On the small index property for AECs with strong amalgamation properties

PALS Seminar - Boulder, Colorado

Andrés Villaveces - *Universidad Nacional de Colombia - Bogotá*

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“Logicians and category theorists seem to have resisted each others’ ideas to a large extent.”

M. Makkai - G. Reyes (1977)

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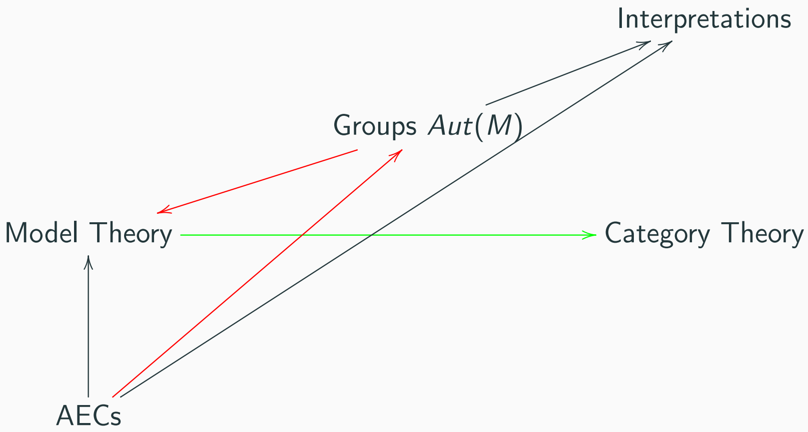
- (with G. Reyes) a crucial book (First Order Categorical Logic, Lecture Notes in Mathematics 611, 1977) - with the precisely descriptive subtitle `Model Theoretic Methods in the Theory of Topoi and related categories`.

Those early days (Makkai-Reyes, Makkai-Shelah)

- The influence of large cardinals on structural properties of abstract elementary classes (originally strongly compact, later other people continued this line),
- The internal logic of a topos – again, structural properties of objects linked originally to Grothendieck constructions, with extreme influence outside of their original realm.

Contents

Some neighbours



Plan

Reconstructing models / Interpretations

- The reconstruction problem
- Interpretations, category-theoretically

From $Aut(M)$ to Int_M: SIP

- Elementary musings and countable issues
- Smoothing SIP beyond $\aleph_0$ - Lascar-Shelah

SIP beyond first order

- Strong amalgamation classes
- SIP for homogeneous AEC
- Examples: quasiminimal classes, the Zilber field, j-invariants

Interpretations between AECs

A "classical enigma": reconstructing from symmetry.

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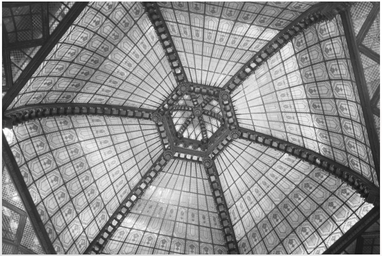

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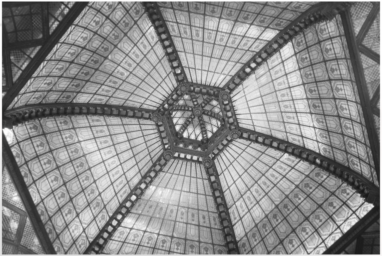

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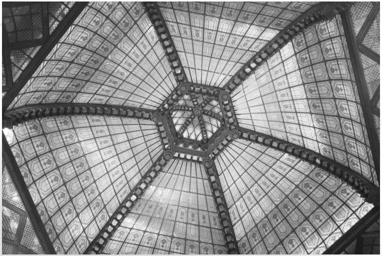

Tell me what is M!

Reconstructing models?

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- a more reasonable question: if for some (First Order) structure M we are given $Aut(M)$, what can we say about $Th(M)$?
- an even more reasonable question: if for some (FO) structure M we are given $Aut(M)$, when can we recover all models biinterpretable with M?

Reconstructing structures: Ahlbrandt-Ziegler

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- If M is $\aleph_0$-categorical, any open subgroup of $Aut(M)$ is a stabilizer $Aut_\alpha(M)$ for some imaginary α. Also $Aut(M) \curvearrowright \{H \leq Aut(M) \mid H \text{ open}\}$ (conjugation).

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- The action $Aut(M) \curvearrowright$ is (almost) $\approx$ to $Aut(M) \curvearrowright M^{eq}$. So, we have recovered the action of $Aut(M)$ on M^{eq} from the topology of $Aut(M)$... so, if M, N are countable $\aleph_0$-categorical structures, TFAE:
  - There is a bicontinuous isomorphism from $Aut(M)$ onto $Aut(N)$
  - M and N are bi-interpretable.

Interpretation between FO theories - Models as functors

(Makkai-Reyes)

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- Morphisms correspond to definable functions: if $A :: \phi(x)$ and $B :: \psi(y)$, a definable morphism $f : A \to B$ is a definable $f :: \chi(x, y)$ such that $T \models \forall x \forall y(\chi(x, y) \to \varphi(x) \wedge \psi(y))$ and $T \models \forall x(\varphi(x) \to \exists y \chi(x, y))$.

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- Given any L-structure $\mathfrak{M}$ and a formula $\varphi(x)$, the **solution set** is $\varphi(\mathfrak{M}) = \{a \in M_x \mid \mathfrak{M} \models \varphi(x)\}$.
- With this, **we regard models of T as functors** from $\mathcal{T}$ to Set: $\mathfrak{M}(A) = \varphi(\mathfrak{M})$. Natural transformations $\equiv$ elementary maps.

Interpretation between FO theories - Models as functors

The category $\mathcal{T} = Def(T)$ is Boolean (regular, and with Boolean algebras of subobjects) and extensive (co-products exist and they form an equivalence between the categories $Sub(X) \times Sub(Y)$ and $Sub(X \sqcup Y)$).

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Boolean categories $\quad \longleftrightarrow \quad$ First Order

An **interpretation** between T_0 and T is a Boolean and extensive morphism

$$\iota : \mathcal{T}_0 \to \mathcal{T}$$

between the categories $\mathcal{T}_0$ and $\mathcal{T}$ (in the vocabularies L_0 and L).

(ι preserves finite limits, induces homomorphisms of Boolean algebras in subobjects and respects images - and respects co-products)

Interpretation functor between classes of models

We lift the interpretation to classes of models:

Given $\iota : \mathcal{T}_0 \to \mathcal{T}$,

$$\iota^* : Mod(T) \to Mod(T_0)$$

$$\mathfrak{M} \models T \mapsto \iota^*(\mathfrak{M}) = \mathfrak{M}_0$$

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where

$$\mathfrak{M}_0 = \mathfrak{M} \circ \iota : \mathcal{T} \to \mathsf{Set}$$

and if $\sigma : \mathfrak{N} \to \mathfrak{M}$ is an elementary embedding ($\sigma = (\sigma_Y)_{Y \in \mathcal{T}}$) then

$$\iota^*(\sigma) : \mathfrak{N}_0 \to \mathfrak{M}_0 : \iota^* \sigma_X = \sigma_{\iota X}$$

for each $X \in \mathcal{T}_0$.

Examples - ACF, RCF

An interpretation we have known for some 200 years is the following:

$$\iota : Def(ACF) \to Def(RCF)$$

$$\iota(K) = R^2 \text{ , componentwise sum}$$

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if $R \models RCF$

$$\iota^*(R) = R[\sqrt{-1}].$$

Many other natural examples: retracts, boolean algebras in boolean rings, etc.

Stable Interpretations - a bit on Galois theory

The notion of stability is reflected in a natural way in interpretations:

Remember a theory T is stable if no formula can define an infinite linear order (in tuples).

An interpretation $\iota : \mathcal{T}_0 \to \mathcal{T}$ is **stable** if for each model $\mathfrak{M}$ of T, the "expanded interpretation" $\iota^{\mathfrak{M}} : \mathcal{T}_0^{\mathfrak{M}_0} \to \mathcal{T}^{\mathfrak{M}}$ is an immersion. This means each definable in ιX ($X \in \mathcal{T}_0$) using parameters from $\mathcal{M}$ is the image of a definable set in X using parameters from $\mathcal{M}_0$.

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If T is a stable theory and $\iota : \mathcal{T}_0 \to \mathcal{T}$ is an interpretation, then ι is a stable interpretation and T_0 is a stable theory.

Hrushovski and Kamensky went as far as reframing a "Galois theory" of model theory for internal covers - Galois theory à la Grothendieck (Exposé IV).

The Galois group of a first order theory

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where M is a saturated model of T and

$$Autf(M) = \langle \bigcup_{A \subset N \prec M} Aut_N(M) \rangle$$

This is an invariant of the theory, allowing a Galois connection between definably closed submodels of M and closed subgroups of the Galois group.

Plan

SIP - the link between algebra and topology

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Now, to the main property of the group $Aut(M)$ that enables us to capture its topology...

The classical topology.

Fix M for now a countable structure. The classical way of making $Aut(M)$ into a topological space is by decreeing that basic open sets around 1_M are pointwise stabilizers of finite subsets $A_{fin} \subset M$, $Aut_A(M) = \{f \in Aut(M) \mid f \restriction A = 1_A\}$.

This gives $Aut(M)$ the structure of a Polish space.

Definition (Small Index Property - SIP)
Let M be a countable structure. M has the small index property if for any subgroup H of $Aut(M)$ of index less than $2^{\aleph_0}$, there exists a finite set $A \subset M$ such that $Aut_A(M) \subset H$.

The Small Index Property (countable version)

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In other words, if G is "large algebraically speaking" then it is also "large topologically speaking".

Basic facts on countable SIP

SIP allows us to recover the topological structure of $Aut(M)$ from its pure group structure:
Open neighborhoods of 1 in pointwise convergence topology $=$ Subgroups containing pointwise stabilizers $Aut_A(M)$ for some finite A.

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Subgroups containing pointwise stabilizers $Aut_A(M)$ for some finite A.

- SIP holds for random graph, infinite set, DLO, vector spaces over finite fields, generic relational structures, $\aleph_0$-categorical $\aleph_0$-stable structures, etc.
- It fails e.g. for $M \models ACF_0$ with ∞ transc. degree.

Galois group (of a theory)

The Galois group of a model M,

$$Gal(M) := Aut(M)/Autf(M),$$

is invariant across saturated models of a theory[1].

Possible failures of SIP are encoded in this quotient.

[1] **Lascar, Daniel.** Automorphism Groups of Saturated Structures, ICM 2002, Vol. III - 1-3.
Lascar, Daniel. Les automorphismes d'un ensemble fortement minimal, JSL, vol. 57, n. 1. March 1992.

SIP for uncountable structures

We now switch focus to the uncountable, first order, case.

Fix $\lambda = \lambda^{<\lambda}$ an uncountable cardinal, and fix M a saturated model of cardinality λ.

[2] **Sy-David Friedman, Tapani Hyttinen and Vadim Kulikov,** Generalized descriptive set theory and classification theory, Memoirs of the American Mathematical Society, 2014; Volume 230, Number 1081

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We now use the topology $\mathcal{T}^\lambda$ on $Aut(M)$, whose basic open sets around 1_M are stabilizers of subsets of size $< \lambda$ - as before $Aut_A(M)$ but now $A \subset M$ with $|A| < \lambda$.

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$Aut(M)$ with this topology is of course no longer a Polish space. The techniques from Descriptive Set Theory that have been used for the countable case need to be replaced (Friedman, Hyttinen and Kulikov's Descriptive Set Theory for some uncountable cardinalities might become relevant to this[2]).

[2] **Sy-David Friedman, Tapani Hyttinen and Vadim Kulikov,** Generalized descriptive set theory and classification theory, Memoirs of the American Mathematical Society, 2014; Volume 230, Number 1081

Theorem (Lascar-Shelah: Uncountable saturated models have the SIP)
Let M be saturated, of cardinality $\lambda = \lambda^{<\lambda}$ and let G be a subgroup of $Aut(M)$ such that $[Aut(M) : G] < 2^\lambda$. Then there exists $A \subset M$ with $|A| < \lambda$ such that $Aut_A(M) \subset G$.

[3] **Daniel Lascar and Saharon Shelah,** Uncountable Saturated Structures have the Small Index Property, Bull. London Math. Soc. 25 (1993) 125-131.

Lascar-Shelah's Theorem

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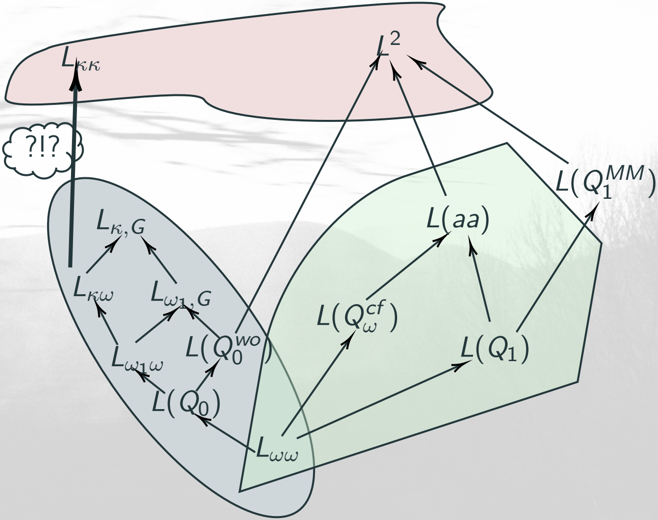
The proof[3] consists of building directly (assuming that G does not contain any open set $Aut_A(M)$ around the identity) a **binary tree** of height λ of automorphisms of M in such a way that every two of them are not conjugate. This is enough but requires two crucial notions: **generic** and **existentially closed (sequences of) automorphisms**. These are obtained by assuming that G is not open.

[3] **Daniel Lascar and Saharon Shelah,** Uncountable Saturated Structures have the Small Index Property, Bull. London Math. Soc. 25 (1993) 125-131.

Plan

Now, beyond First Order

Logics - AECs?



L^2
$L_{\kappa\kappa}$
?!?
$L_{\kappa,G}$
$L_{\kappa\omega}$
$L_{\omega_1,G}$
$L_{\omega_1\omega}$
$L(Q_0^{wo})$
$L(Q_0)$
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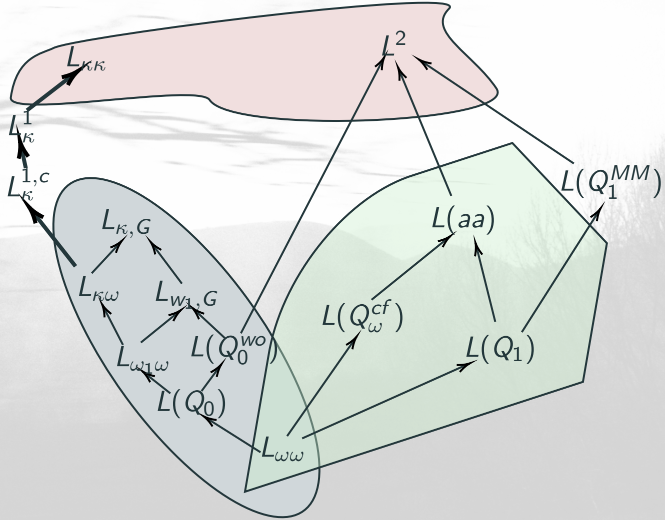
$L(aa)$
$L(Q_1)$
$L(Q_1^{MM})$

New Logics and AECs



Beyond First Order

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- Abstract Elementary Classes with a well-behaved closure notion, and the particular case:
- Quasiminimal (qm excellent) Classes.

The main result: SIP for homogeneous AEC.

With **Ghadernezhad** we have proved[4]:

Theorem (SIP for $(Aut(M), \mathcal{T}^{cl})$ **- Ghadernezhad, V.)**
"Strong" amalgamation classes have the SIP (in homogeneous models).

(Reasonable conditions to begin a Galois theoretical analysis of AECs)

[4] **Ghadernezhad, Zaniar and Villaveces, Andrés.** The Small Index Property for Homogeneous AEC's, Archive for Mathematical Logic, February 2018, Volume 57, Issue 1–2, pp 141–157.

Example: quasiminimal classes, "Zilber field"

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Example: quasiminimal classes, “Zilber field”

- $\mathcal{Q}$ qm pregeom. class $\to$ for every model M of $\mathcal{Q}$, $Aut(M)$ has SIP,
- The “Zilber field” has SIP.
- The j-invariant has the SIP.

Plan

Toward interpretation between AECs

We already have some ingredients:

- A good and solid (category-theoretical) way of dealing with interpretation, leading to a Galois theory in the sense of Grothendieck.

So, where can we go? Interpretation is a natural way.

Toward interpretation between AECs

We already have some ingredients:

- A good and solid (category-theoretical) way of dealing with interpretation, leading to a Galois theory in the sense of Grothendieck.
- A criterion for reconstruction (the SIP) lifting to some AECs and their homogeneous models.

So, where can we go? Interpretation is a natural way.

Ultimate goal: reconstruction

With the ultimate goal of reconstruction in mind (what properties of an AEC are reflected by the automorphims of a large homogeneous model?) it is natural to study interpretations in various different ways.

Using specific logics to build the interpretation

Boney-Vasey have used a logic harking back to Stavi (structural logic) to capture AECs with intersections. These classes are closely related to our strong amalgamation classes with closures.

They prove that AECs with intersections correspond to classes axiomatizable by universal theories in that logic.

Other AECs can be axiomatized by sentences in infinitary logics $\mathbb{L}_{(2^\kappa)^+,\kappa^+}$, for $\kappa = LS(\mathcal{K})$ (Shelah-V., 2020; independently, Leung (in a logic with a game quantifier)).

Interpretation of a $\mathbb{L}^{\kappa-struct}$-axiomatizable classes

Given $\iota : Def_{\psi_0} \to Def_{\psi}$,

$$\iota^* : \mathcal{K} \to \mathcal{K}_0$$

$$\mathfrak{M} \models T \mapsto \iota^*(\mathfrak{M}) = \mathfrak{M}_0$$

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where (again)

$$\mathfrak{M}_0 = \mathfrak{M} \circ \iota : \mathcal{T} \to \mathrm{Set}$$

and if $\sigma : \mathfrak{N} \to \mathfrak{M}$ is an $\mathbb{L}^{\kappa-struct}$-elementary embedding $(\sigma = (\sigma_Y)_{Y \in Def_{\psi}}$ then

$$\iota^*(\sigma) : \mathfrak{N}_0 \to \mathfrak{M}_0 : \iota^* \sigma_X = \sigma_{\iota X}$$

for each $X \in Def_{\psi_0}$.

Using types to build the interpretation

A more direct approach may either use Morleyization of the vocabulary (expanding by adding all orbital types as predicates), or use Shelah's Presentation Theorem (but dealing with omitting types functorially will require additional understanding):

Theorem (Shelah)
Let $(\mathcal{K}, \leq_K)$ be an AEC in a language L. Then there exist

- *A language $L' \supset L$, with size $LS(\mathcal{K})$,*
- *A (first order) theory T' in L' and*
- *A set of T'-types, Γ', such that*

$$\mathcal{K} = PC(L, T', \Gamma') := \{M' \upharpoonright L \mid M' \models T', M' \textit{ omits } \Gamma'\}.$$

Moreover, if $M', N' \models T'$, they both omit Γ', $M = M' \upharpoonright L$ and $N = N' \upharpoonright L$,

$$M' \subset N' \Leftrightarrow M \leq_K N.$$

The Galois group of an AEC

This is well defined in Strong Amalgamation AECs:

$N \in \mathcal{K}$, $\mathcal{K}$

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where M is a homogeneous model in $\mathcal{K}$, $N \prec_{\mathcal{K}} M$ is small and as before

$$Autf(M) = \langle \bigcup_{N \prec_{\mathcal{K}} N' \prec M} Aut_{N'}(M) \rangle$$

This is an invariant of $\mathcal{K}$.

A Galois connection between definably closed submodels of M and closed subgroups of the Galois group...

Thank you for your attention!