Higher Kiss terms

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$$\exists \text{ exactly one } i \in n(f(i) \neq g(i)).$$
- A (0)-dimensional cube is a single vertex, a (1)-dimensional cube is a pair connected by an edge, a (2)-dimensional cube is a square, a (3) dimensional cube is a cube, etc.

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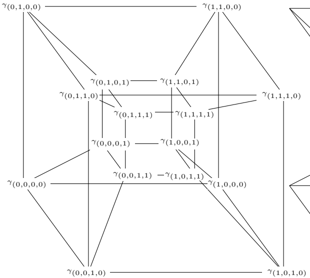
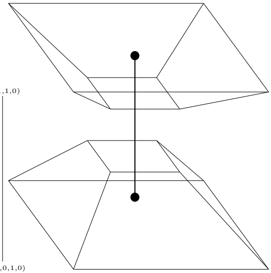
- If $Q = \{i\}$ is a singleton set, we call this map Faces_i. If $n \setminus Q = \{i\}$ is a singleton set, we call this map Lines_i

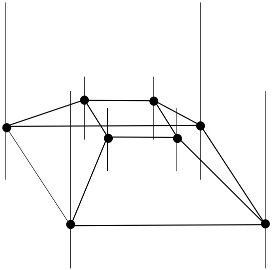
Example pictures:



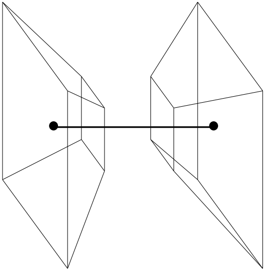
$\gamma \in A^{2^4}$



$\mathrm{Cut}_{\{1\}}(\gamma)$ or $\mathrm{Faces}_1(\gamma)$



$\mathrm{Cut}_{\{0,2,3\}}(\gamma)$ or $\mathrm{Lines}_1(\gamma)$



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\begin{tikzcd} c \arrow[r, dash] \arrow[d, dash] & d \arrow[d, dash] \\ a \arrow[r, dash] & b \end{tikzcd} \in R \implies \begin{tikzcd} d \arrow[r, dash] \arrow[d, dash] & c \arrow[d, dash] \\ b \arrow[r, dash] & a \end{tikzcd}, \begin{tikzcd} a \arrow[r, dash] \arrow[d, dash] & b \arrow[d, dash] \\ c \arrow[r, dash] & d \end{tikzcd} \in R.
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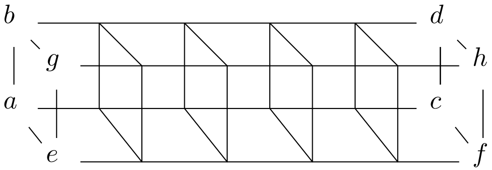
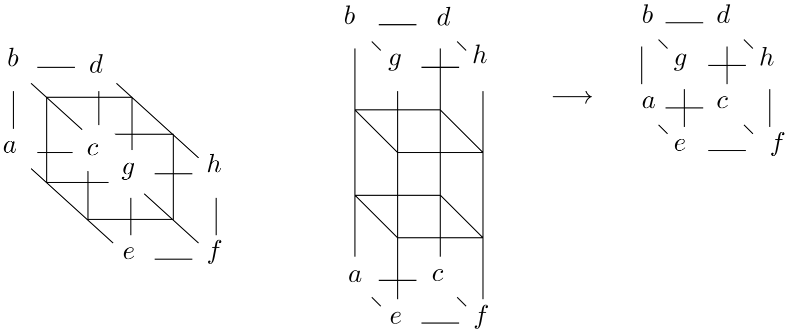
- (n)-transitive if $\mathrm{Faces}_i(R)$ is transitive for each $i \in R$.

Here's the $n = 3$ picture of (n)-transitivity:





Each cube is an element of a (3)-transitive relation R.

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Note: The (1)-dimensional congruences form a bigger collection (in general) than the congruences of an algebra. This follows from the definitions. The (0)-dimensional congruences of an algebra are subalgebras of $\mathbb{A}$ and the (1)-dimensional congruences are the congruences of subalgebras of $\mathbb{A}$.

Mal'cev complexes

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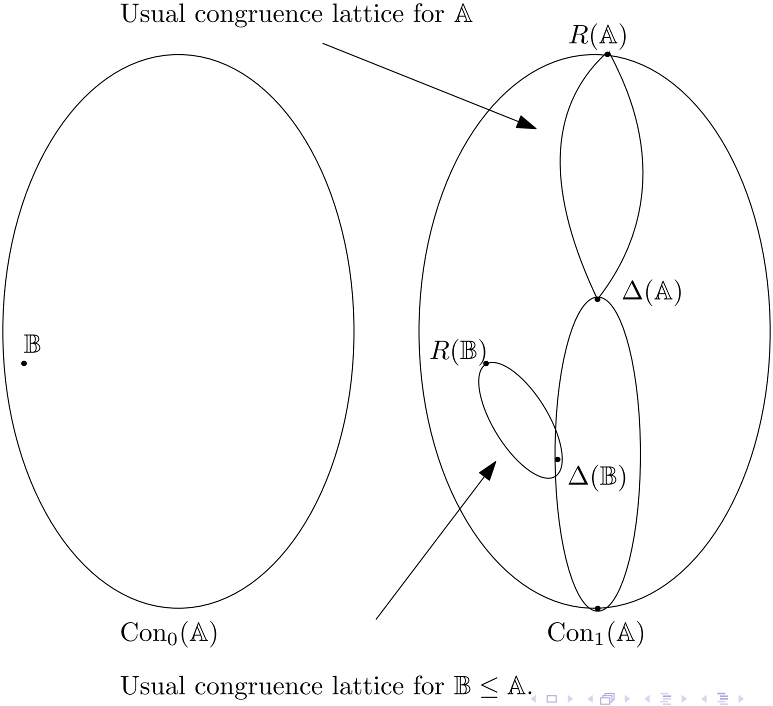
Usual congruence lattice for $\mathbb{A}$
$R(\mathbb{A})$
$\Delta(\mathbb{A})$
$\mathbb{B}$
$R(\mathbb{B})$
$\Delta(\mathbb{B})$
$\mathrm{Con}_0(\mathbb{A})$
$\mathrm{Con}_1(\mathbb{A})$
Usual congruence lattice for $\mathbb{B} \leq \mathbb{A}$.

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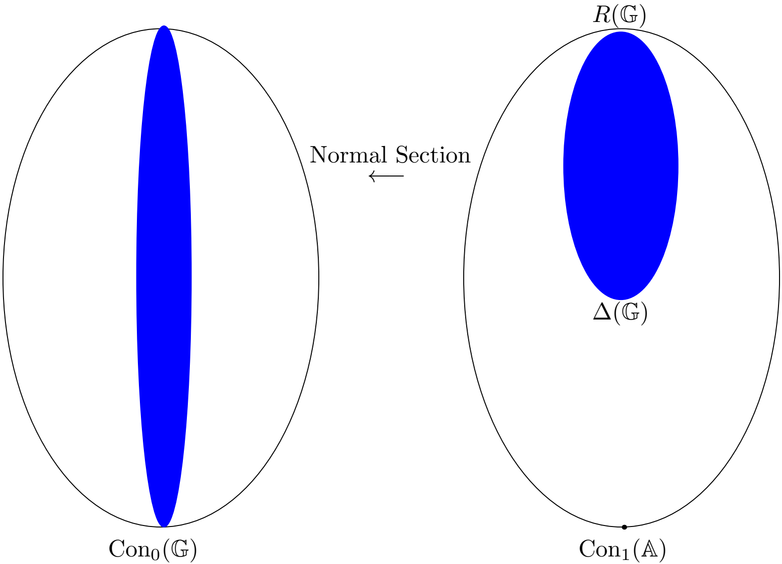
Example: Groups



$$
\begin{aligned}
N(\alpha) &= \{g \in G : \langle 1, g\rangle \in \alpha\} \\
&= \{x^{-1}y : \langle x, y\rangle \in \alpha\}
\end{aligned}
$$

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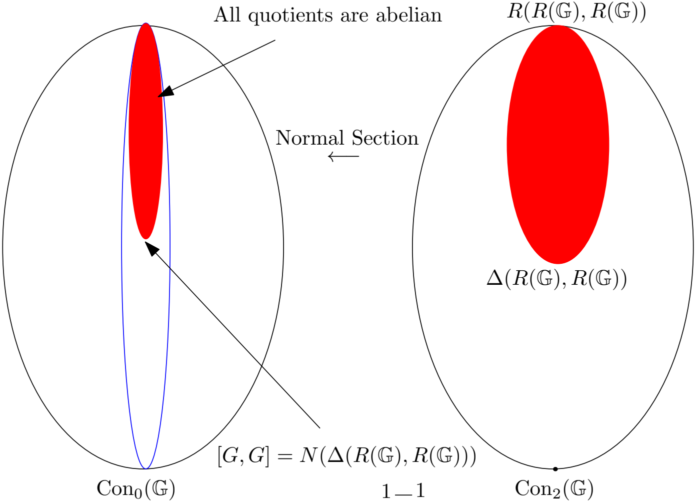

$$N(\beta) = \{g \in G : \begin{array}{ccc} 1 & — & 1 \\ | & & | \\ 1 & — & g \end{array} \in \beta\}$$

$$= \{c^{-1}ab^{-1}d : \begin{array}{ccc} a & — & b \\ | & & | \\ c & — & d \end{array} \in \beta\}$$

Example: Groups

- For $\mathbb{G}$ a group, we can define a 'normal section' map from any higher dimensional congruence lattice to $\mathsf{Con}_0(\mathbb{G})$: for a particular $\alpha \in \mathsf{Con}_n(\mathbb{G})$, take $N(\alpha)$ to be the set of all g that color a vertex in a cube belonging to α that is colored by the identity everywhere else.

Example: Groups

- For example, for $\alpha \in \mathrm{Con}_3(G)$,

$$N(\alpha) = \left\{ g \in G : \begin{tikzcd}[column sep=small, row sep=small] 1 \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & 1 \arrow[dr, dash] \arrow[dd, dash] & \\ & 1 \arrow[rr, dash] \arrow[dd, dash] & & g \arrow[dd, dash] \\ 1 \arrow[rr, dash] \arrow[dr, dash] & & 1 \arrow[dr, dash] & \\ & 1 \arrow[rr, dash] & & 1 \end{tikzcd} \in \alpha \right\}$$

$$= \left\{ hf^{-1}eg^{-1}ca^{-1}bd^{-1} : \begin{tikzcd}[column sep=small, row sep=small] c \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & d \arrow[dr, dash] \arrow[dd, dash] & \\ & g \arrow[rr, dash] \arrow[dd, dash] & & h \arrow[dd, dash] \\ a \arrow[rr, dash] \arrow[dr, dash] & & b \arrow[dr, dash] & \\ & e \arrow[rr, dash] & & f \end{tikzcd} \in \alpha \right\}$$

In this case, the subgroups that are the image of this map are those subgroups that one can quotient (a particular subgroup) by to obtain a 2-step nilpotent group.

Example: Groups

- Main idea: The method of obtaining a normal subgroup from a congruence of a group generalizes to a method of obtaining more special kinds of normal subgroups (quotients are abelian, (2)-step nilpotent, etc.) of a group from a higher dimensional congruence. Of particular importance are the $\Delta - R$ intervals, which are higher dimensional analogues of 'regular' congruence lattices.

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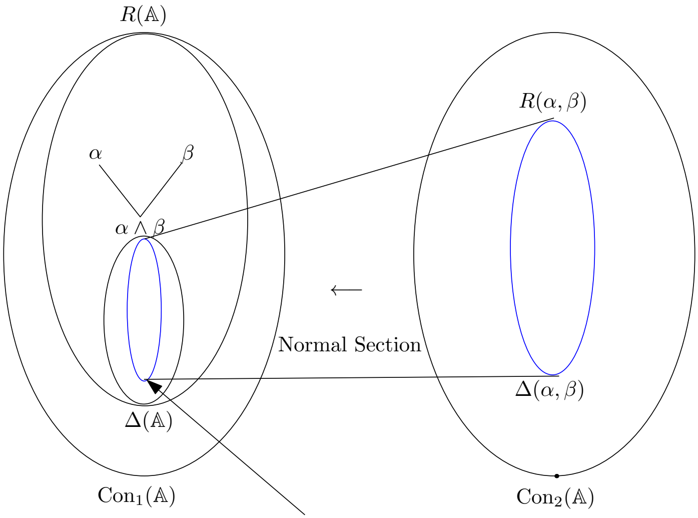
- For a general algebra $\mathbb{A}$, there is no analogue of a normal subgroup. We can still define a normal section map to the (1)-dimensional congruence lattice that generalizes the normal section map for groups. For $n \geq 2$, we set

$$N : \mathrm{Con}_n(\mathbb{A}) \to \mathrm{Con}_1(\mathbb{A})$$
$$\alpha \mapsto \{\langle x, y\rangle : \exists \text{cube} \in \alpha \text{ with } 2^n - 1 \text{ vertices colored by } x, \text{ and } 1 \text{ vertex colored by } y\}$$



$N(\Delta(\alpha, \beta))$ is $[\alpha, \beta]_H$, the 'hypercommutator' of α and β.

Definition

Let $\mathbb{A}$ be an algebra, $n \geq 0$, and let $\theta_0, \ldots, \theta_{n-1}$ be (ordinary) congruences of $\mathbb{A}$. Set the *hypercommutator* of these congruences to be

$$[\theta_0, \ldots, \theta_{n-1}]_H = N(\Delta(\theta_0, \ldots, \theta_{n-1})).$$

We say a congruence θ is (n)-step supernilpotent if

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NB It is known that there exist abelian Taylor algebras with nonabelian quotients. A natural question is: which quotients of abelian Taylor algebras are abelian? The answer is: exactly those quotients whose kernel is the normal section of a (2)-dimensional congruence belonging to $I_{\Delta,R}(R(\mathbb{A}), R(\mathbb{A}))$.

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- Each $I_{\Delta,R}(\theta_0, \ldots, \theta_n - 1)$ is an interval in $\mathrm{Tol}_n(\mathbb{A})$. The top element is again $R(\theta_0, \ldots, \theta_{n-1})$ and we call the bottom element $M(\theta_0, \ldots, \theta_{n-1})$. It is called the **algebra of** $(\theta_0, \ldots, \theta_{n-1})$**-matrices**.

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- The term condition was generalized to higher arity by Bulatov. Using our terminology, we say that $C_{TC}(\theta_0, \ldots, \theta_{n-1}; \delta)$ holds if there is no cube γ belonging to $M(\theta_0, \ldots, \theta_{n-1})$ with exactly $2^{n-2} - 1$ vertices of $\mathrm{Lines}_{n-1}(\gamma)$ colored by δ-pairs.

- We then set the *term condition commutator* to be

$$[\theta_0, \ldots, \theta_{n-1}]_{TC} = \bigwedge \{\delta : C_{TC}(\theta_0, \ldots, \theta_{n-1}; \delta)\}$$

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- For example, take three congruences $\theta_0, \theta_1, \theta_2$. We have that

$$M(\theta_0, \theta_1, \theta_2) = \mathrm{Sg}_{\mathbb{A}^{2^3}} \left(\left\{ \begin{tikzcd}[row sep=small, column sep=small] x \arrow[rr, dash] \arrow[dd, dash] \arrow[dr, dash] & & y \arrow[dd, dash] \arrow[dr, dash] & \\ & x \arrow[rr, dash] \arrow[dd, dash] & & y \arrow[dd, dash] \\ x \arrow[rr, dash] \arrow[dr, dash] & & y \arrow[dr, dash] & \\ & x \arrow[rr, dash] & & y \end{tikzcd} : \langle x, y \rangle \in \theta_0 \right\} \cup \right.$$

$$\left\{ \begin{tikzcd}[row sep=small, column sep=small] y \arrow[rr, dash] \arrow[dd, dash] \arrow[dr, dash] & & y \arrow[dd, dash] \arrow[dr, dash] & \\ & y \arrow[rr, dash] \arrow[dd, dash] & & y \arrow[dd, dash] \\ x \arrow[rr, dash] \arrow[dr, dash] & & x \arrow[dr, dash] & \\ & x \arrow[rr, dash] & & x \end{tikzcd} : \langle x, y \rangle \in \theta_1 \right\} \cup$$

$$\left. \left\{ \begin{tikzcd}[row sep=small, column sep=small] x \arrow[rr, dash] \arrow[dd, dash] \arrow[dr, dash] & & x \arrow[dd, dash] \arrow[dr, dash] & \\ & y \arrow[rr, dash] \arrow[dd, dash] & & y \arrow[dd, dash] \\ x \arrow[rr, dash] \arrow[dr, dash] & & x \arrow[dr, dash] & \\ & y \arrow[rr, dash] & & y \end{tikzcd} : \langle x, y \rangle \in \theta_2 \right\} \right)$$

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c \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & d \arrow[dr, dash] \arrow[dd, dash] & \\
& g \arrow[rr, dash] \arrow[dd, dash] & & h \arrow[dd, dash] \\
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- We could have defined the hypercommutator with a similar kind of condition, but quantified over Δ instead of matrices: $C_H(\theta_0, \dots, \theta_{n-1}; \delta)$ holds if there is no cube γ belonging to $\Delta(\theta_0, \dots, \theta_{n-1})$ with exactly $2^{n-2} - 1$ vertices of $\mathrm{Lines}_{n-1}(\gamma)$ colored by δ-pairs.

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$$[\theta, \dots, \theta]_{TC} = [\theta, \dots, \theta]_H$$

for a congruence θ of a Taylor algebra $\mathbb{A}$.

Modular Varieties

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- An aspect of this was understood early on in the development of commutator theory. Freese and McKenzie define $\Delta(\alpha, \beta)$ to be congruence of β (as a subalgebra of $\mathbb{A}^2$) generated by $M(\alpha, \beta)$ (interpreted as a compatible binary relation on $\beta \leq \mathbb{A}^2$).

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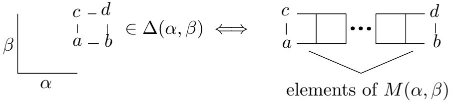

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- We can restate this result in a slightly more general way: In a modular variety, every (2)-dimensional tolerance that is transitively closed in one direction is already transitively closed in the other.
- In general, it is only necessary in a modular variety to take $(n-1)$-many transitive closures (in different directions) of an (n)-dimensional tolerance to produce an (n)-dimensional congruence.

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$$\begin{array}{c} \beta \uparrow \ \xrightarrow{\ \alpha\ } \end{array} \quad \begin{tikzcd}[column sep=small] c \arrow[d, dash] & \\ a \arrow[r, dash] & b \end{tikzcd} \ \text{completes to} \ \begin{tikzcd} p(c,a,a) \arrow[r, dash] \arrow[d, dash] & p(c,a,b) \arrow[d, dash] \\ p(a,a,a) \arrow[r, dash] & p(a,a,b) \end{tikzcd} \ = \ \begin{tikzcd}[column sep=small] c \arrow[r, dash] \arrow[d, dash] & p(c,a,b) \arrow[d, dash] \\ a \arrow[r, dash] & b \end{tikzcd}$$

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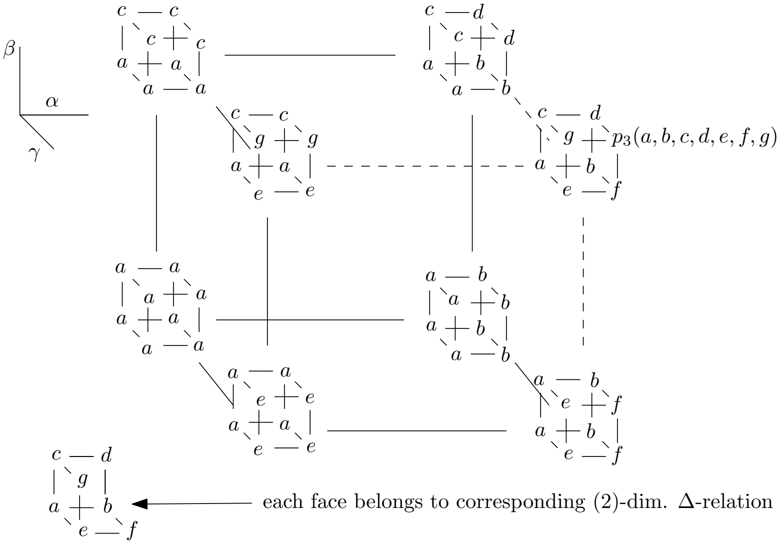
$$p(x,y,x) \approx p(x,x,y) \approx y$$

and now recursively define the **strong cube terms**

$$\begin{aligned} p_2(x,y,z) &= p(x,y,z), \text{ and} \\ p_n(x_0,x_1,\ldots,x_{2^n-2}) &= \\ & p(p_{n-1}(x_0,\ldots,x_{2^{n-1}-2}), x_{2^{n-1}-1}, p_{n-1}(x_{2^{n-1}},\ldots,x_{2^n-2})) \end{aligned}$$


β
α
γ
p3(a, b, c, d, e, f, g)
each face belongs to corresponding (2)-dim. Δ-relation

- Opršal uses this property of strong cube terms to show that, for a Mal'cev algebra $\mathbb{A}$, the congruences and all higher commutator operations determine the Δ-relations, and vice versa. So, for any Mal'cev algebra $\mathbb{A}$ there is a greatest clone that shares the same Mal'cev operation, congruences,and higher commutator operations with $\mathbb{A}$ (just look at the clone of polymorphisms of all $\Delta(\theta_0, \ldots, \theta_{n-1})$ for all sequences of congruences of $\mathbb{A}$).

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- Kiss discovered that every modular variety has a special (4)-ary term, now called a Kiss term, which can be used to modify elements of $R(\alpha, \beta)$ and produce elements of $\Delta(\alpha, \beta)$.

$$\beta \qquad \begin{array}{ccc} c & - & d \\ | & & | \\ a & - & b \end{array} \quad \text{completes to} \quad \begin{array}{ccc} c & - & q(a,b,c,d) \\ | & & | \\ a & - & b \end{array}$$

$$\alpha$$

- We recursively define a sequence of 'higher dimensional' Kiss terms:

$$q_2(x, y, z, u) = q(x, y, z, u) \text{ and}$$
$$q_n(x_0, \dots, x_{2^n-1}) =$$
$$q(q_{n-1}(x_0, \dots, x_{2^{n-1}-1}), x_{2^{n-1}-1}, q_{n-1}(x_{2^{n-1}}, \dots, x_{2^n-1}), x_{2^n-1})$$

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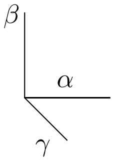
- These higher Kiss terms modify labeled cubes with the property that every face belongs to the appropriate lower dimensional Δ to produce an element of the correct dimension Δ. For example:

If



$$\begin{tikzcd}[arrows=dash, row sep=small, column sep=small]
c \arrow[r] \arrow[dd] \arrow[dr] & d \arrow[dd] \arrow[dr] & \\
& g \arrow[r] \arrow[dd] & h \arrow[dd] \\
a \arrow[r] \arrow[dr] & b \arrow[dr] & \\
& e \arrow[r] & f
\end{tikzcd}$$

is such that

$$\begin{tikzcd}[arrows=dash, row sep=small, column sep=small]
c \arrow[r] \arrow[d] & d \arrow[d] \\
a \arrow[r] & b
\end{tikzcd}
\quad
\begin{tikzcd}[arrows=dash, row sep=small, column sep=small]
g \arrow[r] \arrow[d] & h \arrow[d] \\
e \arrow[r] & f
\end{tikzcd}
\in \Delta(\alpha, \beta)$$

$$\begin{tikzcd}[arrows=dash, row sep=small, column sep=small]
c \arrow[r] \arrow[dr] & d \arrow[dr] & \\
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\end{tikzcd}
\quad
\begin{tikzcd}[arrows=dash, row sep=small, column sep=small]
a \arrow[r] \arrow[dr] & b \arrow[dr] & \\
& e \arrow[r] & f
\end{tikzcd}
\in \Delta(\alpha, \gamma)$$

$$\begin{tikzcd}[arrows=dash, row sep=small, column sep=small]
c \arrow[d] \arrow[dr] & \\
a \arrow[dr] & g \arrow[d] \\
& e
\end{tikzcd}
\quad
\begin{tikzcd}[arrows=dash, row sep=small, column sep=small]
d \arrow[d] \arrow[dr] & \\
b \arrow[dr] & h \arrow[d] \\
& f
\end{tikzcd}
\in \Delta(\beta, \gamma)$$

Then,

$$\begin{tikzcd}[arrows=dash, row sep=small, column sep=small]
c \arrow[r] \arrow[dd] \arrow[dr] & d \arrow[dd] \arrow[dr] & \\
& g \arrow[r] \arrow[dd] & q_3(a,b,c,d,e,f,g,h) \arrow[dd] \\
a \arrow[r] \arrow[dr] & b \arrow[dr] & \\
& e \arrow[r] & f
\end{tikzcd}
\in \Delta(\alpha, \beta, \gamma)$$

- We can now mimic Opršal's argument. If an algebra has Day terms, then the congruences and all higher commutator operations determine the Δ-relations and vice versa. So, for any algebra $\mathbb{A}$ that generates a modular variety, there exists a largest clone sharing Day terms, congruences, and higher commutator operations with $\mathbb{A}$.

Some questions:

- When is the normal section map injective?

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- Fix an algebraic signature τ. What is an implication base for the class of τ structures that embed into a reduct of the clone of polynomials of some supernilpotent Mal'cev algebra?

Thank you!