (1) are pushouts, and **CSemiRings** is co-extensive.

Left Co-extensivity

We say \mathcal{C} is left co-extensive when for any f, g the following diagram is a pushout

$$\begin{array}{ccc} X \times Y & \xrightarrow{\pi_X} & X \\ \downarrow f \times g & & \downarrow f \\ A \times B & \xrightarrow{\pi_A} & A \end{array}$$

Again, by the pasting law this is equivalent to requiring that the following is a pushout for any X, Y :

$$\begin{array}{ccc} 0 \times 0 & \xrightarrow{\pi_1} & 0 \\ \downarrow !_X \times !_Y & & \downarrow !_X \\ X \times Y & \xrightarrow{\pi_X} & X \end{array}$$

Characterisation of left co-extensivity

A variety of universal algebras \mathcal{C} is left co-extensive if and only if there exist $(n + m)$ -ary terms u_0, \dots, u_k , unary terms $t_1, \dots, t_m, t'_1, \dots, t'_m \in F(\{x\})$, and constants $e_1, \dots, e_n, e'_1, \dots, e'_n, e''_1, \dots, e''_n \in F(\emptyset)$ such that for all $0 \leq i < k$ the following identities hold:

$$u_0(t'_1, t'_2, \dots, t'_m, e''_1, e''_2, \dots, e''_n) = x$$

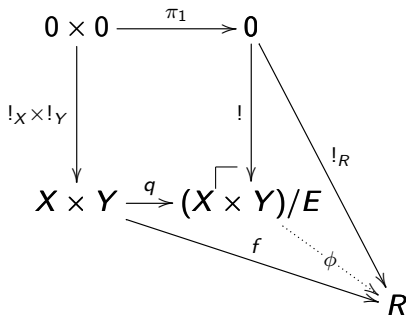
$$u_i(t_1, t_2, \dots, t_m, e_1, e_2, \dots, e_n) = x$$

$$u_i(t'_1, t'_2, \dots, t'_m, e'_1, e'_2, \dots, e'_n) = u_{i+1}(t'_1, t'_2, \dots, t'_m, e''_1, e''_2, \dots, e''_n)$$

$$u_k(t'_1, t'_2, \dots, t'_m, e'_1, e'_2, \dots, e'_n) = 0$$

pushout diagram

Let $g\pi_2 = fh$ for some homomorphisms f, g .



Where E is the congruence generated by $(e, e') \simeq_E (e, e'')$ for all constants $e, e', e'' \in F(\emptyset)$

Then the unique $\phi : (X \times Y)/E \rightarrow R$ is given by:
 $\phi([(x, y)]_E) = f((x, y))$.

$(X \times Y)/E \simeq X \iff (x, y) \simeq_E (x, z)$ for all $x \in X$ and $y, z \in Y$

Left Co-extensivity

- ▶ $((x, y), (x, z)) \in E$ for all $x \in X$ and $y, z \in Y$
- ▶ Let $X = Y = F(\{x\})$ and fix some constant $0 \in F(\emptyset)$
- ▶ $((x, x), (x, 0)) \in E$
- ▶ E is the transitive homomorphic symmetric reflexive closure of R , where $(e, e') \simeq_R (e, e'')$ for all $e, e', e'' \in F(\emptyset)$
- ▶ A sequence $(x, x) \simeq_Q (x, a_1) \simeq_Q \cdots \simeq_Q (x, a_k) \simeq_Q (x, 0)$
Where Q is the homomorphic symmetric reflexive closure of R
- ▶ For each $i < k$ there exists a term u_i such that:

$$(x, a_i) = u_i((t_1, t'_1), \dots, (t_{m_i}, t'_{m_i}), (e_1, e''_1), \dots, (e_{n_i}, e''_{n_i}))$$
$$(x, a_{i+1}) = u_i((t_1, t'_1), \dots, (t_{m_i}, t'_{m_i}), (e_1, e'_1), \dots, (e_{n_i}, e'_{n_i}))$$

for some unary terms $t_0, \dots, t_{m_i}, t'_0, \dots, t'_{m_i} \in F(\{x\})$, and constants $e_0, \dots, e_{n_i}, e'_0, \dots, e'_{n_i}, e''_0, \dots, e''_{n_i} \in F(\emptyset)$

Left Co-extensivity

Finally:

$$(x, a_0) = (x, x)$$

$$(x, a_i) = u_i((t_1, t'_1), \dots, (t_{m_i}, t'_{m_i}), (e_1, e''_1), \dots, (e_{n_i}, e''_{n_i}))$$

$$(x, a_{i+1}) = u_i((t_1, t'_1), \dots, (t_{m_i}, t'_{m_i}), (e_1, e'_1), \dots, (e_{n_i}, e'_{n_i}))$$

$$(x, a_k) = (x, 0)$$

Becomes:

$$u_0(t'_1, t'_2, \dots, t'_m, e''_1, e''_2, \dots, e''_n) = x$$

$$u_i(t_1, t_2, \dots, t_m, e_1, e_2, \dots, e_n) = x$$

$$u_i(t'_1, t'_2, \dots, t'_m, e'_1, e'_2, \dots, e'_n) = u_{i+1}(t'_1, t'_2, \dots, t'_m, e''_1, e''_2, \dots, e''_n)$$

$$u_k(t'_1, t'_2, \dots, t'_m, e'_1, e'_2, \dots, e'_n) = 0$$

As in our characterisation.

Left co-extensive example

For example, the variety of algebras with a binary operation \cdot , constants $0, 1$ and satisfying $x \cdot 1 = x$ and $x \cdot 0 = 0$.

Then for $k = 0, n = m = 1, u_0 = \cdot, t_1 = x, e_1 = e_1'' = 1$, and $e_1' = 0$ we have:

$$u_0(t_1', e_1'') = x \cdot 1 = x$$

$$u_i(t_1, e_1) = x \cdot 1 = x$$

$$u_k(t_1', e_1') = x \cdot 0 = 0$$

Equivalently: $(x, y) = (x, y) \cdot (1, 1) \simeq (x, y) \cdot (1, 0) = (x, 0)$

Diagonalising term

Any coextensive variety contains, for some $k \geq 1$, a $(k + 2)$ -ary term t and constants $e_1, \dots, e_k, e'_1, \dots, e'_k \in F(\emptyset)$ such that the identities hold:

$$t(x, y, e_1, \dots, e_k) = x$$

$$t(x, y, e'_1, \dots, e'_k) = y$$

For $h : X \times Y \longrightarrow Z$ we define the map $\delta : Z \times Z \longrightarrow Z$

$$\delta(x, y) = t(x, y, h(e_1, e'_1), \dots, h(e_k, e'_k))$$

When $h = f \times g : X \times Y \longrightarrow A \times B$ we have:

$$\delta((a, b), (c, d)) = (t(a, c, e_1, \dots, e_k), t(b, d, e'_1, \dots, e'_k)) = (a, d)$$

in CRing we have $\delta(x, y) = x \cdot (1, 0) + y \cdot (0, 1)$

Diagonalising term

Let $A \subseteq F(\{x, y\})^2$ be generated by $F(\emptyset)^2 \cup \{(x, x), (y, y)\}$.

Consider the pushouts:

$$\begin{array}{ccccc} F(\emptyset) & \xleftarrow{\pi_1} & F(\emptyset) \times F(\emptyset) & \xrightarrow{\pi_2} & F(\emptyset) \\ \downarrow \subset & & \downarrow \subset & & \downarrow \subset \\ F(\{x, y\}) & \xleftarrow{p_1} & A & \xrightarrow{p_2} & F(\{x, y\}) \end{array}$$

Then since both squares are pushouts the bottom row is a product diagram. In particular we have $(x, y) \in A$ and so for some t :

$$\begin{aligned} (x, y) &= t((x, x), (y, y), (e_1, e'_1), \dots, (e_k, e'_k)) \\ &= (t(x, y, e_1, \dots, e_k), t(x, y, e'_1, \dots, e'_k)) \end{aligned}$$

Diagonalising term

Let \mathcal{C} be a variety of universal algebras containing a diagonalizing term. The following conditions hold:

1. \mathcal{C} is left co-extensive
2. for any $h : X \times Y \rightarrow Z$ the map (p_1, p_2) is surjective.
3. For surjective h , the map (p_1, p_2) is an isomorphism, so $h \simeq h_1 \times h_2$.

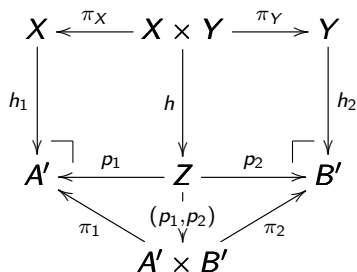
$$\begin{array}{ccccc} X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \\ \downarrow h_1 & & \downarrow h & & \downarrow h_2 \\ A' & \xleftarrow{p_1} & Z & \xrightarrow{p_2} & B' \\ & \swarrow \pi_1 & \downarrow (p_1, p_2) & \searrow \pi_2 & \\ & & A' \times B' & & \end{array}$$

Diagonalising term

\mathcal{C} is left coextensive by our characterisation since we can fix any constant as 0 and have the equalities:

$$t(x, 0, e_1, \dots, e_k) = x$$

$$t(x, 0, e'_1, \dots, e'_k) = 0$$



- ▶ p_1, p_2 are surjective
- ▶ so for $(a, b) \in A' \times B'$ there exist $z_1, z_2 \in Z$ such that $p_1(z_1) = a$ and $p_2(z_2) = b$
- ▶ $p_i(\delta(z_1, z_2)) = t(p_i(z_1), p_i(z_2), p_i(e_1, e'_1), \dots, p_i(e_k, e'_k)) = p_i(z_i)$
- ▶ $(p_1, p_2)(\delta(z_1, z_2)) = (p_1(z_1), p_2(z_2)) = (a, b)$

So (p_1, p_2) is surjective

Lattices

A (bounded) lattice can be considered as a tuple $(L, \vee, \wedge, 0, 1)$ where L is a set, \vee and \wedge are binary operations on L , and 0 and 1 are constants. Any lattice satisfies the associative, commutative, and absorption identities, and in particular the identities:

$$1 \wedge x = x \quad 0 \vee x = x \quad 0 \wedge x = 0 \quad 1 \vee x = 1$$

The category of lattices **Lat** is then a variety of universal algebras, with a diagonalising term $t(a, b, c, d) = (a \wedge c) \vee (b \wedge d)$ since:

$$t(x, y, 1, 0) = (x \wedge 1) \vee (y \wedge 0) = x \vee 0 = x$$

$$t(x, y, 0, 1) = (x \wedge 0) \vee (y \wedge 1) = 0 \vee y = y$$

Lat is not co-extensive, but the category **DLat** of Distributive lattices is coextensive as it is a full subcategory in **CSemiRings** closed under products.

Characterisation

1. \mathcal{C} has a diagonalising term t with constants $e_1, \dots, e_k, e'_1, \dots, e'_k \in F(\emptyset)$
2. There exist terms $\alpha_0, \dots, \alpha_n, \beta_0, \dots, \beta_n \in F(X + U(F(\emptyset)^2))$, operations $u_0, \dots, u_m \in F(X)$ and for all operations $s \in F(X)$ there exist $u_0^{(s)}, \dots, u_m^{(s)}$ s.t

$$u_0(\beta_0, \dots, \beta_n) = \delta(x, x).$$

$$u_0^{(s)}(\beta_0, \dots, \beta_n) = \delta(s(x_1, \dots, x_l), s(x'_1 \dots x'_l))$$

$$u_m(\alpha_0, \dots, \alpha_n) = x.$$

$$u_m^{(s)}(\alpha_0, \dots, \alpha_n) = s(\delta(x_1, x'_1), \dots, \delta(x_l, x'_l))$$

$$u_i(\alpha_0, \dots, \alpha_n) = u_{i+1}(\beta_0, \dots, \beta_n). \quad u_i^{(s)}(\alpha_0, \dots, \alpha_n) = u_{i+1}^{(s)}(\beta_0, \dots, \beta_n)$$

3. For each $j < n$ one of the following is true:

3.1 $\alpha_j = \beta_j$

3.2 $\alpha_j = (\omega, \omega')$ and $\beta_j = \delta(\omega, \omega')$ for some constants ω, ω'

3.3 $\beta_j = (\omega, \omega')$ and $\alpha_j = \delta(\omega, \omega')$ for some constants ω, ω'

3.4 $\alpha_j = v(\delta(\omega_0, \omega'_0), \dots, \delta(\omega_l, \omega'_l))$ and $\beta_j = \delta(v(\omega_0, \dots, \omega_l), v(\omega'_0, \dots, \omega'_l))$ for some constants $\omega_0, \dots, \omega_l, \omega'_0, \dots, \omega'_l$

3.5 $\beta_j = v(\delta(\omega_0, \omega'_0), \dots, \delta(\omega_l, \omega'_l))$ and $\alpha_j = \delta(v(\omega_0, \dots, \omega_l), v(\omega'_0, \dots, \omega'_l))$ for some constants $\omega_0, \dots, \omega_l, \omega'_0, \dots, \omega'_l$

From δ to co-extensivity

In a category with a diagonalising term, coextensivity is reduced to the injectivity of (p_1, p_2) for any set X in the diagram:

$$\begin{array}{ccccc} F(\emptyset) & \xleftarrow{\pi_1} & F(\emptyset) \times F(\emptyset) & \xrightarrow{\pi_2} & F(\emptyset) \\ \downarrow h_1 & & \downarrow h & & \downarrow h_2 \\ F(X) & \xleftarrow{p_1} & F(X) + F(\emptyset)^2 & \xrightarrow{p_2} & F(X) \\ & \swarrow \pi_1 & \downarrow (p_1, p_2) & \searrow \pi_2 & \\ & & F(X) \times F(X) & & \end{array}$$

Recall, however, that δ is defined such that:

$$\begin{aligned} p_i(\delta(x_1, x_2)) &= p_i(t(x_1, x_2, (e_1, e'_1), \dots, (e_k, e'_k))) \\ &= t(p_i(x_1), p_i(x_2), p_i((e_1, e'_1)), \dots, p_i((e_k, e'_k))) \\ &= p_i(x_i) \\ \implies (p_1, p_2)(\delta(x_1, x_2)) &= (p_1(x_1), p_2(x_2)) \end{aligned}$$

Injectivity of (p_1, p_2)

From $(p_1, p_2)(\delta(x_1, x_2)) = (p_1(x_1), p_2(x_2))$, it can be shown that (p_1, p_2) is injective if and only if:

1. $\delta(x, x) = x$ for any $x \in F(X) + F(\emptyset)^2$
2. $u(\delta(x_1, y_1), \dots, \delta(x_n, y_n)) = \delta(u(x_1, \dots, x_n), u(y_1, \dots, y_n))$ for any operation u and any $x_i, y_i \in F(X) + F(\emptyset)^2$

If $F(X) + F(\emptyset)^2$ is a product, then we can write $x = (x', x'')$ and $y = (y', y'')$, so $\delta(x, y) = (x', y'')$ and these conditions become:

1. $(x', x'') = (x', x'')$
2. $u((x'_1, y''_1), \dots, (x'_n, y''_n)) = (u(x'_1, \dots, x'_n), u(y''_1, \dots, y''_n))$

Which are true for any product

Characterisation

To produce a characterisation we need to consider $F(X) + F(\emptyset)^2$ as a quotient on some free algebra.

- ▶ $F(X) + F(\emptyset)^2 \simeq F(X + U(F(\emptyset)^2))/E$
- ▶ Here E is generated by $(\omega, \omega') \simeq \delta(\omega, \omega')$ and $v(\delta(\omega_0, \omega'_0), \dots, \delta(\omega_l, \omega'_l)) \simeq \delta(v(\omega_0, \dots, \omega_l), v(\omega'_0, \dots, \omega'_l))$ for any constants ω_i, ω'_i
- ▶ We need $\delta(x, x) \simeq_E x$ and for any operation s $s(\delta(x_1, y_1), \dots, \delta(x_n, y_n)) \simeq_E \delta(s(x_1, \dots, x_n), s(y_1, \dots, y_n))$ for all x_i, y_i (not just constants)
- ▶ again we consider E as the transitive homomorphic symmetric reflexive closure of this relation.
- ▶ Sequences $\delta(x, x) \simeq_Q a_1 \simeq_Q \dots \simeq_Q a_k \simeq_Q x$
- ▶ $s(\delta(x_1, y_1), \dots, \delta(x_n, y_n)) \simeq_Q a_1^{(s)} \simeq_Q \dots \simeq_Q a_k^{(s)} \simeq_Q \delta(s(x_1, \dots, x_n), s(y_1, \dots, y_n))$ for each operation s
- ▶ Here Q is the homomorphic symmetric reflexive closure.

Characterisation

1. \mathcal{C} has a diagonalising term t with constants $e_1, \dots, e_k, e'_1, \dots, e'_k \in F(\emptyset)$
2. There exist terms $\alpha_0, \dots, \alpha_n, \beta_0, \dots, \beta_n \in F(X + U(F(\emptyset)^2))$, operations $u_0, \dots, u_m \in F(X)$ and for all operations $s \in F(X)$ there exist $u_0^{(s)}, \dots, u_m^{(s)}$ s.t

$$u_0(\beta_0, \dots, \beta_n) = \delta(x, x).$$

$$u_0^{(s)}(\beta_0, \dots, \beta_n) = \delta(s(x_1, \dots, x_l), s(x'_1 \dots x'_l))$$

$$u_m(\alpha_0, \dots, \alpha_n) = x.$$

$$u_m^{(s)}(\alpha_0, \dots, \alpha_n) = s(\delta(x_1, x'_1), \dots, \delta(x_l, x'_l))$$

$$u_i(\alpha_0, \dots, \alpha_n) = u_{i+1}(\beta_0, \dots, \beta_n). \quad u_i^{(s)}(\alpha_0, \dots, \alpha_n) = u_{i+1}^{(s)}(\beta_0, \dots, \beta_n)$$

3. For each $j < n$ one of the following is true:

3.1 $\alpha_j = \beta_j$

3.2 $\alpha_j = (\omega, \omega')$ and $\beta_j = \delta(\omega, \omega')$ for some constants ω, ω'

3.3 $\beta_j = (\omega, \omega')$ and $\alpha_j = \delta(\omega, \omega')$ for some constants ω, ω'

3.4 $\alpha_j = v(\delta(\omega_0, \omega'_0), \dots, \delta(\omega_l, \omega'_l))$ and $\beta_j = \delta(v(\omega_0, \dots, \omega_l), v(\omega'_0, \dots, \omega'_l))$ for some constants $\omega_0, \dots, \omega_l, \omega'_0, \dots, \omega'_l$

3.5 $\beta_j = v(\delta(\omega_0, \omega'_0), \dots, \delta(\omega_l, \omega'_l))$ and $\alpha_j = \delta(v(\omega_0, \dots, \omega_l), v(\omega'_0, \dots, \omega'_l))$ for some constants $\omega_0, \dots, \omega_l, \omega'_0, \dots, \omega'_l$