Syntactical characterization of co-extensive varieties of universal algebras

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Overview

- Characterisation Example
- Products of commutative semirings
- Coextensivity
- Characterisation of left coextensivity
- Diagonalising terms
- Characterisation of coextensivity

Famous Example

A variety $\mathcal{C}$ is called **Mal'cev** if it satisfies the following equivalent conditions:

1. Any reflexive homomorphic relation in $\mathcal{C}$ is a congruence
2. $\mathcal{C}$ is congruence permutable: $R \circ S = S \circ R$
3. $\mathcal{C}$ has a Mal'cev term $t(x, z, z) = x, t(x, x, z) = z$

Where a relation R on an algebra A is called **homomorphic** if it is a subalgebra of $A \times A$. A category $\mathcal{C}$ is called **Mal'cev** if:

1. Any reflexive internal relation in $\mathcal{C}$ is an equivalence.

When $\mathcal{C}$ is regular it is possible to define composition of internal relations and so $\mathcal{C}$ is Mal'cev if and only if:

2. $\mathcal{C}$ is congruence permutable: $R \circ S = S \circ R$

Commutative semirings

A **commutative semiring** is a system $A = (A, 0, +, 1, \cdot)$ in which:

1. $(A, 0, +)$ and $(A, 1, \cdot)$ are commutative monoids;
2. $a \cdot 0 = 0$ and $a \cdot (b + c) = a \cdot b + a \cdot c$ (distributivity)

The category **CSemiRings** of commutative semirings forms a variety of universal algebras. Then the product $A \times B$ is, as usual, the product of the underlying sets $A \times B$ with component-wise operations:

$$(a, b) + (a', b') = (a + a', b + b')$$
$$(a, b) \cdot (a', b') = (a \cdot a', b \cdot b')$$

Note that $(1, 0) \cdot (0, 1) = 0$ and $(1, 0) + (0, 1) = 1$

Commutative semirings

To consider a commutative semiring S as a product is to give two such elements $e_1, e_2 \in S$ with:

$$e_1 e_2 = 0 \text{ and } e_1 + e_2 = 1$$

- $e_i = e_i \cdot 1 = e_i(e_1 + e_2) = e_i e_1 + e_i e_2 = e_i^2$
- $e_i S = \{e_i s | s \in S\}$ is a commutative semiring with $1 = e_i$ and inheriting all other operations from S

1. $e_i(e_i s) = (e_i e_i)s = e_i s$ (multiplicative unit)
2. $e_i s + e_i s' = e_i(s + s') \in e_i S$ (addition)
3. $(e_i s)(e_i s') = (e_i e_i)(ss') = e_i(ss') \in e_i S$ (multiplication)

Commutative semirings

We define maps:

$$\phi : S \longrightarrow e_1 S \times e_2 S \qquad \phi(s) = (e_1 s, e_2 s)$$
$$\psi : e_1 S \times e_2 S \longrightarrow S \qquad \psi(s, t) = s + t$$

Then $S \simeq e_1 S \times e_2 S$:

$$\psi\phi(s) = e_1 s + e_2 s = (e_1 + e_2)s = 1 \cdot s = s$$
$$\phi\psi(e_1 s, e_2 t) = (e_1(e_1 s + e_2 t), e_2(e_1 s + e_2 t)) = (e_1 s, e_2 t)$$

In particular for $S = A \times B$ we have:

$$e_1 = (1, 0), e_2 = (0, 1)$$
$$(1, 0)S = \{(a, 0) \in A \times B\} \simeq A$$
$$(0, 1)S = \{(0, b) \in A \times B\} \simeq B$$

So every product is of the form $S \simeq e_1 S \times e_2 S$.

Coextensivity Overview

- Products
- Pushouts
- Initial objects
- Coextensivity of commutative semirings

Products

For objects A, B in a category $\mathcal{C}$, the product $A \times B$ is an object equipped with morphisms π_1, π_2 such that:

- for all C, f, g there exists a unique (f, g) such that $\pi_1(f, g) = f$ amd $\pi_2(f, g) = g$

$$
\begin{tikzcd}
& C \arrow[dl, "f"'] \arrow[d, dotted, "{(f,g)}"] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B
\end{tikzcd}
$$

When $\mathcal{C}$ is a variety, $A \times B$ is the usual product of two algebras:

- $\pi_1((a, b)) = a$
- $\pi_2((a, b)) = b$

Pushouts

We say the square (1) is a pushout if:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "(1)"] & B \arrow[d, "\overline{g}"'] \arrow[ddr, "h"] & \\
C \arrow[r, "\overline{f}"] \arrow[drr, "k"'] & D \arrow[dr, dotted, "\phi"] & \\
& & E
\end{tikzcd}$$

- $\overline{g}f = \overline{f}g$
- for all h, k such that $hf = kg$ there exists a unique ϕ such that $\phi\overline{g} = h$ and $\phi\overline{f} = k$

Pasting Law

One useful result about pushouts is the pasting law. In a diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "(2)"] & C \arrow[d] \\ D \arrow[r] & E \arrow[r] & F \end{tikzcd}$$

If square (1) is a pushout then square (2) is a pushout if and only if the rectangle (1)+(2) is a pushout

Co-extensivity

We say a category $\mathcal{C}$ is Co-extensive, if for any diagram:

$$\begin{tikzcd}
X \arrow[d, "f"'] & X \times Y \arrow[l, "\pi_X"'] \arrow[r, "\pi_Y"] \arrow[d, "h"'] & Y \arrow[d, "g"] \\
A' & S \arrow[l, "p'_1"] \arrow[r, "p'_2"'] & B'
\end{tikzcd}$$

the bottom row of the following diagram is a product diagram if and only if both squares are pushouts.

Initial objects

We say that an object 0 is **initial** in some category $\mathcal{C}$, if for any object $A \in \mathcal{C}$ there exists a unique morphism $!_A : 0 \longrightarrow A$.

In a variety, the initial object is the free algebra generated on the empty set $F(\emptyset)$, whose elements are constant terms.

For example, in **CSemiRings**, we have constants $0, 1$ and binary operations $+, \cdot$, so the initial object 0 is $F(\emptyset) \simeq \mathbb{N}$

Co-extensivity Simplified

Assume the condition holds when both objects in the top row are initial. Then (1) and (3) are pushouts in the diagram:

$$\begin{tikzcd}
0 \arrow[d, "!_X"'] & 0 \times 0 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "!_X \times !_Y"] \arrow[dl, phantom, "(1)"] \arrow[dr, phantom, "(3)"] & 0 \arrow[d, "!_Y"] \\
X \arrow[d, "f"'] & X \times Y \arrow[l, "\pi_X"'] \arrow[r, "\pi_Y"] \arrow[d, "h"'] \arrow[dl, phantom, "(2)"] \arrow[dr, phantom, "(4)"] & Y \arrow[d, "g"] \\
A' & S \arrow[l, "p'_1"] \arrow[r, "p'_2"'] & B'
\end{tikzcd}$$

So, by the pasting law, (2) and (4) are pushouts $\iff$ (1)+(2) and (3)+(4) are pushouts $\iff$ the bottom row is a product diagram. Therefore this is equivalent to co-extensivity.

Co-extensivity of commutative semirings

To find a homomorphism $h : 0 \times 0 \longrightarrow S$ is exactly to find two elements $e_1 = h(1,0), e_2 = h(0,1)$ with:

$$e_1 \cdot e_2 = h(1,0) \cdot h(0,1) = h(0,0) = 0$$
$$e_1 + e_2 = h(1,0) + h(0,1) = h(1,1) = 1$$

So we have $S \simeq h(1,0)S \times h(0,1)S$. It remains to show that the squares (1) and (2) are pushouts:

$$\begin{tikzcd}
0 \arrow[d, "h_1"'] \arrow[dr, phantom, "(1)"] & 0 \times 0 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "h"'] \arrow[dr, phantom, "(2)"] & 0 \arrow[d, "h_2"] \\
h(1,0)S & S \arrow[l, "p_1"] \arrow[r, "p_2"'] & h(0,1)S
\end{tikzcd}$$

pushout diagram

Let $g\pi_2 = fh$ for some homomorphisms f, g.

$$
\begin{tikzcd}
0 \times 0 \arrow[r, "\pi_2"] \arrow[d, "h"'] \arrow[dr, phantom, "(2)"] & 0 \arrow[d, "h_2"'] \arrow[ddr, "g"] & \\
S \arrow[r, "p_2"] \arrow[drr, "f"'] & h(0,1)S \arrow[dr, dotted, "\phi"] \arrow[ul, phantom, "\ulcorner", very near start] & \\
& & R
\end{tikzcd}
$$

Then the unique $\phi : h(0,1)S \longrightarrow R$ is given for all $s \in S$ by $\phi(h(0,1)s) = f(h(0,1)s) = f(s)$.

Therefore (2) and similarly (1) are pushouts, and **CSemiRings** is co-extensive.

Left Co-extensivity

We say $\mathcal{C}$ is left co-extensive when for any f, g the following diagram is a pushout

$$\begin{tikzcd} X \times Y \arrow[r, "\pi_X"] \arrow[d, "f \times g"'] & X \arrow[d, "f"] \\ A \times B \arrow[r, "\pi_A"'] & A \end{tikzcd}$$

Again, by the pasting law this is equivalent to requiring that the following is a pushout for any X, Y:

$$\begin{tikzcd} 0 \times 0 \arrow[r, "\pi_1"] \arrow[d, "!_X \times !_Y"'] & 0 \arrow[d, "!_X"] \\ X \times Y \arrow[r, "\pi_X"'] & X \end{tikzcd}$$

Characterisation of left co-extensivity

A variety of universal algebras $\mathcal{C}$ is left co-extensive if and only if there exist $(n+m)$-ary terms $u_0, \ldots, u_k$, unary terms $t_1, \ldots, t_m, t'_1, \ldots, t'_m \in F(\{x\})$, and constants $e_1, \ldots, e_n, e'_1, \ldots, e'_n, e''_1, \ldots, e''_n \in F(\emptyset)$ such that for all $0 \leq i < k$ the following identities hold:

$$\begin{aligned}
u_0(t'_1, t'_2, \ldots, t'_m, e''_1, e''_2, \ldots, e''_n) &= x \\
u_i(t_1, t_2, \ldots, t_m, e_1, e_2, \ldots, e_n) &= x \\
u_i(t'_1, t'_2, \ldots, t'_m, e'_1, e'_2, \ldots, e'_n) &= u_{i+1}(t'_1, t'_2, \ldots, t'_m, e''_1, e''_2, \ldots, e''_n) \\
u_k(t'_1, t'_2, \ldots, t'_m, e'_1, e'_2, \ldots, e'_n) &= 0
\end{aligned}$$

pushout diagram

Let $g\pi_2 = fh$ for some homomorphisms f, g.

$$\begin{tikzcd}
0 \times 0 \arrow[r, "\pi_1"] \arrow[d, "!_X \times !_Y"'] & 0 \arrow[d, "!"'] \arrow[ddr, "!_R"] & \\
X \times Y \arrow[r, "q"] \arrow[drr, "f"] & (X \times Y)/E \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dotted, "\phi"] & \\
& & R
\end{tikzcd}$$

Where E is the congruence generated by $(e, e') \simeq_E (e, e'')$ for all constants $e, e', e'' \in F(\emptyset)$

Then the unique $\phi : (X \times Y)/E \longrightarrow R$ is given by:
$\phi([(x, y)]_E) = f((x, y))$.

$(X \times Y)/E \simeq X \iff (x, y) \simeq_E (x, z)$ for all $x \in X$ and $y, z \in Y$

Left Co-extensivity

- $((x, y), (x, z)) \in E$ for all $x \in X$ and $y, z \in Y$
- Let $X = Y = F(\{x\})$ and fix some constant $0 \in F(\emptyset)$
- $((x, x), (x, 0)) \in E$
- E is the transitive homomorphic symmetric reflexive closure of R, where $(e, e') \simeq_R (e, e'')$ for all $e, e', e'' \in F(\emptyset)$
- A sequence $(x, x) \simeq_Q (x, a_1) \simeq_Q \cdots \simeq_Q (x, a_k) \simeq_Q (x, 0)$ Where Q is the homomorphic symmetric reflexive closure of R
- For each $i < k$ there exists a term u_i such that:

$$(x, a_i) = u_i((t_1, t_1'), \ldots, (t_{m_i}, t_{m_i}'), (e_1, e_1''), \ldots, (e_{n_i}, e_{n_i}''))$$
$$(x, a_{i+1}) = u_i((t_1, t_1'), \ldots, (t_{m_i}, t_{m_i}'), (e_1, e_1'), \ldots, (e_{n_i}, e_{n_i}'))$$

for some unary terms $t_0, \ldots, t_{m_i}, t_0', \ldots, t_{m_i}' \in F(\{x\})$, and constants $e_0, \ldots, e_{n_i}, e_0', \ldots, e_{n_i}', e_0'', \ldots, e_{n_i}'' \in F(\emptyset)$

Left Co-extensivity

Finally:

$$\begin{aligned}
(x, a_0) &= (x, x) \\
(x, a_i) &= u_i((t_1, t_1'), \ldots, (t_{m_i}, t_{m_i}'), (e_1, e_1''), \ldots, (e_{n_i}, e_{n_i}'')) \\
(x, a_{i+1}) &= u_i((t_1, t_1'), \ldots, (t_{m_i}, t_{m_i}'), (e_1, e_1'), \ldots, (e_{n_i}, e_{n_i}')) \\
(x, a_k) &= (x, 0)
\end{aligned}$$

Becomes:

$$\begin{aligned}
u_0(t_1', t_2', \ldots, t_m', e_1'', e_2'', \ldots, e_n'') &= x \\
u_i(t_1, t_2, \ldots, t_m, e_1, e_2, \ldots, e_n) &= x \\
u_i(t_1', t_2', \ldots, t_m', e_1', e_2', \ldots, e_n') &= u_{i+1}(t_1', t_2', \ldots, t_m', e_1'', e_2'', \ldots, e_n'') \\
u_k(t_1', t_2', \ldots, t_m', e_1', e_2', \ldots, e_n') &= 0
\end{aligned}$$

As in our characterisation.

Left co-extensive example

For example, the variety of algebras with a binary operation $\cdot$, constants $0, 1$ and satisfying $x \cdot 1 = x$ and $x \cdot 0 = 0$.

Then for $k = 0, n = m = 1$, $u_0 = \cdot, t_1 = x, e_1 = e_1'' = 1$, and $e_1' = 0$ we have:

$$
\begin{aligned}
u_0(t_1', e_1'') &= x \cdot 1 = x \\
u_i(t_1, e_1) &= x \cdot 1 = x \\
u_k(t_1', e_1') &= x \cdot 0 = 0
\end{aligned}
$$

Equivalently: $(x, y) = (x, y) \cdot (1, 1) \simeq (x, y) \cdot (1, 0) = (x, 0)$

Diagonalising term

Any coextensive variety contains, for some $k \geq 1$, a $(k+2)$-ary term t and constants $e_1, \ldots, e_k, e'_1, \ldots, e'_k \in F(\emptyset)$ such that the identities hold:

$$t(x, y, e_1, \ldots, e_k) = x$$
$$t(x, y, e'_1, \ldots, e'_k) = y$$

For $h : X \times Y \longrightarrow Z$ we define the map $\delta : Z \times Z \longrightarrow Z$

$$\delta(x, y) = t(x, y, h(e_1, e'_1), \ldots, h(e_k, e'_k))$$

When $h = f \times g : X \times Y \longrightarrow A \times B$ we have:

$$\delta((a, b), (c, d)) = (t(a, c, e_1, \ldots, e_k), t(b, d, e'_1, \ldots, e'_k)) = (a, d)$$

in CRing we have $\delta(x, y) = x \cdot (1, 0) + y \cdot (0, 1)$

Diagonalising term

Let $A \subseteq F(\{x, y\})^2$ be generated by $F(\emptyset)^2 \cup \{(x, x), (y, y)\}$.
Consider the pushouts:

$$\begin{tikzcd}
F(\emptyset) \arrow[d, "\subset"'] & F(\emptyset) \times F(\emptyset) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\subset"'] & F(\emptyset) \arrow[d, "\subset"'] \\
F(\{x, y\}) & A \arrow[l, "p_1"'] \arrow[r, "p_2"] & F(\{x, y\})
\end{tikzcd}$$

Then since both squares are pushouts the bottom row is a product diagram. In particular we have $(x, y) \in A$ and so for some t:

$$\begin{aligned}
(x, y) &= t((x, x), (y, y), (e_1, e_1'), \dots, (e_k, e_k')) \\
&= (t(x, y, e_1, \dots, e_k), t(x, y, e_1', \dots, e_k'))
\end{aligned}$$

Diagonalising term

Let $\mathcal{C}$ be a variety of universal algebras containing a diagonalizing term. The following conditions hold:

1. $\mathcal{C}$ is left co-extensive
2. for any $h : X \times Y \longrightarrow Z$ the map (p_1, p_2) is surjective.
3. For surjective h, the map (p_1, p_2) is an isomorphism, so $h \simeq h_1 \times h_2$.

$$\begin{tikzcd}
X \arrow[d, "h_1"'] & X \times Y \arrow[l, "\pi_X"'] \arrow[r, "\pi_Y"] \arrow[d, "h"'] & Y \arrow[d, "h_2"] \\
A' \arrow[ur, phantom, "\urcorner", very near start] & Z \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "{(p_1,p_2)}"] & B' \arrow[ul, phantom, "\ulcorner", very near start] \\
& A' \times B' \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

Diagonalising term

$\mathcal{C}$ is left coextensive by our characterisation since we can fix any constant as 0 and have the equalities:

$$t(x, 0, e_1, \ldots, e_k) = x$$

$$t(x, 0, e'_1, \ldots, e'_k) = 0$$

$$\begin{tikzcd}
X \arrow[d, "h_1"'] & X \times Y \arrow[l, "\pi_X"'] \arrow[r, "\pi_Y"] \arrow[d, "h"] & Y \arrow[d, "h_2"] \\
A' \arrow[ur, phantom, "\urcorner", very near start] & Z \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, dashed, "{(p_1,p_2)}"] & B' \arrow[ul, phantom, "\ulcorner", very near start] \\
& A' \times B' \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

- p_1, p_2 are surjective
- so for $(a, b) \in A' \times B'$ there exist $z_1, z_2 \in Z$ such that $p_1(z_1) = a$ and $p_2(z_2) = b$
- $p_i(\delta(z_1, z_2)) = t(p_i(z_1), p_i(z_2), p_i(e_1, e'_1), \ldots, p_i(e_k, e'_k)) = p_i(z_i)$
- $(p_1, p_2)(\delta(z_1, z_2)) = (p_1(z_1), p_2(z_2)) = (a, b)$

So (p_1, p_2) is surjective

Lattices

A (bounded) lattice can be considered as a tuple $(L, \vee, \wedge, 0, 1)$ where L is a set, $\vee$ and $\wedge$ are binary operations on L, and 0 and 1 are constants. Any lattice satisfies the associative, commutative, and absorption identities, and in particular the identities:

$$1 \wedge x = x \qquad 0 \vee x = x \qquad 0 \wedge x = 0 \qquad 1 \vee x = 1$$

The category of lattices **Lat** is then a variety of universal algebras, with a diagonalising term $t(a, b, c, d) = (a \wedge c) \vee (b \wedge c)$ since:

$$t(x, y, 1, 0) = (x \wedge 1) \vee (y \wedge 0) = x \vee 0 = x$$
$$t(x, y, 0, 1) = (x \wedge 0) \vee (y \wedge 1) = 0 \vee y = y$$

Lat is not co-extensive, but the category **DLat** of Distributive lattices is coextensive as it is a full subcategory in **CSemiRings** closed under products.

Characterisation

1. $\mathcal{C}$ has a diagonalising term t with constants $e_1, \ldots, e_k, e'_1, \ldots, e'_k \in F(\emptyset)$
2. There exist terms $\alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n \in F(X + U(F(\emptyset)^2))$, operations $u_0, \ldots, u_m \in F(X)$ and for all operations $s \in F(X)$ there exist $u_0^{(s)}, \ldots, u_m^{(s)}$ s.t

$$u_0(\beta_0, \ldots, \beta_n) = \delta(x, x). \qquad u_0^{(s)}(\beta_0, \ldots, \beta_n) = \delta(s(x_1, \ldots, x_l), s(x'_1 \ldots x'_l))$$

$$u_m(\alpha_0, \ldots, \alpha_n) = x. \qquad u_m^{(s)}(\alpha_0, \ldots, \alpha_n) = s(\delta(x_1, x'_1), \ldots, \delta(x_l, x'_l))$$

$$u_i(\alpha_0, \ldots, \alpha_n) = u_{i+1}(\beta_0, \ldots, \beta_n). \quad u_i^{(s)}(\alpha_0, \ldots, \alpha_n) = u_{i+1}^{(s)}(\beta_0, \ldots, \beta_n)$$

3. For each $j < n$ one of the following is true:
   - 3.1 $\alpha_j = \beta_j$
   - 3.2 $\alpha_j = (\omega, \omega')$ and $\beta_j = \delta(\omega, \omega')$ for some constants ω, ω'
   - 3.3 $\beta_j = (\omega, \omega')$ and $\alpha_j = \delta(\omega, \omega')$ for some constants ω, ω'
   - 3.4 $\alpha_j = v(\delta(\omega_0, \omega'_0), \ldots, \delta(\omega_l, \omega'_l))$ and $\beta_j = \delta(v(\omega_0, \ldots, \omega_l), v(\omega'_0, \ldots, \omega'_l))$ for some constants $\omega_0, \ldots, \omega_l, \omega_0, \ldots, \omega'_l$
   - 3.5 $\beta_j = v(\delta(\omega_0, \omega'_0), \ldots, \delta(\omega_l, \omega'_l))$ and $\alpha_j = \delta(v(\omega_0, \ldots, \omega_l), v(\omega'_0, \ldots, \omega'_l))$ for some constants $\omega_0, \ldots, \omega_l, \omega_0, \ldots, \omega'_l$

From δ to co-extensivity

In a category with a diagonalising term, coextensivity is reduced to the injectivity of (p_1, p_2) for any set X in the diagram:

$$
\begin{tikzcd}
F(\emptyset) \arrow[d, "h_1"'] & F(\emptyset) \times F(\emptyset) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "h"'] & F(\emptyset) \arrow[d, "h_2"] \\
F(X) & F(X) + F(\emptyset)^2 \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "{(p_1,p_2)}"] & F(X) \\
 & F(X) \times F(X) \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] &
\end{tikzcd}
$$

Recall, however, that δ is defined such that:

$$
\begin{aligned}
p_i(\delta(x_1, x_2)) &= p_i(t(x_1, x_2, (e_1, e_1'), \ldots, (e_k, e_k'))) \\
&= t(p_i(x_1), p_i(x_2), p_i((e_1, e_1')), \ldots, p_i((e_k, e_k'))) \\
&= p_i(x_i) \\
\Longrightarrow (p_1, p_2)(\delta(x_1, x_2)) &= (p_1(x_1), p_2(x_2))
\end{aligned}
$$

Injectivity of (p_1, p_2)

From $(p_1, p_2)(\delta(x_1, x_2)) = (p_1(x_1), p_2(x_2))$, it can be shown that (p_1, p_2) is injective if and only if:

1. $\delta(x, x) = x$ for any $x \in F(X) + F(\emptyset)^2$
2. $u(\delta(x_1, y_1), \ldots, \delta(x_n, y_n)) = \delta(u(x_1, \ldots, x_n), u(y_1, \ldots, y_n))$ for any operation u and any $x_i, y_i \in F(X) + F(\emptyset)^2$

If $F(X) + F(\emptyset)^2$ is a product, then we can write $x = (x', x'')$ and $y = (y', y'')$, so $\delta(x, y) = (x', y'')$ and these conditions become:

1. $(x', x'') = (x', x'')$
2. $u((x'_1, y''_1), \ldots, (x'_n, y''_n)) = (u(x'_1, \ldots, x'_n), u(y''_1, \ldots, y''_n))$

Which are true for any product

Characterisation

To produce a characterisation we need to consider $F(X) + F(\emptyset)^2$ as a quotient on some free algebra.

- $F(X) + F(\emptyset)^2 \simeq F(X + U(F(\emptyset)^2))/E$
- Here E is generated by $(\omega, \omega') \simeq \delta(\omega, \omega')$ and $v(\delta(\omega_0, \omega'_0), \ldots, \delta(\omega_l, \omega'_l)) \simeq \delta(v(\omega_0, \ldots, \omega_l), v(\omega'_0, \ldots, \omega'_l))$ for any constants ω_i, ω'_i
- We need $\delta(x, x) \simeq_E x$ and for any operation s $s(\delta(x_1, y_1), \ldots, \delta(x_n, y_n)) \simeq_E \delta(s(x_1, \ldots, x_n), s(y_1, \ldots, y_n))$ for all x_i, y_i (not just constants)
- again we consider E as the transitive homomorphic symmetric reflexive closure of this relation.
- Sequences $\delta(x, x) \simeq_Q a_1 \simeq_Q \cdots \simeq_Q a_k \simeq_Q x$
- $s(\delta(x_1, y_1), \ldots, \delta(x_n, y_n)) \simeq_Q a_1^{(s)} \simeq_Q \cdots \simeq_Q a_k^{(s)} \simeq_Q \delta(s(x_1, \ldots, x_n), s(y_1, \ldots, y_n))$ for each operation s
- Here Q is the homomorphic symmetric reflexive closure.

Characterisation

1. $\mathcal{C}$ has a diagonalising term t with constants $e_1, \ldots, e_k, e'_1, \ldots, e'_k \in F(\emptyset)$
2. There exist terms $\alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n \in F(X + U(F(\emptyset)^2))$, operations $u_0, \ldots, u_m \in F(X)$ and for all operations $s \in F(X)$ there exist $u_0^{(s)}, \ldots, u_m^{(s)}$ s.t

$$
\begin{aligned}
u_0(\beta_0, \ldots, \beta_n) &= \delta(x, x). & u_0^{(s)}(\beta_0, \ldots, \beta_n) &= \delta(s(x_1, \ldots, x_l), s(x'_1 \ldots x'_l)) \\
u_m(\alpha_0, \ldots, \alpha_n) &= x. & u_m^{(s)}(\alpha_0, \ldots, \alpha_n) &= s(\delta(x_1, x'_1), \ldots, \delta(x_l, x'_l)) \\
u_i(\alpha_0, \ldots, \alpha_n) &= u_{i+1}(\beta_0, \ldots, \beta_n). & u_i^{(s)}(\alpha_0, \ldots, \alpha_n) &= u_{i+1}^{(s)}(\beta_0, \ldots, \beta_n)
\end{aligned}
$$

3. For each $j < n$ one of the following is true:
   3.1 $\alpha_j = \beta_j$
   3.2 $\alpha_j = (\omega, \omega')$ and $\beta_j = \delta(\omega, \omega')$ for some constants ω, ω'
   3.3 $\beta_j = (\omega, \omega')$ and $\alpha_j = \delta(\omega, \omega')$ for some constants ω, ω'
   3.4 $\alpha_j = v(\delta(\omega_0, \omega'_0), \ldots, \delta(\omega_l, \omega'_l))$ and $\beta_j = \delta(v(\omega_0, \ldots, \omega_l), v(\omega'_0, \ldots, \omega'_l))$ for some constants $\omega_0, \ldots, \omega_l, \omega_0, \ldots, \omega'_l$
   3.5 $\beta_j = v(\delta(\omega_0, \omega'_0), \ldots, \delta(\omega_l, \omega'_l))$ and $\alpha_j = \delta(v(\omega_0, \ldots, \omega_l), v(\omega'_0, \ldots, \omega'_l))$ for some constants $\omega_0, \ldots, \omega_l, \omega_0, \ldots, \omega'_l$