Algebras, Lattices, Varieties I–N
(or Ralph, tell us all that you know)

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Panglobal **A**lgebra and **L**ogic **S**eminar
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Outline

1. Algebras, Lattices, Varieties I–N
   - Our Project in 1977
   - The Field in 1980
   - The Field in 1988
   - The Field in 2018 or Our Inclusion/Exclusion Princple
2. Mal'tsev Started a Story that Continues
3. Theorems Spanning Chapters 6 and 11

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The Books on General Algebra Available in 1977

- *Universal Algebra* by Paul M. Cohn, 1965 333 + xv pp.
- *Lattice Theory*, 3^{rd} *ed.* by Garrett Birkhoff, 1967 418 + vi pp.
- *Introduction to the Theory of Abstract Algebras* by Richard Pierce, 1968 148 + viii pp.
- *Universal Algebra* by George Grätzer, 1968 329 + xii pp.
- *Algebraic Systems* by A. I. Mal'tsev 1973 317 + xii pp.
- *Topics in Universal Algebra* by Bjarni Jónsson 1972 220 + vi pp.

What Was the Field Like in 1977?

- The first issue of *Algebra Universalis* was published in 1971.
- There were research centers of intense activity in North America, Europe, and the Soviet Union.
- There were also isolated but productive reseachers.
- The commutator in congruence modular varieties was well developed.
- Understanding of Mal'tsev conditions was solidly developed.
- The first results of tame congruence theory had emerged.
- There were regular international conferences.

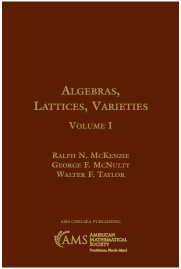
What Was Needed in 1977?

A **new book** to address all that new material as well as older results, like Post's description of the lattice of clones on a two-element set.

The Field in 1980

1. Three more books!
   - *Einführung in die allgemeine Algebra* by Heinrich Werner 1978 146 pp.
   - *Funktionen- und Relationenalgebren* Reinhard Pöschel and Lev A. Kalužnin, 1979 259 pp.
   - *A Course in Universal Algebra* by S. Burris and H. Sankappanavar 1981 276 + xvi pp.
2. Progress!
   - Heinz-Peter Gumm gave us a geometric perspective of the commutator.
   - Tame congruence theory was at our finger tips.

Algebras, Lattices, Varieties I–III



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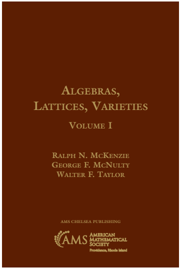
Algebras, Lattices, Varieties I–III



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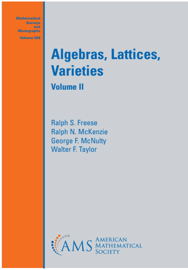

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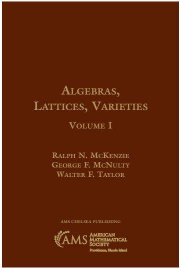
Algebras, Lattices, Varieties I–III



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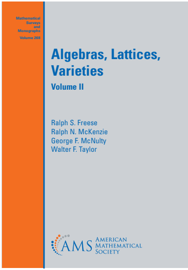

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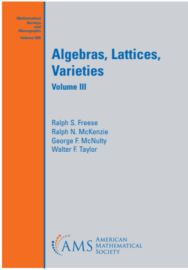

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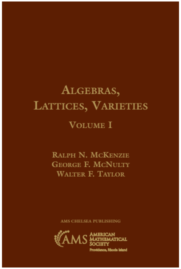
Algebras, Lattices, Varieties I–III



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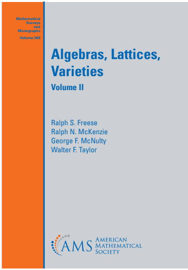

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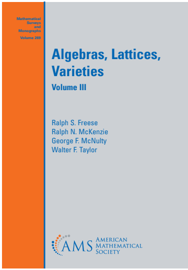

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https://bookstore.ams.org/

Algebras, Lattices, Varieties I

Algebras, Lattices, Varieties II

The Field in 1988

1. Six More Books!
   - *A Survey on Congruence Lattice Representation* by E. Tamás Schmidt, 115 pp.
   - *Clones in Universal Algebra* by Ágnes Szendrei 1986, 166 pp.
   - *Algebras, Lattices, Varieties I* by Ralph McKenzie, George McNulty, and Walter Taylor, 1987, 361 + xii pp.
   - *Commutator Theory for Congruence Modular Varieties* by Ralph Freese and Ralph McKenzie, 1987, 227 pp.
   - *Allgemeine Algebra* by Thomas Ihringer, 1988, 180 pp.
   - *The Structure of Finite Algebras* by David Hobby and Ralph McKenzie, 1988, 209 + ix pp.
2. A Lot More Theorems.

The Team in 2018

The Field in 2018 or Our Inclusion/Exclusion Principle

1. What to Include in Volumes II and III.
   - An idiosyncratic selection of the most important ideas and results.
   - Big Results like Rosenberg's Characterization of Maximal Clones on Finite Sets, McKenzie's Resolution of Tarski's Finite Basis Problem, and J. B. Nation's Counterexample to the Finite Height Conjecture.
2. What to Exclude from Volumes II and III.
   - Tame Congruence Theory, Constraint Satisfaction Problems. (But there is supposed to be Volume IV.)
   - Theorems That Require Notions or Methods Beyond Those Familiar to First Year Graduate Students, like the Oates-Powell Theorem and Ralph Freese's Theorem that the Equational Theory of Modular Lattices is Undecidable.
   - Results Whose Proofs Would Require Too Much Space, like Jezek's Description of Subsets of Lattices of Equational Theories Definable in First-Order Logic. (Again a candidate for Volume IV.)

The Story Mal'tsev Started: From Volume I.

Dedekind 1900 discovered that groups are congruence-permutable and that congruence permutability implies modularity of the congruence lattices. (This discovery is important for theorems like the Jordan-Hölder Theorem.)

Theorem 4.141 in Volume I, A. I. Mal'tsev 1954

All the algebras in a variety $\mathcal{V}$ have permuting congruences iff there is a ternary term p of the signature of $\mathcal{V}$ such that the equation-set

$$\{p(x,y,y) \approx x, \quad p(x,x,y) \approx y\} \tag{$\star$}$$

is valid in $\mathcal{V}$.

The Story Mal'tsev Started: Walter Taylor

Theorem 10.149 in Volume III, Walter Taylor 1973

A non-empty class K of varieties is Mal'tsev-definable if and only if K satisfies all of the following five conditions:

1. K is closed under the formation of equivalent varieties;
2. K is closed under the formation of subvarieties;
3. K is closed under the formation of finite products of varieties;
4. if $\mathcal{V} \in K$ and $\mathcal{V}$ is generated by all reducts of members of $\mathcal{W}$ to the signature of $\mathcal{V}$, then $\mathcal{W} \in K$;
5. if the equations Σ define a variety in K of signature τ, then there exist finite subsets Σ_0 and τ_0 of Σ and τ with Σ_0 defining a variety in K of signature τ_0.

The Story Mal'tsev Started: Walter Taylor

Theorem 10.152 in Volume III, Walter Taylor 1975

There are $2^{\aleph_0}$ Mal'tsev conditions.

The Story Mal’tsev Started: 1980’s

Many properties were found to be Mal’tsev-definable.

$$
\begin{tikzcd}
 & & & \mathrm{E} \arrow[dr] & \mathrm{SD}_\wedge \arrow[dr] & \\
\mathrm{NU} \arrow[rr] \arrow[drr] & & \mathrm{D} \arrow[ur] \arrow[r] \arrow[dr] & \mathrm{SD}_\vee \arrow[r] \arrow[ur] & \mathrm{HM} \arrow[r] & \mathrm{TT} \\
 & & \mathrm{CT} \arrow[r] & \mathrm{M} \arrow[ur] & & \\
\mathrm{U} \arrow[r] & \mathrm{WU} \arrow[r] & \mathrm{R} \arrow[ur] \arrow[dr] & & & \\
\mathrm{C} \arrow[r] \arrow[ur] & \mathrm{P} \arrow[r] & \mathrm{P}_3 \arrow[r] \arrow[uur] & \mathrm{P}_* \arrow[r] \arrow[uuur] & \mathrm{NL} & \\
 & \cdots \arrow[r] & \mathrm{K}_3 \arrow[r] & \mathrm{K}_2 \arrow[ur] & &
\end{tikzcd}
$$

What do those letters mean?

K_n is the class of varieties in which no algebra has a universe which can be written as the union of n proper subuniverses. The table below is a key for the other properties.

U	:	congruence uniformity	C	:	coherence
WU	:	weak uniformity	R	:	regularity
P_3	:	3-permutability	P	:	permutable congruences
P_*	:	k-permutability for some k	D	:	distributive congruences
HM	:	has a Hobby-McKenzie term	M	:	modular congruences
$SD_\wedge$	:	congruence meet semidistributive	NL	:	no compatible linear order
$SD_\vee$	:	congruence join semidistributive	TT	:	has a Taylor term
E	:	congruences satisfy	NU	:	has a near unanimity term
		$x \wedge (y \vee z) \leq p(x, y, z) \vee q(x, y, z)$	CT	:	has a cube term

where $p(x, y, z) = [x \wedge (y \vee (x \wedge (z \vee (x \wedge (y \vee (x \wedge z))))))]$
and $q(x, y, z) = [x \wedge (z \vee (x \wedge (y \vee (x \wedge (z \vee (x \wedge y))))))]$.

The Story Mal'tsev Started: Tschantz

The simplicity of Mal'tsev's two equations does not necessarily mean that permutable varieties are easy to recognize.

An Example due to Steven T. Tschantz, 1986

Let $\mathcal{T}$ be the variety based on the set of two equations (in one binary operation symbol, denoted by juxtaposition) $\{xx \approx x, (xy)(yx) \approx y\}$. One easily checks that the ternary $\mathcal{T}$-term

$$p(x,v,w) = \Big[\Big([v(vx)][(vx)w]\Big)\Big((vw)x\Big)\Big]\Big[\Big(x(wv)\Big)\Big([w(xv)][(xv)v]\Big)\Big]$$

is a Mal'tsev term for $\mathcal{T}$. Tschantz found this term with a computer search. Further calculations confirmed that no shorter term will serve as a Mal'tsev term for $\mathcal{T}$.

The Story Mal'tsev Started: McNulty

In fact the difficulty in recognizing permutable varieties is, in a sense, insurmountable.

Theorem 7.71 in Volume II, George McNulty, 1976

Let τ be a computable signature that provides at least one operation symbol of rank at least 2. Let $\mathcal{P}$ be a collection of finite sets of equations of signature τ such that

1. $\mathcal{P}$ is not empty,
2. If $\Gamma \in \mathcal{P}$ and Σ is a finite set of equations so that $\mathsf{Mod}\,\Gamma = \mathsf{Mod}\,\Sigma$, then $\Sigma \in \mathcal{P}$, and
3. For each $\Gamma \in \mathcal{P}$ there is a term t in which both v_0 and v_1 occur such that $\Gamma \vdash t \approx v_0$.

Then $\mathcal{P}$ is not decidable.

The Story Mal'tsev Started: McNulty

Consequence

When $\mathcal{P}$ is the collection of all finite sets of equations that are bases of congruence permutable varieties, then $\mathcal{P}$ will be undecidable. The same conclusion obtains for many of the classes of varieties that we consider.

The Story Mal'tsev Started: Polin's Variety

Polin's variety is generated by two two-element algebras, $\mathbf{A}_0$ and $\mathbf{A}_1$. Each of these (like any two-element algebra) has permuting congruences. Their product does not. Moreover each $\mathbf{A}_i$ generates a residually small variety, whereas their product $\mathbf{A}_0 \times \mathbf{A}_1$ generates Polin's variety, which is residually large.

The Story Mal'tsev Started: Other Products

The situation is different if there exists a congruence-modular variety $\mathcal{V}$ such that algebras $\mathbf{A}_0$ and $\mathbf{A}_1$ both belong to $\mathcal{V}$. In this case, if $\mathbf{A}_0$ and $\mathbf{A}_1$ both have permuting congruences, then $\mathbf{A}_0 \times \mathbf{A}_1$ has permuting congruences. And if $\mathbf{A}_0$ and $\mathbf{A}_1$ each generate a residually small subvariety of $\mathcal{V}$, then the variety generated by $\mathbf{A}_0 \times \mathbf{A}_1$ will be residually small as well.

The Story Mal'tsev Started: Permutability, Modularity, and the Arguesian Equation

Theorem 4.67 in Volume I, Bjarni Jónnson, 1953

Every lattice of 3-permuting equivalence relations is modular. If an algebra $\mathbf{A}$ has 3-permuting congruence relations, then $\mathbf{A}$ has a modular congruence lattice.

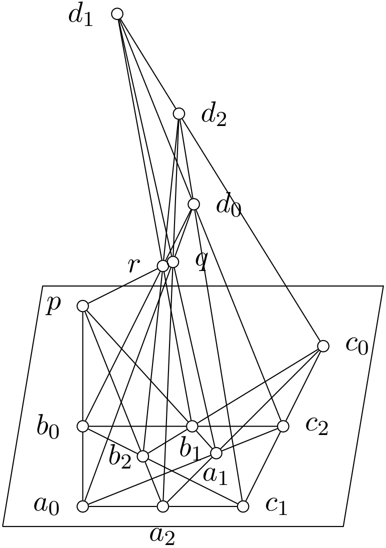
Theorem 6.104 in Volume II, Bjarni Jónnson, 1953

Every lattice of permuting equivalence relations obeys the Arguesian equation. This equation, an abstract form of the Desargues' Theorem of projective geometry, is stronger than modularity.

Theorem 6.109 in Volume II, Freese and Jónnson, 1976

Let $\mathcal{V}$ be congruence modular variety. Then $\mathcal{V}$ is congruence Arguesian.

The Story Mal'tsev Started: The Arguesian Equation



The Story Mal'tsev Started: J D H Smith

- If **J** and **K** are normal subgroups of a group, then we may form their **commutator** $[J, K] := \langle \{a^{-1}b^{-1}ab \mid a \in J, b \in K\}\rangle$, which supports a rich chapter of group theory (solvable groups, nilpotent groups, and so on).
- In 1976 J. D. H. Smith generalized the commutator from groups (and rings) to a binary operation $\langle \varphi, \psi\rangle \longmapsto [\varphi, \psi]$ defined on the congruence lattice of an algebra in a congruence-permutable variety.

The Story Mal'tsev Started: From permutable to modular

- In 1979 J Hagemann and C Herrmann showed how to define the commutator on all modular varieties. Gumm 1983 and Freese and McKenzie 1987 gave other definitions and showed they were equivalent for congruence modular varieties (but not for all varieties).
- The commutator defined using the term condition has turned out to be the most useful definition and is now the standard. The term condition was also discovered by William A. Lampe about the same time.

The Story Mal'tsev Started: difference terms

A **difference term for** $\mathcal{V}$ (p. II.46, footnote 7; p. III.290) is a ternary term $p(x, y, z)$, satisfying, for each $\mathbf{A} \in \mathcal{V}$, each $\theta \in \operatorname{Con} \mathbf{A}$, and each pair $\langle a, b \rangle \in \theta$,

$$p^{\mathbf{A}}(a, a, b) \;=\; b \quad \text{and} \quad p^{\mathbf{A}}(a, b, b)\; [\theta, \theta]\; a, \tag{$\star$}$$

where $[\cdot, \cdot]$ is the (term-condition) commutator operation described above. It turns out that the class of varieties having a difference term is a Mal'tsev class and this class includes all congruence modular varieties.

The Story Mal'tsev Started: Taylor terms

Let M and N be $n \times n$ matrices of the letters x and y, such that M has x's all along its diagonal, and N has y's; and let f be an n-ary term in the language of $\mathcal{V}$. The (M, N, f)-Taylor equations are $f(x, x, \dots) \approx x$ and $f(M) \approx f(N)$ (one equation for each row of the matrices). If these equations hold in a variety $\mathcal{V}$ for suitable M and N, then we say that f is a **Taylor term** for $\mathcal{V}$.

The Story Mal'tsev Started: Taylor terms

For example, if $\mathcal{V}$ is a permutable variety, with $p(x, y, z)$ a Mal'tsev term for $\mathcal{V}$, then the equations

$$p(x, \ldots, x) \approx x; \qquad p\begin{pmatrix} x & x & y \\ x & x & y \\ y & x & x \end{pmatrix} \approx p\begin{pmatrix} y & y & y \\ y & y & y \\ y & y & y \end{pmatrix}$$

assert that p is a Taylor term for $\mathcal{V}$, since the display above unfolds as

$$p(x, \ldots, x) \approx x; \qquad \begin{array}{rcl} p(x, x, y) & \approx & p(y, y, y) \\ p(x, x, y) & \approx & p(y, y, y) \\ p(y, x, x) & \approx & p(y, y, y) \end{array}$$

(Fnding a Taylor term for $\mathcal{V}$ is not always so easy.)

The Story Mal'tsev Started: Olšák 2020

There is a weakest Taylor equation, which means: If variety $\mathcal{V}$ has any Taylor term, then there exists a 12-ary $\mathcal{V}$-term q, such that $\mathcal{V}$ satisfies $q(x, \cdots, x) \approx x$ and $q(M_0) \approx q(N_0)$, where

$$M_0 = \begin{pmatrix} x & x & x & x & x & x & y & y & y & y & y & y \\ x & y & x & x & y & y & x & y & x & x & y & y \end{pmatrix} \quad \text{and}$$

$$N_0 = \begin{pmatrix} x & x & y & y & y & y & y & y & x & x & x & x \\ y & x & x & y & x & y & y & x & x & y & x & y \end{pmatrix}.$$

The Story Mal'tsev Started: An old open problem

An Old Open Problem

Is it true that every congruence-uniform variety $\mathcal{V}$ is congruence-permutable?

Theorem 11.148 in Volume III, Ralph McKenzie 1982

Let $\mathcal{V}$ be a variety such that the free algebra on two generators is finite. That is $\mathbf{F}_{\mathcal{V}}(x, y)$ is finite. Then, if $\mathcal{V}$ is congruence uniform, it is congruence permutable. In particular, uniformity implies permutability for locally finite varieties.

Theorems Spanning Chapters 6 and 11.

$$
\begin{tikzcd}
 & \mathrm{SD}_\vee \arrow[r] \arrow[ddddr] & \mathrm{SD}_\wedge \arrow[d, leftrightarrow] & & & & \\
{[\bigvee \alpha_i, \beta] = \bigvee [\alpha_i, \beta]} & & \mathrm{N} \arrow[dd] & {[\alpha,\beta] = [\beta,\alpha]} & & & \\
 & & & & & & \\
\mathrm{M} \arrow[uu] \arrow[r, leftrightarrow] & \mathrm{GDT} \arrow[r] \arrow[ddr] & \mathrm{DT} \arrow[r, leftrightarrow] \arrow[uur] & \mathrm{KDT} \arrow[r] & \mathrm{WDT} \arrow[r] \arrow[dd] & \mathrm{Ab{=}Aff} \arrow[dr] & \\
 & & & & & & \mathrm{Ab{=}QAff} \\
 & \mathrm{P}_* \arrow[r] & \mathrm{HM} \arrow[uurr] & & \mathrm{TT} \arrow[r] & {[\alpha,\beta]_s = [\alpha,\beta]_\ell} \arrow[ur] &
\end{tikzcd}
$$

HM	:	has a Hobby-McKenzie term	M	:	congruence modular
GDT	:	has a Gumm difference term	DT	:	has a difference term
KDT	:	has a Kiss difference term	WDT	:	has a weak difference term
$\mathrm{SD}_\wedge$	:	congruence meet semidistributive	TT	:	has a Taylor term
$\mathrm{SD}_\vee$	:	congruence join semidistributive	N	:	congruence neutral
P_*	:	k-permutability for some k			

Theorems Spanning Chapters 6 and 11.

Theorem (6.21)

TFAE for a variety $\mathcal{V}$:

(i) *$\mathcal{V}$ is congruence meet semidistributive.*

(ii)–(v) … *several conditions* …

(vi) *$\mathcal{V}$ satisfies an idempotent Mal'tsev condition that fails in every nontrivial variety of modules.*

(vii) *$\mathcal{V}$ is congruence neutral: the commutator equals the meet: $[\alpha, \beta] = \alpha \wedge \beta$.*

(i) $\Leftrightarrow$ (vii) is Theorem 11.37.
(vi) $\Rightarrow$ (i) is Theorem 11.73.
Note (vi) generalizes the TCT result locally finite varieties with no 1's and no 2's are congruence meet semidistributive.

Theorems Spanning Chapters 6 and 11.

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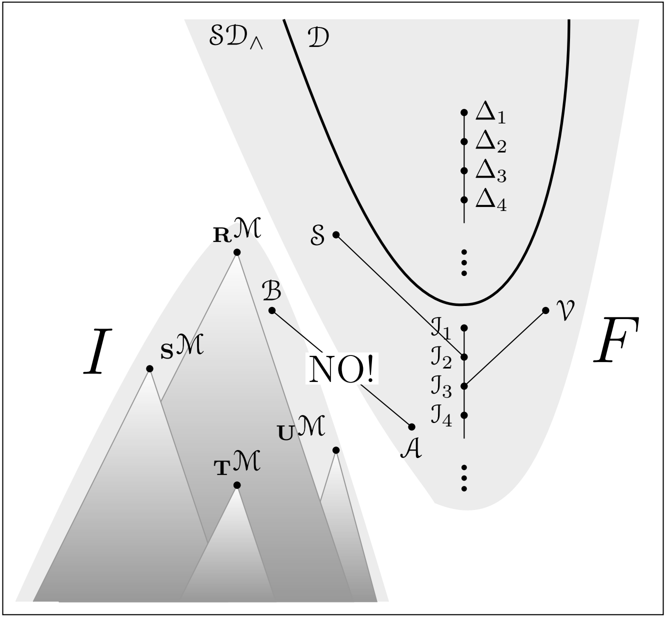
An Illustration of Theorem 6.21 from Chapter 10



Theorems Spanning Chapters 6 and 11.

Theorem (6.24)

TFAE for a variety $\mathcal{V}$:

(i) *$\mathcal{V}$ is congruence semidistributive.*

(ii) *$\mathcal{V}$ is congruence join semidistributive.*

(iiia) $\gamma \cap (\alpha \circ \beta) \subseteq (\alpha \wedge \beta) \vee (\beta \wedge \gamma) \vee (\alpha \wedge \gamma)$.

(iv) *For some k, $\mathcal{V}$ has terms $d_0(x, y, z), \ldots, d_k(x, y, z)$ satisfying*

$$\begin{aligned}
&d_0(x, y, z) \approx x \qquad d_k(x, y, z) \approx z; \\
&d_i(x, y, y) \approx d_{i+1}(x, y, y) \quad \text{if } i \equiv 0 \text{ or } 1 \pmod 3; \\
&d_i(x, y, x) \approx d_{i+1}(x, y, x) \quad \text{if } i \equiv 0 \text{ or } 2 \pmod 3; \\
&d_i(x, x, y) \approx d_{i+1}(x, x, y) \quad \text{if } i \equiv 1 \text{ or } 2 \pmod 3.
\end{aligned}$$

(v) *$\mathcal{V}$ satisfies an idempotent Mal'tsev condition that fails in the variety of semilattices and in every nontrivial variety of modules.*

Theorems Spanning Chapters 6 and 11.

Proof.

- The forward implications are standard Mal'tsev arguments:
- (i) $\Rightarrow$ (ii) is clear.
- (ii) $\Rightarrow$ (iiia) follows from two applications of join semidistributivity and a little lattice theory.
- (iiia) $\Rightarrow$ (iv) is standard.
- (iv) $\Rightarrow$ (v) is easy.
- (v) $\Rightarrow$ (i) is quite hard, is proved in Theorem 11.74, and is due to K. Kearnes and E. Kiss and Á. Szendrei.

$\square$

Related Theorems.

Corollary (11.75)

$\mathcal{V}$ is join semidistributive
iff
it is meet semidistributive and has a Hobby-McKenzie term.

Theorem (Kearnes and Kiss)

If $\mathcal{V}$ has a Hobby-McKenzie term then an $(SD_\vee)$ failure interval is abelian: $\alpha \vee \beta = \alpha \vee \gamma \Rightarrow$ the interval $I[\alpha \vee (\beta \wedge \gamma), \alpha \vee \beta]$ is abelian.

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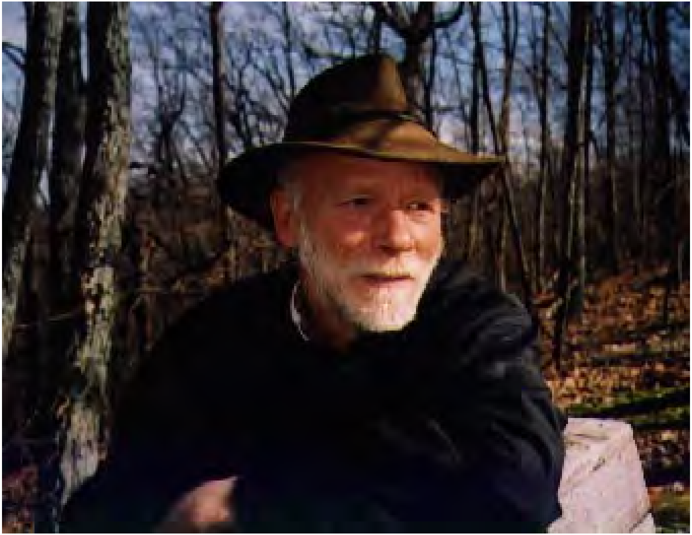
Where is *that* flute?



Ralph, Tell Us All That You Know

Now I came to Berkeley to hear the good word,
An exciting new theory, or so I had heard.
Still Helen's convinced that it's all quite absurd,
So Ralph, tell me all that you know.

Chorus: *Join in the chorus, sing out the song.*
It's really quite short, so it won't take so long.
Pick up your glasses; fill them again.
Let's drink a toast to that congruence tame.

Brian Davey, 1986

Which way is Hamlin?