ON WHEN THE UNION OF TWO ALGEBRAIC SETS IS ALGEBRAIC

Bernardo Rossi (joint work with E. Aichinger and M. Behrisch)

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JOHANNES KEPLER UNIVERSITY LINZ

Introduction

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We know that the solutions sets of systems of equations over a field induce a topology.

We want to characterize clones with similar properties.

Algebraic sets

Definition (Universal algebraic geometry)

Let $\mathcal{C}$ be a clone on a set A.

$B \subseteq A^n$ is $\mathcal{C}$-algebraic if there exist functions $(p_i)_{i \in I}, (q_i)_{i \in I}$ in $\mathcal{C}^{[n]}$ such that

$$B = \{\, \boldsymbol{x} \in A^n \mid \forall i \in I : p_i(\boldsymbol{x}) = q_i(\boldsymbol{x}) \,\}.$$

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For $n \in \mathbb{N}$, $\mathrm{Alg}_n \mathcal{C}$ is the collection of the algebraic subsets of A^n.

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$\mathcal{K} \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{P}(A^n)$ is an algebraic geometry if $\exists \mathcal{D}$ on A such that $\mathcal{K} = \operatorname{Alg} \mathcal{D}$.

Universal Algebraic Geometry

Studying universal algebraic geometries is related to studying the following equivalence relation:

Definition

Two clones $\mathcal{C}$ and $\mathcal{D}$ on A are algebraically equivalent ($\mathcal{C} \sim_{\text{alg}} \mathcal{D}$) if $\operatorname{Alg} \mathcal{C} = \operatorname{Alg} \mathcal{D}$.

Closure properties of algebraic geometries

Properties of $\operatorname{Alg}\mathcal{C}$

All algebraic geometries satisfy the following properties:

- $\forall n \in \mathbb{N}$, $\forall (B_i)_{i\in I}$ from $\operatorname{Alg}_n \mathcal{C}$ we have $\bigcap_{i\in I} B_i \in \operatorname{Alg}_n \mathcal{C}$;
- $\forall n, m \in \mathbb{N}$, $\forall \sigma\colon \{1,\ldots,n\} \to \{1,\ldots,m\}$:
  $B \in \operatorname{Alg}_n \mathcal{C} \Rightarrow \{\, (a_1,\ldots,a_m) \mid (a_{\sigma(1)},\ldots,a_{\sigma(n)}) \in B \,\} \in \operatorname{Alg}_m \mathcal{C}$.

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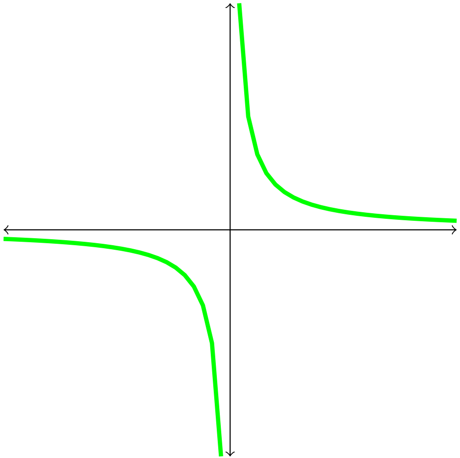

The green line is the algebraic set $\{(x_1, x_2) \mid x_1 \cdot x_2 = 1\}$. Its projection on the first coordinate is not algebraic.

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In general algebraic geometries are not relational clones and are not closed under finite unions.

Adding additional closure properties

Lemma (Tòth and Waldhauser 2017)

Let $\mathcal{C}$ be a clone such that $\mathrm{Alg}\,\mathcal{C}$ is a relational clone. Then $\mathrm{Alg}\,\mathcal{C} = \mathsf{Inv}\,\mathcal{C}^*$.

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Theorem (Burris, Willard 1987)

On a finite set there are only finitely many clones of the form $\mathcal{C}^*$.

Corollary

On a finite set there are only finitely many algebraic geometries that are relational clones.

Geometries on the two-element set

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Equationally additive clones

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A clone $\mathcal{C}$ on a set A is equationally additive if for all $n \in \mathbb{N}$ and for all $B, C \in \operatorname{Alg}_n \mathcal{C}$ we have $B \cup C \in \operatorname{Alg}_n \mathcal{C}$.

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Theorem (Pinus 2017)

On a finite set there are only finitely many equationally additive clones modulo algebraic equivalence.

The number of geometries on finite sets

Theorem (Aichinger and R. 2022)

Let A be a finite set with at least three elements.

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Theorem (Aichinger, Behrisch, R.)

On the two-element set there are exactly $\aleph_0$ equationally additive clones.
On a finite set with at least three elements there are exactly $2^{\aleph_0}$ equationally additive clones.

Number of clones modulo $\sim_{\text{alg}}$

Property	Number of clones	Number of clones modulo $\sim_{\text{alg}}$
all $n = 2$	$\aleph_0$	25
all $n > 2$	$2^{\aleph_0}$	$2^{\aleph_0}$
equationally additive, $n > 2$	$2^{\aleph_0}$	finite

We want to describe algebras whose clone of term functions or polynomial functions is equationally additive.

Known results

Theorem (Daniyarova, Myasnikov, Remeslennikov 2011)

For a commutative associative ring $\mathbf{A}$ with $A \neq 0$ the following are equivalent:

- $\mathbf{A}$ has no zero divisors;
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- If $\mathbf{G}$ is simple and non-Abelian, then $\operatorname{Pol} \mathbf{G}$ is equationally additive.

A special relation

For a set A we define

$$\Delta_A^{(4)} := \{\, \boldsymbol{x} \in A^4 \mid x_1 = x_2 \text{ or } x_3 = x_4 \,\}\,.$$

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Theorem (Daniyarova, Myasnikov, Remeslennikov 2011)

A clone $\mathcal{C}$ on a set A is equationally additive if and only if $\Delta_A^{(4)} \in \mathrm{Alg}_4\,\mathcal{C}$.

Proof

Let $\mathcal{C}$ be a clone on a A with $\Delta_A^{(4)} \in \operatorname{Alg}\mathcal{C}$, let $B, C \subseteq A^n$ and let us suppose that

$$\begin{aligned}\Delta_A^{(4)} &= \{\boldsymbol{a} \in A^4 \mid \forall i \in I\colon \quad p_i(\boldsymbol{a}) = q_i(\boldsymbol{a})\}\\ B &= \{\boldsymbol{a} \in A^n \mid \forall j \in J\colon \; f_j(\boldsymbol{a}) = g_j(\boldsymbol{a})\}\\ C &= \{\boldsymbol{a} \in A^n \mid \forall k \in K\colon h_k(\boldsymbol{a}) = t_k(\boldsymbol{a})\}\end{aligned}$$

for $(p_i)_{i\in I}, (q_i)_{i\in I} \subseteq \mathcal{C}^{[4]}$, $(f_j)_{j\in J}, (g_j)_{j\in J}, (h_k)_{k\in K}, (t_k)_{k\in K} \subseteq \mathcal{C}^{[n]}$. Then we have

$$\begin{aligned}B \cup C = \{&\boldsymbol{a} \in A^n \mid \forall (i,j,k) \in I \times J \times K\colon\\ &p_i(f_j(\boldsymbol{a}), g_j(\boldsymbol{a}), h_k(\boldsymbol{a}), t_k(\boldsymbol{a})) = q_i(f_j(\boldsymbol{a}), g_j(\boldsymbol{a}), h_k(\boldsymbol{a}), t_k(\boldsymbol{a}))\}.\end{aligned}$$

Integral domains

Let $\mathbf{K} = (K; +, -, 0, \cdot)$ be a ring with no zero divisors. Then

$$\begin{aligned}\Delta_K^{(4)} &= \{\boldsymbol{k} \in K^4 \mid k_1 = k_2 \text{ or } k_3 = k_4\} \\ &= \{\boldsymbol{k} \in K^4 \mid (k_1 - k_2) \cdot (k_3 - k_4) = 0\} \\ &= \{\boldsymbol{k} \in K^4 \mid f(\boldsymbol{k}) = 0\},\end{aligned}$$

where $f = (x_1 - x_2) \cdot (x_3 - x_4)$.

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Let $\{\, p_i \mid i \in I \,\}, \{\, q_j \mid j \in J \,\} \subseteq \operatorname{Clo}_n \mathbf{K}$.

Let $A = \{\, \boldsymbol{k} \in K^n \mid \forall i \in I :\, p_i(\boldsymbol{k}) = 0 \,\}$ and
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$A \cup B = \{\, \boldsymbol{k} \in K^n \mid \forall i \in I, \forall j \in J :\, f(p_i, 0, q_j, 0)(\boldsymbol{k}) = 0 \,\}$.

Consequences

Corollary

Let A be a set. The set of equationally addive clones on A is an order filter in the poset of clones on A.

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On the two-element set we can describe the equationally additive clones by giving the generators of the filter.

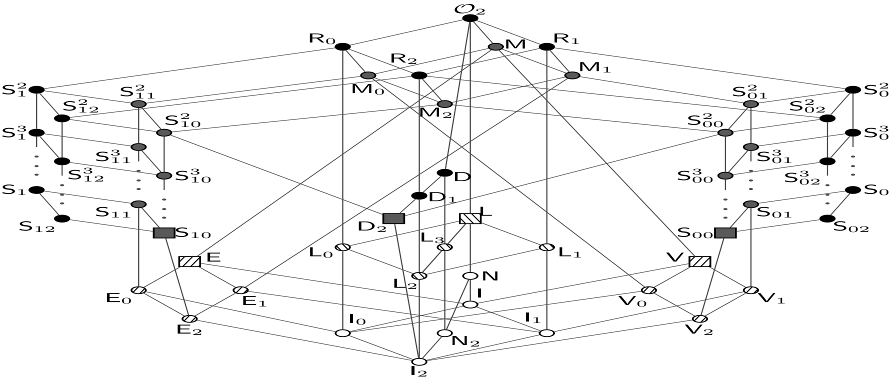
Equationally additive Boolean clones

Theorem (Aichinger, Behrisch, R.)

For a clone $\mathcal{C}$ on the two element set the following are equivalent:

1. $\mathcal{C}$ is equationally additive;
2. $\mathcal{C}$ is above one of the following clones:
    - 2.1 D_2 generated by the majority operation;
    - 2.2 S_{00} generated by $(x, y, z) \mapsto x \vee (y \wedge z)$;
    - 2.3 S_{10} generated by $(x, y, z) \mapsto x \wedge (y \vee z)$.

The Post Lattice



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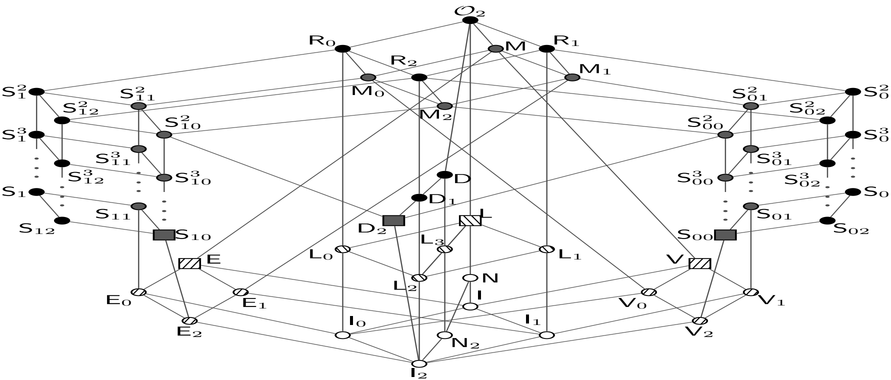

○ clones of TCT-type $\mathbf{1}$ ⦸ clones of TCT-type $\mathbf{2}$ ⊘ clones of TCT-type $\mathbf{5}$

● clones of TCT-type $\mathbf{3}$ ● clones of TCT-type $\mathbf{4}$

Equationally additive boolean clones

Corollary

For a clone $\mathcal{C}$ on the two element set the following are equivalent:

1. $\mathcal{C}$ is equationally additive;
2. $\mathcal{C}$ is of TCT-type $\mathbf{3}$ or $\mathbf{4}$;
3. the algebra $(A;\mathcal{C})$ generates a congruence distributive variety.

Structure of equationally additive algebras

Lemma (Aichinger, Behrisch, R.)

Let A be a finite set, let $\mathcal{C}$ be a clone on A, let $\mathbf{A} = (A; \mathcal{C})$, and let $f \in \mathcal{C}^{[4]}$ and $a \in A$ be such that $\Delta_A^{(4)} = \{\boldsymbol{a} \in A^4 \mid f(\boldsymbol{a}) = a\}$. Then

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- $\langle 0_{\mathbf{A}}, \mu \rangle$ has TCT-type $\mathbf{3}$.

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4. $\mathbf{B}/\mu \cong \mathbf{A} + a$;

   where $\mathbf{A} + a$ is $\mathbf{A}$ expanded with the 4-ary function with constant value a.

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Let A be finite.

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5. If $\mathbf{A}$ is E-minimal, then $\mathbf{A}$ is not of type $\mathbf{1}$.

E-minimal algebras

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Theorem (Aichinger, Behrisch, R.)

The clone of term operations of an E-minimal algebra $\mathbf{A}$ is equationally additive if and only if $\mathbf{A}$ is of TCT-type $\mathbf{3}$ or $\mathbf{4}$.

What can we say about the clone of polynomial functions of a Mal'cev algebra?

Interpolation Lemma

Interpolation Lemma

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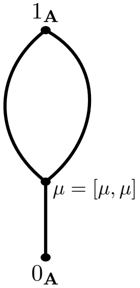

Figure: $\operatorname{Con} \mathbf{A}$

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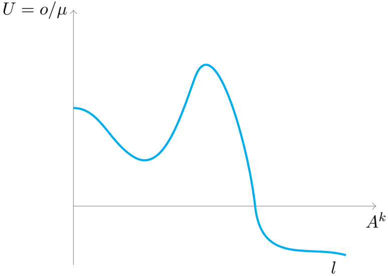

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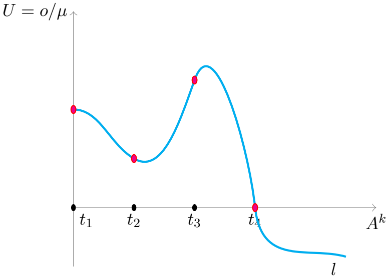

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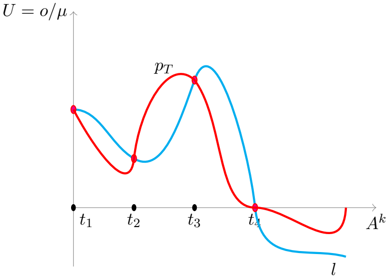
and let $l : A^k \to U$ for $k \in \mathbb{N}$.
Then for all $T \subseteq A^k$ finite,
$\exists p_T \in \mathrm{Pol}_k\, \mathbf{A}$ such that
$\forall \boldsymbol{t} \in T \colon p_T(\boldsymbol{t}) = l(\boldsymbol{t})$,
and $\forall \boldsymbol{x} \in A^k \colon p_T(\boldsymbol{x}) \in U$.



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Let us define $f\colon A^4 \to A$ by

$$f(\boldsymbol{x}) = \begin{cases} a & \text{if } \boldsymbol{x} \in \Delta_A^{(4)}, \\ b & \text{otherwise,} \end{cases}$$

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Then the Interpolation Lemma yields $f \in \operatorname{Pol}_4 \mathbf{A}$, and we have

$$\Delta_A^{(4)} = \{\boldsymbol{x} \in A^4 \mid f(\boldsymbol{x}) = a\}.$$

Main result. Finite case

Theorem (Aichinger, Behrisch, R.)

For a **finite** Mal'cev algebra $\mathbf{A}$ with $|A| \geq 2$ the following are equivalent:

1. $\mathbf{A}$ is subdirectly irreducible and the monolith is non-Abelian.
2. There exists $f \in \operatorname{Pol}_4 \mathbf{A}$ and $a \in A$ such that $\Delta_A^{(4)} = \{\boldsymbol{x} \mid f(\boldsymbol{x}) = a\}$.
3. $\operatorname{Pol} \mathbf{A}$ is equationally additive.

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- $A(5)$ is simple and non-Abelian, thus it satisfies condition (1);
- $\operatorname{Clo} A(5)$ is not equationally additive (cf. Daniyarova, Myasnikov, Remeslennikov 2011).

Expansions of finite Abelian groups

Corollary (Aichinger, Behrisch, R.)

Let $\mathbf{G}$ be a finite Abelian group.

The number of constantive equationally additive expansions of $\mathbf{G}$ is

- finite, if $|G|$ is square free or the square of a prime,
- countably infinite, otherwise.

What can we say about algebras with infinite domain?

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A binary polynomial p is absorbing at $(u_1, u_2) \in A^2$ if $\forall x_1, x_2 \in A$ we have $p(x_1, u_2) = p(u_1, x_2) = p(u_1, u_2)$.

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Absorption Lemma

Let $\alpha = \mathrm{Con}_{\mathbf{A}}(\{(u_1, v_1)\})$ and $\beta = \mathrm{Con}_{\mathbf{A}}(\{(u_2, v_2)\})$. Then

$$[\alpha, \beta] = \{(c(v_1, v_2), c(u_1, u_2)) \mid c \in \mathrm{Pol}_2\,\mathbf{A} \text{ is absorbing at } (u_1, u_2)\}.$$

Consequences of the Absorption Lemma

The Absorption Lemma allows us to prove the following:

Proposition

Let $\mathbf{A}$ be an algebra with a Mal'cev polynomial such that
$\forall \alpha, \beta \in \operatorname{Con} \mathbf{A} \setminus \{0_{\mathbf{A}}\} \colon [\alpha, \beta] > 0_{\mathbf{A}}$.
Then $\operatorname{Pol} \mathbf{A}$ is equationally additive.

Main result

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$\forall \boldsymbol{a} \in A^n \colon \{\boldsymbol{a}\} \in \operatorname{Alg}_n \operatorname{Pol}\mathbf{A}$, since

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Thus, every finite set is algebraic.

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If A is finite, then (1)-(4) are equivalent to the following:

5. $\mathbf{A}$ is subdirectly irreducible and the monolith μ is non-Abelian.
6. There exist $f \in \operatorname{Pol}_4\mathbf{A}$ and $a \in A$ such that $\Delta_A^{(4)} = \{\boldsymbol{x} \mid f(\boldsymbol{x}) = a\}$ and f_γ is constant for all $\gamma \in \operatorname{Con}\mathbf{A} \setminus \{0_{\mathbf{A}}\}$.