Lattices with normal elements

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Introduction

We investigate the structure of a group in the framework of lattice theory, using properties of the associated lattices.

The best known of these is *the lattice* Sub(G) *of all subgroups of a group G* (Dedekind 1897, Ore 1937-38, Baer 1939, Iwasawa 1941, Suzuki 1956, Pàlfy 2001, Schmidt 2011,...).

(Ore, 1937-38)

The lattice Sub(G) *is distributive if and only if G is locally cyclic.*

(Iwasawa, 1941)

If G is a finite group, then Sub(G) *satisfies the Jordan-Dedekind chain condition if and only if G is supersolvable.*

(Bechtell, 1965)

If G is finite, then Sub(G) *is sectionally complemented if and only if every Sylow subgroup of G is elementary abelian.*

Our research of groups is here related to *the lattice of weak congruences* (Vojvodić and Šešelja, 1988).

These are symmetric and transitive subuniverses of the square of a group, forming an algebraic lattice under inclusion.

The subgroup lattice is isomorphic to the ideal generated by the diagonal relation in the weak-congruence lattice of a group.

The lattices of normal subgroups of all subgroup are, up to the corresponding embeddings, interval sublattices, partitioning the whole lattice.

In order to model weak-congruence lattices of groups, we define a special class of algebraic lattices.

We give a list of postulates and using them we define so called normal elements, special sublattices, lattice commutators, particular lattice-congruences, quotients, endomorphisms and closures, ...

We prove basic properties of these notions together with induced structural properties of such lattices.

All is purely lattice-theoretic.

Next we show that weak-congruence lattices of groups are lattices with normal elements.

Consequently, all new defined lattice-theoretic notions and their properties correspond to analogue objects in groups.

Many group properties are proved to hold as consequences of introduced lattice postulates, without being connected to the corresponding algebraic counterparts.

Our final results:

We give necessary and sufficient conditions for the weak congruence lattice of a group under which this group is
- Dedekind,
- Hamiltonian,
- Abelian,
- solvable,
- perfect,
- supersolvable,
- finite nilpotent.

All characterizations are firstly proved as particular lattice properties in the corresponding lattices with normal elements.

Special elements in a lattice

Ore (1935, 1942), Birkhoff (1940), Grätzer (1959, 1978), Grätzer and Schmidt (1961), Hashimoto and Kinugawa (1963), Reilley (1984)...

An element a of a lattice L is **distributive** if for every $x, y \in L$,

$$a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y).$$

A dual notion is a **codistributive** element.
An element a of a lattice L is **neutral** if for all $x, y \in L$,

$$(a \vee x) \wedge (a \vee y) \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y) \vee (x \wedge y).$$

An element $a \in L$ is **cancellable**, if
from $x \wedge a = y \wedge a$ and $x \vee a = y \vee a$ it follows that $x = y$.

Proposition

Let L be a lattice and $a \in L$. Then a is a codistributive element if and only if the mapping $m_a : L \longrightarrow \downarrow a$ defined by $m_a(x) = a \wedge x$ is an endomorphism on L.

We denote by φ_a the kernel of m_a:

for $x, y \in L$, $x\varphi_a y$ of and only if $x \wedge a = y \wedge a$.

In our research L is an algebraic lattice.

$T_a \subseteq L$ is the set of top elements of φ_a-classes (in an algebraic lattice, the top elements of φ_a-classes exist):

$$T_a = \{\overline{x} \mid x \in L,\ \overline{x} = \bigvee [x]_{\varphi_a}\}.$$

T_a is a lattice under the order from L, it is closed under meets in L, but not necessarily under joins.

The classes under φ_a are particular intervals in L, namely for every $x \in L$ there is $b \in \downarrow a$, such that $[x]_{\varphi_a} = [b, \overline{b}]$.

If L is algebraic, we say that $a \in L$ is a **full** codistributive element of L if T_a is closed under joins, i.e., if T_a is a complete, hence algebraic sublattice of L.

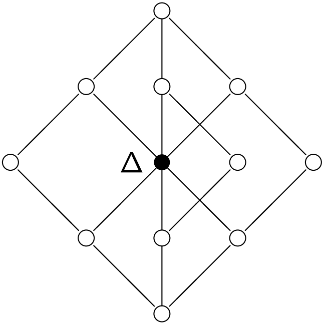
Proposition

The following are equivalent for an element a in a lattice L:
(i) *a is neutral;*
(ii) *a is distributive, codistributive and cancellable;*
(iii) *L is embeddable into the product $\downarrow a \times \uparrow a$.*

An element a is **infinitely distributive** in L if for every family $\{x_i \mid i \in I\}$, $a \vee (\bigwedge_{i \in I} x_i) = \bigwedge_{i \in I}(a \vee x_i)$.

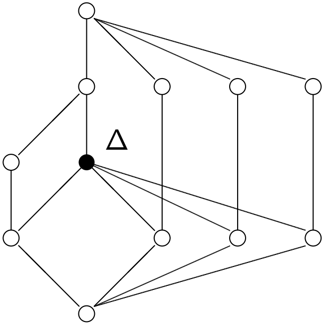
Weak congruences on groups

A **weak congruence** of a group G is a symmetric and transitive subuniverse of G^2.
Equivalently, it is a congruence on a subgroup of G, considered as a relation on G.
The collection $\mathrm{Wcon}(G)$ of all weak congruences on G is a set union of all congruences (equivalently of all normal subgroups) of all subgroups of G.
It contains also the subgroup lattice as the principal ideal generated by the diagonal of G.
In addition, every weak congruence on G can be represented as an ordered pair (H, K) of $(\mathrm{Sub}(G))^2$, so that $H \lhd K$.

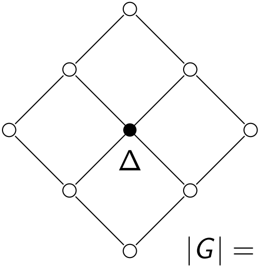


$\mathrm{Con_w}(\mathcal{K})$

Klein four-group

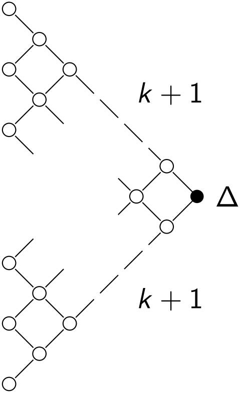


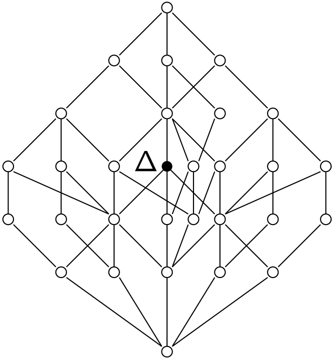
$\mathrm{Con_w}(\mathcal{S}_3)$



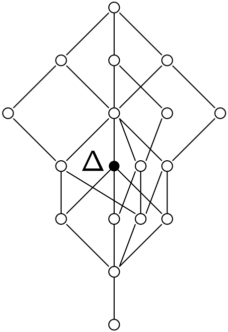
$|G| = p \cdot q$

$|G| = p^k$





a) *dihedral group of order 8*



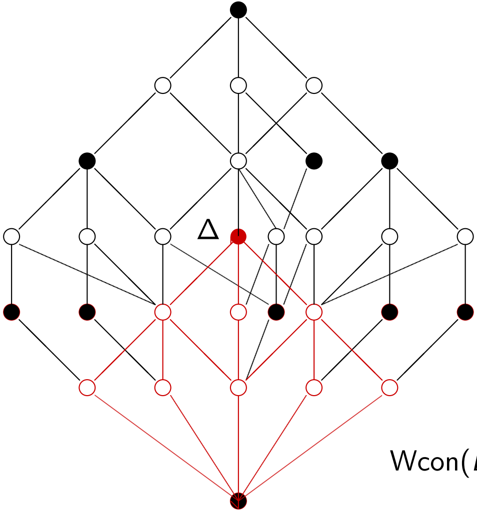
b) *quaternion group*

Proposition

The diagonal Δ *of a group* G *is a full codistributive element in* $\mathrm{Wcon}(G)$*:*

$$\Delta \wedge (\rho \vee \theta) = (\Delta \wedge \rho) \vee (\Delta \wedge \theta),$$

for any two weak congruences ρ, θ *on* G *(which are ordinary congruences on two subgroups of* G*, determined by the corresponding diagonals).*
The set T_Δ *of top elements of kernel-classes for* $\rho \mapsto \rho \wedge \Delta$ *consists of all squares of all subgroups of* G*. This is a sublattice of* $\mathrm{Wcon}(G)$*, isomorphic with* $\downarrow\Delta$ *i.e., with the lattice* $\mathrm{Sub}(G)$*.*



$\mathrm{Wcon}(D_8)$

Recall that an algebra $\mathcal{A}$ satisfies the **Congruence Extension Property** (the **CEP**) if for any congruence ρ on a subalgebra $\mathcal{B}$ of $\mathcal{A}$, there is a congruence θ on $\mathcal{A}$, such that $\rho = B^2 \cap \theta$.

An algebra $\mathcal{A}$ satisfies the CEP if and only if Δ is a cancallable element of the lattice $\mathrm{Wcon}(\mathcal{A})$.

A group G has the **Congruence Intersection Property**, the **CIP**, if for any two subgroup H, K of G,

$$(H \cap K)^G = H^G \cap K^G,$$

where H^G is the **normal closure** of H in G.
G satisfies the $*$**CIP** if

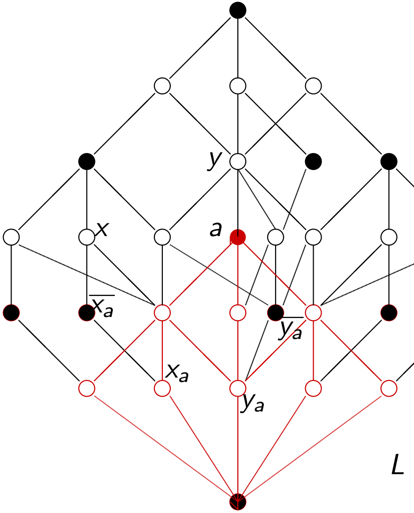
$$(\bigcap_{i \in I} H_i)^G = \bigcap_{i \in I} H_i^G,$$

for every family $\{H_i \mid i \in I\}$ of subgroups.

A group G satisfies the CIP if and only if Δ is a distributive element and the $*$CIP if and only if Δ is an infinitely distributive element of the lattice $\mathrm{Wcon}(\mathcal{A})$.

Let L be an algebraic lattice, in which a is a fixed full codistributive element, we say that a is the **main** codistributive element in L.
For every $x \in L$, we denote by x_a the element from $\downarrow a$ given by:

$$x_a := \bigvee(y \in \downarrow a \mid \overline{y} \leqslant x).$$



Lattice with normal elements - Postulates

Let L be an algebraic lattice in which a is the main codistributive element and let the following hold:

(Pi) For every $x \in \downarrow a$, $[x]_{\varphi_a}$ is a modular lattice; in particular, $[0]_{\varphi_a} = \{0\}$, as the only one-element φ_a-class.

(Pii) For every $x \in L$, $x = \overline{x_a} \vee (x \wedge a)$.

(Piii) For all $x, y \in \downarrow a$, if $x < y$ and $x \neq z_a$ for each $z \in [y, \overline{y}]$, there are $x_i \in \downarrow a, i \in I$, forming the antichain $x, x_i, i \in I$ so that:

- **(a)** $\overline{x_i} \vee y = \overline{x} \vee y$ for all $i \in I$;
- **(b)** $x \wedge (\bigwedge_{i \in I} x_i) = z_a$ for some $z \in [y, \overline{y}]$.

(Piv) The map $f : L \to T_a$, such that $f(x) = \overline{x_a}$ satisfies the following:

- **(a)** if $\chi = \ker f$, then χ-classes are closed under joins in L.
- **(b)** f is compatible with joins in φ_a-classes.

Proposition

For every $b \in \downarrow a$, the map $g_b : [b, \overline{b}] \to \downarrow a$, such that $g_b(x) := x_a$, is an embedding.

Let $n, b \in \downarrow a$, $n \leqslant b$. We say that n is **normal in** $\downarrow b$, if $n = x_a$, for some $x \in [b, \overline{b}]$.

Lemma

Let L be a lattice fulfilling conditions $(Pi) - (Piv)$ and let $n, b \in \downarrow a$, $n \leqslant b$. Then n is normal in $\downarrow b$ if and only if $[\overline{n}, \overline{n} \vee b] \cap T_a = \{\overline{n}\}$.

If n is normal in $\downarrow b$, then we denote it by $n \lhd b$.
In particular, 0 and a are normal elements in $\downarrow a$ $(0 \lhd a,\ a \lhd a)$.

From now on, we call a lattice L a **lattice with normal elements** if it is an algebraic lattice fulfilling conditions $(Pi) - (Piv)$, and in which a is the main codistributive element.

Proposition

Let L be a lattice with normal elements and $b, c, d, e \in \downarrow a$. The following hold:
(a) If $b \lhd c$, then $b \lhd d$ for every $d \in [b, c]$.
(b) If $b \lhd b \vee c$, then $b \wedge c \lhd c$
(c) If $b \lhd c$ and $d \lhd e$, then $b \wedge d \lhd c \wedge e$.
(d) If $b \lhd c$ and $b \lhd d$, then $b \lhd c \vee d$.
(e) if $b \lhd d$ and $c \lhd d$ then $b \vee c \lhd d$.

As straightforward consequences of (c), we have:
(g) $b \lhd c$ implies $b \wedge d \lhd c \wedge d$;
(h) if $b \lhd c$ and $b \lhd d$ then $b \lhd c \wedge d$;
(j) if $b \lhd c$ and $d \lhd c$ then $b \wedge d \lhd c$.

Proposition

Let L be a lattice with normal elements and $\{b_i, c_i \mid i \in I\} \subseteq \downarrow a$. If for every i, $b_i \lhd c_i$, then $\bigwedge_i b_i \lhd \bigwedge_i c_i$.

Theorem

The lattice $\mathrm{Wcon}(G)$ *of a group* G *is a lattice with normal elements in which* Δ *is the main codistributive element. In particular, if* H, K *are subgroups of* G*, then* H *is a normal subgroup of* K *if and only if* $\Delta_H \lhd \Delta_K$ *in the lattice* $\mathrm{Wcon}(G)$*.*

Let $b, c \in \downarrow a$ and $b \lhd c$. Let $c/b := \overline{b} \vee c$. We say that c/b is the **quotient** (of c over b).

Let $b, x \in \downarrow a$, $b \leqslant x$. The **normal closure** of b in $\downarrow x$ is

$$(b)^x := \bigwedge \{w \in \downarrow a \mid b \leqslant w, w \lhd x\}.$$

Proposition

Let $x, y \in L$ such that $x = b/x_a$, and $y = c/y_a$. The following hold:

(a) $x \wedge y = (b \wedge c)/(x_a \wedge y_a)$.

(b) $x \vee y = (b \vee c)/(x_a \vee y_a)^{b \vee c}$.

(c) *In particular, if $b = c$, $x \vee y = b/(x_a \vee y_a)$.*

Proposition

If L is a lattice with normal elements, then for every $b \in \downarrow a$, $\downarrow \overline{b}$ is a lattice with normal elements in which b is the main codistributive element.

Proposition

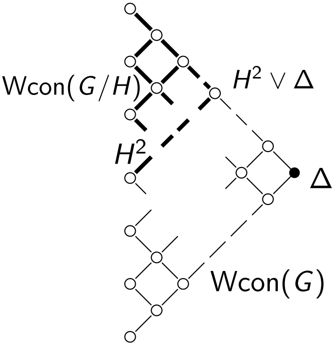
For every $b \in \downarrow a$, $\{x \in \downarrow b \mid x \lhd b\}$ is a modular sublattice of $\downarrow a$.

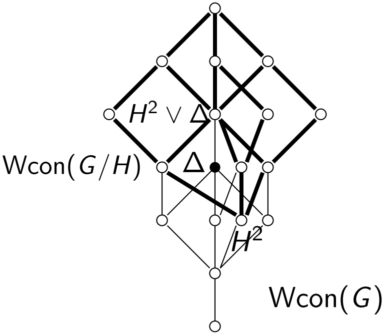
Theorem

If L is a lattice L with normal elements, then for $b, c \in \downarrow a$, $[\overline{b}, \overline{c}]$ is a lattice with normal elements in which c/b is the main codistributive element, if and only if $b \lhd c$.

Proposition

If H is a subgroup of a group G, then $H \lhd G$, if and only if the principal filter $\uparrow(H^2)$ in $\mathrm{Wcon}(G)$ is a lattice with normal elements in which the main codistributive element is $H^2 \vee \Delta$.





Theorem

Let L be a lattice with normal elements. The following are equivalent:
(j) every $n \in \downarrow a$ is normal in $\downarrow a$;
(jj) all kernel-classes of $x \mapsto x \vee a$ have bottom elements;
(jjj) a is an infinitely distributive element in L;
(jv) kernels of $x \mapsto \overline{x_a}$ and $x \mapsto x \vee a$ coincide;
(v) the map $g : \uparrow a \longrightarrow \downarrow a$, $x \mapsto x_a$, is an isomorphism;
(vj) a is a neutral element in L;
(vjj) for every $b \in \downarrow a$, $\uparrow \overline{b}$ is a lattice with normal elements in which a/b is the main codistributive element;
(vjjj) L is a modular lattice.

Corollary

A finite lattice L with normal elements is modular (equivalently: every $n \in \downarrow a$ is normal in $\downarrow a$) if and only if a is a distributive element in L.

A group whose all subgroups are normal is a **Dedekind** group.

A Dedekind group may be **Abelian** or non-Abelian, i.e., **Hamiltonian**.

The latter is a direct product of the quaternion group Q_8, an Abelian group with elements of finite odd order, and an Abelian group of exponent at most 2.

Theorem (Czédli, Erné, Šešelja, Tepavčević, 2009)

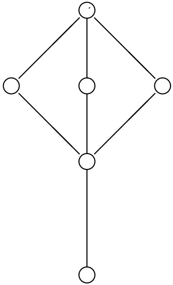
The following are equivalent for a group G:

- *G is a Dedekind group;*
- *the lattice $\mathrm{Wcon}(G)$ of weak congruences of G is modular;*
- *the diagonal relation Δ is a neutral element in the lattice $\mathrm{Wcon}(G)$;*
- *G satisfies the CEP and the CIP.*
- *G satisfies the *CIP.*

Corollary

A finite group G is a Dedekind group if and only if G has the CIP, equivalently
if and only if the diagonal relation Δ is a distributive element in the lattice $\mathrm{Wcon}(G)$.

We call a lattice L an **A-lattice** if it is a modular lattice with normal elements in which $\downarrow a$ does not have an interval-sublattice which is isomorphic with the lattice Q given in figure.



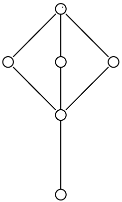
Q

Proposition

If L is an A-lattice, then for every $b \in \downarrow a$, the ideal $\downarrow \overline{b}$ and the filter $\uparrow \overline{b}$ are also A-lattices with main codistributive elements b and $\overline{b} \vee a$ respectively.

Theorem

A group G is Abelian if and only if $\mathrm{Wcon}(G)$ *is an A-lattice.*



Q

Consequently,

A group G is Hamiltonian if and only if the lattice $\mathrm{Wcon}(G)$ *is modular, containing an interval sublattice* $[\Delta_H, \Delta_K]$, $H, K \in \mathrm{Sub}(G)$, *isomorphic to the lattice Q.*

For a group G, the subgroup generated by all the commutators, denoted by G' is the **commutator subgroup** or the **derived subgroup** of G.

A group is **perfect** if it coincides with the commutator subgroup, or equivalently,
if and only if no maximal subgroup is normal.

Let $b \in \downarrow a$. We say that b is the (**lattice**) **commutator** of a, in symbols $b = a'$, if $b = \bigwedge_{i \in I}(b_i \mid b_i \lhd a)$, where $\{b_i \mid i \in I\}$ is the set of all elements in $\downarrow b$, such that for every $i \in I$ the interval $[\overline{b_i}, 1]$ is an A-lattice with the main codistributive element $\overline{b} \vee a$.

Observe that $a' \lhd a$.

.

Corollary

A lattice L with normal elements is an A-lattice if and only if $a' = 0$.

Proposition

The following are equivalent in a finite lattice L with normal elements:
(i) $a' = a$;
(ii) $b \prec a$ *implies* $\overline{b} \prec 1$.

The main codistributive element a in L is **perfect**, if $a' = a$.

Corollary

The main codistributive element a in a lattice L with normal elements is perfect, if and only if for every $b \prec a$, $\uparrow\overline{b}$ is not an A-lattice.

If $b = a'$, then we define $a'' := (a')'$, i.e., a'' is a lattice commutator of b, in the sublattice $\downarrow\overline{b}$ of L, with b being the main codistributive element. Clearly, $a'' \leqslant a' \leqslant a$. Consequently, we define $a^{(n)} := (a^{n-1})'$.

Corollary

The commutator subgroup G' of a group G in the lattice $\mathrm{Wcon}(G)$ is represented by Δ', i.e., by the lattice commutator of the diagonal relation.

Theorem

A group G is:

- *Abelian if and only if $\Delta' = \{(e, e)\}$;*
- *perfect if and only if Δ is a perfect element in $\mathrm{Wcon}(G)$, equivalently, if and only if for every subgroup H of G such that $H \prec G$ in $\mathrm{Sub}(G)$, $\uparrow H^2$ is not an A-lattice in $\mathrm{Wcon}(G)$.*

A finite series of subgroups

$$\{e\} = H_0 < H_1 < \ldots < H_{n-1} < H_n = G$$

such that for every $1 \leqslant i \leqslant n$, $H_{i-1} \lhd H_i$ is a **subnormal series of** G.

Subgroups in a subnormal series need not be normal in G; if they all are, then the above is a **normal series of** G.

A group G is **nilpotent** if there is a **central** series of G, which is a normal series, such that H_{i+1}/H_i is contained in the center of the quotient group G/H_i for every i.

A subnormal series is **solvable** if for every $1 \leqslant i \leqslant n$ the quotient group H_i/H_{i-1} is Abelian.
A group is **solvable** if it has a solvable series of subgroups.

A subnormal series of subgroups of a group G is **supersolvable** if for every $1 \leqslant i \leqslant n$ the quotient group H_i/H_{i-1} is cyclic.
A group G is **supersolvable** if it has a supersolvable series of subgroups.

A group is supersolvable if and only if it has a subnormal series whose quotient factors are either cyclic groups of prime order or infinite cyclic groups.

Let L be a lattice with normal elements.

Consider a finite series of elements

$$0 = b_0 < b_1 < \ldots < b_{n-1} < b_n = a$$

such that for every $1 \leqslant i \leqslant n$, $b_{i-1} \lhd b_i$.

The intervals $[\overline{b_{i-1}}, \overline{b_i}]$ are lattices with normal elements, with main codistributive elements $\overline{b_{i-1}} \vee b_i$.
Then

$$[0, \overline{b_1}], [\overline{b_1}, \overline{b_2}], \ldots, [\overline{b_{n-1}}, 1]$$

is a **central series of intervals** in L induced by the above series.

Clearly, such a series of intervals exists in every lattice with normal elements, induced by at least the series $0, a$.

Theorem

A group G is solvable if and only if the lattice $\mathrm{Wcon}(G)$ has a central series of intervals consisting of A-lattices,
if and only if the sublattice $\uparrow H^2$ of $\mathrm{Wcon}(G)$ is an A-lattice, where $\Delta_H = \Delta^{(n)}$ for some $n \in \mathbb{N}$.

Consider the lattice

$$(\{(m,n) \mid m,n \in \mathbb{N}_0, n \text{ divides } m\}, \sqsupseteq),$$

where $\mathbb{N}_0$ is the set of nonnegative integers and the order $\sqsupseteq$ is the componentwise applied dual of divisibility relation: *contains as a factor*; observe also that for every $n \in \mathbb{N}_0$, we have $0 \sqsupseteq n$.
We call this lattice ***Z*-lattice**.

Z-lattice is isomorphic to the weak-congruence lattice of the additive group $(\mathbb{Z}, +)$ of integers.
Therefore, Z-lattice is a lattice with normal elements.

Theorem

A group G is supersolvable if and only if the weak-congruence lattice of G has a central series of intervals consisting of three-element chains or Z-lattices.

Corollary

A finite group G is supersolvable if and only if the weak-congruence lattice of G has a central series of intervals which is a (finite) chain.

A lattice L is **lower semimodular** if for all $x, y \in L$,

$$x \prec x \vee y \text{ implies } x \wedge y \prec y.$$

Theorem

A finite group G is nilpotent if and only if the lattice $\mathrm{Wcon}(G)$ of weak congruences of G is lower semimodular.

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Thanks for listening!