The Number of Countable Subdirect Powers of Finite Unary Algebras

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Joint work with Nik Ruškuc

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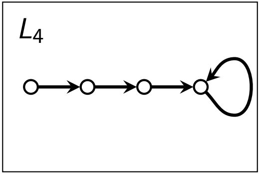
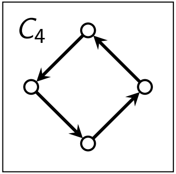
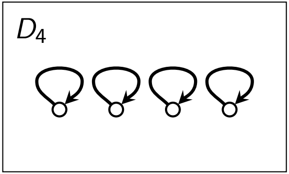
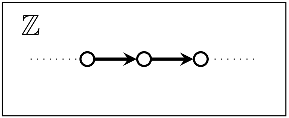
Introductory Definition: Unary Algebras

A set A and a collection $\mathcal{F}$ of unary operations on A.
Can be represented as a directed graph.









Introductory Definitions: Unary Algebras

If we have isomorphic algebras, than the corresponding graphs are isomorphic.

Connected components are subalgebras, and the isomorphism type of an algebra is completely determined by the number of connected components of each isomorphism type.

If A_1, A_2 are subalgebras of A, then the union $B = A_1 \cup A_2$ is also a subalgebra. Furthermore if $A_1 = \langle S_1 \rangle$ and $A_2 = \langle S_2 \rangle$ then $B = \langle S_1 \cup S_2 \rangle$.

Useful ideas

Let $a = (a_x)_{x \in X} \in A^X$.

The content of a is $\{a_x : x \in X\}$.

The format of a is an equivalence relation on X with (x, y) is in the format iff $a_x = a_y$.

Introductory Definition: Subdirect Products

A special type of subalgebra of the direct product.
Projection Maps:

- $\pi_1 : A \times B \to A$, $(a, b) \mapsto a$
- $\pi_2 : A \times B \to B$, $(a, b) \mapsto b$

A subdirect product is a subalgebra P of $A \times B$, such that $\pi_1|_P, \pi_2|_P$ are surjective.
Can be extended to an arbitrary number of factors.

Universality

Theorem

(Birkhoff) Every algebra is a subdirect product of its subdirectly irreducible quotients.

An algebra is subdirectly irreducible if whenever it is expressed as a subdirect product of $\Pi_{i \in I} A_i$, then some projection π_i is an isomorphism.

Fiber Products

For algebras A, B, Q and surjective homomorphisms $\phi : A \to Q$ and $\psi : B \to Q$,

$$\{(a, b) \in A \times B : \phi(a) = \psi(b)\}$$

is a subdirect product of $A \times B$. This is called the fiber product of A and B with respect to ϕ, ψ.

Theorem

(Fleischer's Lemma) Every subdirect product of two algebras in a congruence permutable variety is a fiber product.

Boolean Powers

Let A be a finite algebra and B be a boolean algebra of subsets of S. Then the boolean power A^{B} is the set of tuples $a \in A^S$ such that every equivalence class in the format of a is in B.
A^{B} is a subdirect product of A^S.

How many subdirect powers are there?

Theorem

(Hickin, Plotkin 1981, Mckenzie 1982) A finite group G has countably many non-isomorphic countable subdirect powers iff G is abelian.

Theorem

(Clayton, Ruškuc, in prep) A finite commutative semigroup S has countably many non-isomorphic countable subdirect powers iff S is an abelian group or a zero semigroup.

Theorem

(Ruškuc, de Witt) A finite unary algebra $(A, \mathcal{F})$ has countably many non-isomorphic countable subdirect powers iff each $f \in \mathcal{F}$ is either a bijection or a constant map.

Monounary case

Lemma

Let (A, f) be a finite monounary algebra, and let f be a bijection. Then A has countably many non-isomorphic countable subdirect powers.

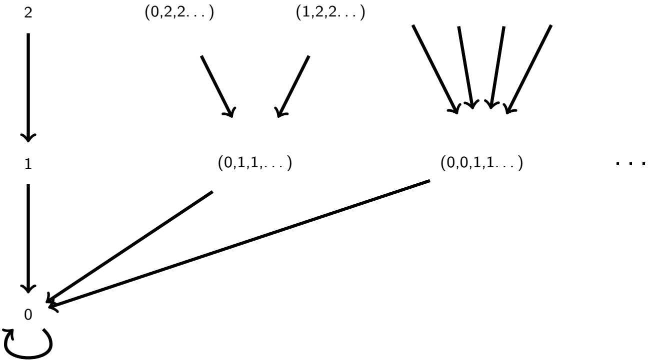
Lemma

Let (A, f) be a finite monounary algebra, and let f be a constant function. Then A has countably many non-isomorphic countable subdirect powers.

Monounary case

Lemma

All other finite monounary algebras have uncountably many non-isomorphic countable subdirect powers.



Monounary case

Lemma

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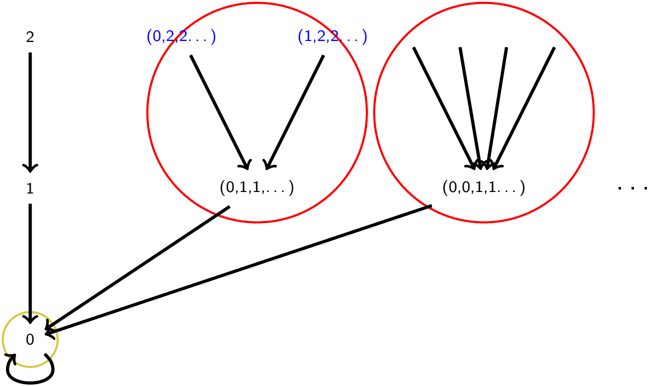
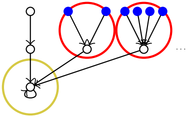


Tools for Unary



Definition

Let $(A, \mathcal{F})$ be a unary algebra. Then we define the following:

1. $B \subseteq A$ is a bottom level component if it is strongly connected and for all $a \in B$ and $f \in \mathcal{F}$, we have $f(a) \in B$.
2. for a bottom level component B, an outer section of A with respect to B is a connected component of the graph $A \setminus B$.
3. $T \subseteq A$ is a top level component if it is strongly connected and there does not exist $a \in A \setminus T$ and $f \in \mathcal{F}$ such that $f(a) \in T$.

Lemma

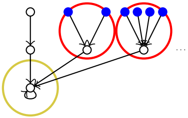
The above are preserved under isomorphism.

Tools for Uncountable Type

Lemma

Let $(A, \mathcal{F})$ *be a finite unary algebra, and* $\mathrm{a} \in A^{\mathbb{N}}$ *be a tuple with* $\mathrm{cont}(\mathrm{a}) = A$*. Then the set* $\{f_1 \circ \cdots \circ f_n(\mathrm{a}) : f_1, \ldots, f_n$ *are bijections in* $\mathcal{F}, n \in \mathbb{N}\}$ *is a top level component of* $A^{\mathbb{N}}$*.*

Proof outline



Let $\mathrm{Mon}(A)$ be the monoid of functions on A generated by $\mathcal{F}$,and pick an $g \in \mathrm{Mon}(A)$ such that $|g(A)| > 1$ is minimal.
Pick a nice sequence of tuples $b_1, b_2, \ldots$ with elements from $g(A)$.
For a each tuple b_k find tuples $t_{k,1}, \ldots, t_{k,k}$ which are contained in distinct top level components, such that $g(t_{k,i}) = b_k$.
This gives a collection of subalgebras $S_n = \langle t_{n,1}, \ldots, t_{n,n} \rangle$ which are all non-isomorphic, and whose pairwise intersections are either all empty or all a bottom level component of the diagonal.
Take arbitrary unions of the S_n, and add in the diagonal to ensure subdirectness, giving uncountably many subdirect powers.

Infinite examples

Let $C_{\mathbb{N}}$ be a monounary algebra, containing precisely one cycle of every length.
$C_{\mathbb{N}}$ has uncountably many non-isomorphic countable subdirect powers.

Let T_2 be a unary algebra on a countable set, whose operations are the bijections which are the identity on all but two points.
T_2 has uncountably many non-isomorphic countable subdirect powers.

$(\mathbb{N}, +1)$ has countably many non-isomorphic subdirect powers.

Related Questions

Question

Does an algebra have countably many countable subdirect powers if and only if it is abelian?

An algebra A is abelian is for all term operators $t(x, \bar{y})$, and for all elements $a, b \in A$ and tuples $\bar{c}, \bar{d}$ from A, we have $t(a, \bar{c}) = t(a, \bar{d}) \Rightarrow t(b, \bar{c}) = t(b, \bar{d})$.

Question

Is being boolean separating algebras equivalent to having uncountably many countable subdirect powers?

Related Questions

For finite groups we know the answer:

$$
\begin{tikzcd}[column sep=large, row sep=large]
\begin{array}{c}\text{Countably many}\\ \text{subdirect powers}\end{array} \arrow[rr, Rightarrow, shift left] \arrow[dr, Rightarrow, shift left] & & \begin{array}{c}\text{Non-Boolean}\\ \text{Separating}\end{array} \arrow[ll, Rightarrow, shift left, "/" marking] \arrow[dl, Rightarrow, shift left, "/" marking, "\text{Lawrence,1981}"] \\
& \text{Abelian} \arrow[ul, Rightarrow, shift left] \arrow[ur, Rightarrow, shift left] &
\end{tikzcd}
$$

Related Questions

Using our results we have the following for the general case:

$$
\begin{tikzcd}
\begin{array}{c}\text{Countably many}\\\text{subdirect powers}\end{array} \arrow[rr, phantom, "\substack{\Rightarrow \\ \not\Leftarrow}"] & & \begin{array}{c}\text{Non-Boolean}\\\text{Separating}\end{array} \arrow[dl, Rightarrow, "/" marking, color=blue] \\
& \text{Abelian} \arrow[ul, Rightarrow, "/" marking, color=red] &
\end{tikzcd}
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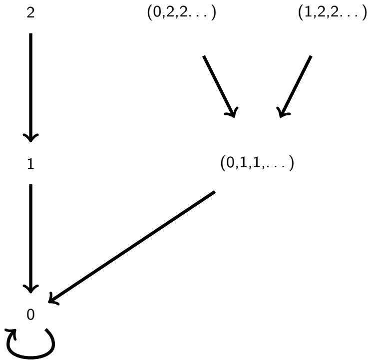
Boolean Separation: the 2-line



Thank you for listening