Forcing $\aleph_1$-Free Groups to Be Free

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The Whitehead Problem

- Are there any non-free abelian groups A with $\mathrm{Ext}(A, \mathbb{Z}) = 0$?
- Abelian groups A satisfying $\mathrm{Ext}(A, \mathbb{Z}) = 0$ are called *Whitehead groups*.
- Every Whitehead group is $\aleph_1$-free, that is, each of its countable subgroups is free.

Theorem (Shelah 1973)

It is undecidable from ZFC whether every Whitehead group of cardinality $\aleph_1$ is free. In particular, it is true assuming $V = L$, and false assuming $MA + \neg CH$.

$\aleph_1$-Free Groups

Theorem (Dugas and Göbel 1982, Corner and Göbel 1985)

Any ring with free additive structure can be realized as the endomorphism ring of some $\aleph_1$-free group.

- This test of ring realization indicates that the class of $\aleph_1$-free groups exhibits a high degree of algebraic complexity.

Theorem (Herden and P. 2021)

If H is $\aleph_1$-free in some transitive model of ZFC, it is $\aleph_1$-free in any transitive model of ZFC.

- In other words, $\aleph_1$-freeness is absolute, suggesting that the class of $\aleph_1$-free groups is somewhat set theoretically simple.

Forcing

- Invented by Paul Cohen in 1963 to resolve the Continuum Hypothesis as undecidable.
- Starting from a ground model $\mathbf{M}$, a partial order $\mathcal{P}$ in $\mathbf{M}$, and a generic filter G over $\mathcal{P}$, forcing defines a model extension $\mathbf{M}[G]$ whose properties correspond to combinatorial properties of $\mathcal{P}$.
- Cardinality may change in generic extensions $\mathbf{M}[G]$.

Forcing $\aleph_1$-Free Groups to Be Free

Example (The Baer-Specker Group)

The direct product of countably many infinite cyclic groups, known as the Baer-Specker group $\mathbb{Z}^\omega$, is $\aleph_1$-free but not free.

- Question: When can we force an $\aleph_1$-free group to be free? (Answer: Always. Apply a forcing which collapses the cardinality of the group to be countable.)
- New question: When can we force an $\aleph_1$-free group to be free while preserving the cardinality of the group?

Relativization

- Let **M** be any class. Then for any formula ϕ, we define $\phi^{\mathbf{M}}$, the *relativization* of ϕ to **M**, inductively as follows:
  - $(x = y)^{\mathbf{M}}$ is $x = y$
  - $(x \in y)^{\mathbf{M}}$ is $x \in y$
  - $(\phi \wedge \psi)^{\mathbf{M}}$ is $\phi^{\mathbf{M}} \wedge \psi^{\mathbf{M}}$
  - $(\neg\phi)^{\mathbf{M}}$ is $\neg(\phi^{\mathbf{M}})$
  - $(\exists x\ \phi)^{\mathbf{M}}$ is $\exists x\ (x \in \mathbf{M} \wedge \phi^{\mathbf{M}})$
- For a sentence ϕ, "ϕ is true in **M**" means that $\phi^{\mathbf{M}}$ is true.

Absoluteness

- We say that a formula ϕ is *absolute* for **M** if and only if $\phi^{\mathbf{M}} \iff \phi$.
- The following are absolute for any transitive model **M** of ZFC (with $x, y, G \in \mathbf{M}$):
  - Ordered pairs (x, y)
  - Set union $\cup x$
  - Set inclusion $x \subseteq y$
  - ω
  - "x is finite"
  - "G is an abelian group"
  - "G is torsion-free"

$\aleph_1$-Freeness is Absolute

- A subgroup H of a torsion-free abelian group G is said to be a *pure subgroup* if and only if $x = ny \implies y \in H$ for all $x \in H, y \in G$ and non-zero $n \in \mathbb{N}$. For torsion-free groups, $\langle S \rangle_*$ denotes the minimal pure subgroup containing S. $\langle S \rangle_*$ is an absolute notion for transitive models of ZFC.

Pontryagin's Criterion

A countable torsion-free abelian group is free if and only if each of it finite rank subgroups is free.

Theorem (Herden and P. 2021)

$\aleph_1$-freeness is absolute for transitive models of ZFC.

- Proof: If H is an abelian group, H is $\aleph_1$-free if and only if H is torsion-free and for all finite subsets S of H, $\langle S \rangle_*$ is free.

$\aleph_1$-Freeness and Freeness

Theorem (Herden and P. 2021)

If $\mathbf{M}$ is a countable transitive model of ZFC, G is $\aleph_1$-free in $\mathbf{M}$ if and only if G is free in some generic extension of $\mathbf{M}$.

Theorem

If H and G/H are $\aleph_1$-free for abelian groups $H \subseteq G$, then G is $\aleph_1$-free.

- Proof: Let $\mathbf{M}$ be a countable transitive model of ZFC with $H, G \in \mathbf{M}$, and let $\mathbf{N}$ be a generic extension in which G is countable. Then G/H is countable and $\aleph_1$-free in $\mathbf{N}$, so it is free in $\mathbf{N}$. Then in $\mathbf{N}$, $G \cong H \oplus G/H$, so G is free in $\mathbf{N}$. Thus G is $\aleph_1$-free in $\mathbf{M}$.

Posets and Generic Filters

- Let $\mathcal{P}$ be a partially ordered set with a maximal element (refered to as a "poset"). We say that $G \subseteq \mathcal{P}$ is a *filter* over $\mathcal{P}$ if every pair of elements in G has a common extension in G, and for all $p \in G$, $q \geqslant p \implies q \in G$ (G is upward closed).
- We say that $D \subseteq \mathcal{P}$ is *dense* if every $p \in \mathcal{P}$ has an extension in D. If a filter G intersects every dense subset of $\mathcal{P}$, we call G a *generic filter*.

Theorem

Let $\mathbf{M}$ be a transitive model of ZF-P, and $\mathcal{P} \in \mathbf{M}$ be a partial order with a maximal element. If $\mathcal{P}$ is such that for every $p \in \mathcal{P}$, there exist $q < p, r < p$ with q and r having no common extension, then for any generic filter G over $\mathcal{P}$, $G \notin \mathbf{M}$.

Generic Extensions

- For $\mathbf{M}$ a countable transitive model of ZFC, the generic extension $\mathbf{M}[G]$ is the minimal model extension of $\mathbf{M}$ containing G.
- Formally, this generic extension is given by recursively defining $\mathcal{P}$*-names* for elements of $\mathbf{M}[G]$ which describe implicitly how the element is constructed from G and the elements of the ground model.
- If τ is a $\mathcal{P}$-name, we use τ_G to refer to the object in $\mathbf{M}[G]$ described by τ.
- We can represent any element $x \in \mathbf{M}$ canonically by a $\mathcal{P}$-name $\check{x} \in \mathbf{M}$.
- In this way, we construct a forcing language which can be understood from the perspective of $\mathbf{M}$, and which makes statements about $\mathbf{M}[G]$.

Fundamental Theorem of Forcing

- Let $\phi(x)$ be a formula, $\mathbf{M}$ a countable transitive model of ZFC, $\mathcal{P}$ be a poset in $\mathbf{M}$, τ a $\mathcal{P}$-name, and $p \in \mathcal{P}$. We say that p *forces* $\phi(\tau)$, written $p \Vdash \phi(\tau)$ if and only if for all $G \in \mathbf{V}$ such that G is $\mathcal{P}$-generic over $\mathbf{M}$ and $p \in G$, $\phi(\tau_G)$ holds in $\mathbf{M}[G]$.
- In order to decide these types of statements from within $\mathbf{M}$, we can define a new notion, $\Vdash^*$, such that for all ϕ,
$$(p \Vdash \phi) \iff (p \Vdash^* \phi)^{\mathbf{M}}.$$

The Fundamental Theorem of Forcing

Let $\mathbf{M}$ be a countable transitive model for ZFC, $\mathcal{P}$ be a poset in $\mathbf{M}$, τ a P-name, and $G \in \mathbf{V}$ $\mathcal{P}$-generic over $\mathbf{M}$. Then

$$\left[\exists p \in G\,(p \Vdash^* \phi(\tau))^{\mathbf{M}}\right] \iff \left[\exists p \in G\,(p \Vdash \phi(\tau))\right] \iff (\phi(\tau_G))^{\mathbf{M}[G]}.$$

In other words, a proposition holds in $\mathbf{M}[G]$ if and only if some $p \in G$ forces it.

Cardinal Preservation

- The *cardinality* $|A|$ of a set A is the least ordinal α such that there exists a bijection between A and α. We say that α is a *cardinal* if and only if α is an ordinal with $|\alpha| = \alpha$.
- If $\mathcal{P}$ is a poset in $\mathbf{M}$, $\mathcal{P}$ *preserves cardinals* if whenever $G \in \mathbf{V}$ is $\mathcal{P}$-generic over $\mathbf{M}$, then

$$\forall \beta \in \mathbf{M}\ [(\beta \text{ is a cardinal})^{\mathbf{M}} \leftrightarrow (\beta \text{ is a cardinal})^{\mathbf{M}[G]}].$$

- If a cardinal κ is not preserved by a poset $\mathcal{P}$, we say that $\mathcal{P}$ *collapses* κ.
- An *antichain* in $\mathcal{P}$ is a subset $A \subseteq \mathcal{P}$ such that

$$\forall p, q \in A\ (p \neq q \rightarrow p \perp q).$$

Chain Conditions and Closure Conditions

- A poset $\mathcal{P}$ has the θ*-chain condition* if and only if every antichain in $\mathcal{P}$ has cardinality $< \theta$.

Theorem

Assume $\mathcal{P}$ is a poset in **M**, and that in **M**, θ is a cardinal, $\mathcal{P}$ has the θ-chain condition, and θ is regular. Then $\mathcal{P}$ preserves cardinals $\geqslant \theta$.

- A partial order $\mathcal{P}$ is λ*-closed* if and only if whenever $\alpha < \lambda$ and $\{p_\beta : \beta < \alpha\}$ is a decreasing sequence of elements of $\mathcal{P}$, then

$$\exists q \in \mathcal{P}\ \forall \beta < \alpha\ (q \leqslant p_\beta).$$

Theorem

Assume $\mathcal{P}$ is a poset in **M**, and that in **M**, λ is a cardinal, and $\mathcal{P}$ is λ-closed. Then $\mathcal{P}$ preserves cardinals $\leqslant \lambda$.

Adding a Basis to an $\aleph_1$-Free Group

Let $\lambda > \aleph_0$ be a regular cardinal and H an $\aleph_1$-free group of size λ.

- $\mathcal{P}_1 = \{S \subset H : S \text{ is linearly independent } \wedge |S| < \lambda \wedge \langle S \rangle \subseteq_* H\}^{\mathbf{M}}$, ordered by $S' \leqslant S \iff S \subseteq S'$.
  - $\bigcup G$ is not a basis for H, for G a $\mathcal{P}_1$-generic filter.
  - $\mathcal{P}_1$ does not satisfy the λ-chain condition.
  - $\mathcal{P}_1$ is λ-closed.
- $\mathcal{P}_2 = \{S \subset H : S \text{ lin. indep. } \wedge |S| < \lambda \wedge H/\langle S \rangle \text{ is } \aleph_1\text{-free}\}^{\mathbf{M}}$, ordered by $S' \leqslant S \iff S \subseteq S'$.
  - $\bigcup G$ is a basis for H, for G a $\mathcal{P}_2$-generic filter.
  - $\mathcal{P}_2$ does not satisfy the λ-chain condition.
  - $\mathcal{P}_2$ is not λ-closed.

The Γ-Invariant

- Let α be a limit ordinal. A subset C of α is called *club* (in α) if C is closed in α (that is, for all $Y \subseteq C, \sup Y \in \alpha \implies \sup Y \in C$) and C is unbounded in α (that is, $\sup C = \alpha$).
- We can define an equivalence relation on subsets of α by $X \sim Y$ iff there is a club C in α such that $X \cap C = Y \cap C$.
- Let G be an abelian group of cardinality $\aleph_1$. An $\aleph_1$*-filtration* of G is a sequence $\{G_\alpha : \alpha \in \aleph_1\}$ of subgroups of G whose union is G and which satisfies for all $\alpha, \beta < \aleph_1$: $|G_\alpha|$ is countable, $\alpha \leqslant \beta \implies G_\alpha \subseteq G_\beta$, and $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$ for α a limit ordinal.

The Γ-Invariant

- Let G be an $\aleph_1$-free abelian group of cardinality $\aleph_1$. Let $\{G_\alpha : \alpha < \aleph_1\}$ be an $\aleph_1$-filtration of G. Let
$$E = \{\alpha < \aleph_1 : G/G_\alpha \text{ is not } \aleph_1\text{-free}\}.$$
The Γ*-invariant* of G, denoted $\Gamma(G)$ is defined to be the equivalence class $[E]$ of E with respect to $\sim$.
- Note that $\Gamma(G)$ does not depend upon the choice of filtration.

The Eklof-Shelah Criterion

If G is an $\aleph_1$-free group of size $\aleph_1$, then G is free if and only if $\Gamma(G) = [\varnothing]$.

- In fact, Eklof and Meckler give a more general result concerning $\leqslant \kappa$-generated κ-free modules, for κ a regular uncountable cardinal.

Forcing a Club into a Stationary Set

- Let α be a limit ordinal. A subset S of α is said to be stationary if it intersects every club in α.
- Note that G is free iff some/every representative of the Γ-invariant of G is stationary.

Theorem (Baumgartner, Harrington and Kleinberg 1976)

Let $\mathbf{M}$ be a countable transitive model of ZFC, with $A \subseteq \aleph_1$ stationary in $\mathbf{M}$. Then there exists a generic extension $\mathbf{N}$ of $\mathbf{M}$ preserving $\aleph_1$ in which there exists a club C such that $C \subseteq A$.

Theorem (Abraham and Shelah)

Let $\mathbf{M}$ be a countable transitive model of ZFC, with $A \subseteq \aleph_1$ stationary in $\mathbf{M}$. Then there exists a generic extension $\mathbf{N}$ of $\mathbf{M}$ which preserves cardinals and in which there exists a club C such that $C \subseteq A$.

Forcing an $\aleph_1$-free Group to Become Free

With Cardinal Preservation

Theorem (Herden and P. 2021)

Let $\mathbf{M}$ be a transitive model of ZFC and G a (non-free) $\aleph_1$-free abelian group with cardinality $\aleph_1$ and $\Gamma(G) = [\aleph_1]$ in $\mathbf{M}$. If $\mathbf{N}$ is a transitive model of ZFC containing $\mathbf{M}$ with G free in $\mathbf{N}$, then ${\aleph_1}^{\mathbf{M}} \neq {\aleph_1}^{\mathbf{N}}$.

- Proof: Suppose $\Gamma(G) = [E] = [\aleph_1]$. Then there exists some C club in $\aleph_1$ with $E \cap C = \aleph_1 \cap C = C$, and thus $C \subseteq E$. Let $\mathbf{N}$ be a transitive model of ZFC with $\mathbf{M} \subseteq \mathbf{N}$, and assume ${\aleph_1}^{\mathbf{M}} = {\aleph_1}^{\mathbf{N}}$. Then by absoluteness, and in particular using the absoluteness of $\aleph_1$-freeness, $([E] = \Gamma(G))^{\mathbf{N}}$.
  Lastly, note that as C is club in $\aleph_1$ in $\mathbf{M}$, then C is club in $\aleph_1$ in $\mathbf{N}$. Let C' be club in $\mathbf{N}$. Then $\emptyset \neq C \cap C' \subseteq E \cap C'$. So E is stationary in $\mathbf{N}$, that is $([E] \neq [\emptyset])^{\mathbf{N}}$, and thus G is not free in $\mathbf{N}$.

Forcing an $\aleph_1$-free Group to Become Free

With Cardinal Preservation

Theorem (Herden and P. 2021)

Let $\mathbf{M}$ be a countable transitive model of ZFC and in $\mathbf{M}$, let G be an $\aleph_1$-free abelian group of cardinality $\aleph_1$ with $\Gamma(G) \neq [\aleph_1]$. Then there exists a generic extension $\mathbf{N}$ of $\mathbf{M}$ which preserves the cardinality of G with G free in $\mathbf{N}$.

- Suppose $\Gamma(G) = [E] \neq [\aleph_1]$ in $\mathbf{M}$. Then $\aleph_1 - E$ is stationary in $\mathbf{M}$. For if $\aleph_1 - E$ is not stationary, then there exists some C club in $\aleph_1$ with $C \subseteq E$. Thus $E \cap C = C = \aleph_1 \cap C$, and so $[E] = [\aleph_1]$. Since $\aleph_1 - E$ is stationary, we can apply the BHK or Abraham and Shelah forcing to produce a generic extension $\mathbf{N}$ of $\mathbf{M}$ with containing some C club in $\aleph_1$ with $C \subseteq \aleph_1 - E$, and with ${\aleph_1}^{\mathbf{M}} = {\aleph_1}^{\mathbf{N}}$. Thus E is not stationary in $\mathbf{N}$. So as $([\varnothing] = [E] = \Gamma(G))^{\mathbf{N}}$, G is free in $\mathbf{N}$.

Forcing an $\aleph_1$-free Group to Become Free

With Cardinal Preservation

- Combining the two previous results gives us the following necessary and sufficient condition for when an $\aleph_1$-free group of cardinality $\aleph_1$ can be forced to be free with cardinal preservation.

Theorem (Herden and P. 2021)

Let **M** be a countable transitive model of ZFC and G an $\aleph_1$-free abelian group of size $\aleph_1$ in **M**. Then there exists some transitive model **N** of ZFC extending **M** in which the cardinality of G is preserved and G is free if and only if $\Gamma(G) \neq [\aleph_1]$ in **M**.

- We call a group G with $\Gamma(G) = [\aleph_1]$ a *turbid group*.
- The Baer-Specker group is turbid assuming CH.

Partial Basis Forcing Revisited

- Recall the partial basis forcing:
  $\mathcal{P} = \{S \subset H : S \text{ lin. indep. } \wedge |S| < \lambda \wedge H/\langle S \rangle \text{ is } \aleph_1\text{-free}\}^{\mathbf{M}}$,
  ordered by $S' \leqslant S \iff S \subseteq S'$.

Theorem (Herden and P. 2021)

Let **M** be a countable transitive model of ZFC, and let H be a non-turbid $\aleph_1$-free abelian group of cardinality $\aleph_1$. Then forcing with $\mathcal{P}$ makes H free and preserves $\aleph_1$.

Proof:

- Fix a (strictly increasing) filtration $\{H_\alpha : \alpha < \aleph_1\}$ for H and define $E = \{\alpha < \aleph_1 : H/H_\alpha \text{ is not } \aleph_1\text{-free } \}$.
- We will work in $\mathcal{P}' = \{p \in \mathcal{P} : \exists\alpha \in \aleph_1 - E \text{ with } p \text{ a basis for } H_\alpha\}$, which is dense in $\mathcal{P}$.
- $p \in \mathcal{P}'$. Define the "height", $h(p)$, of p to be the unique ordinal with $\langle p \rangle = H_{h(p)}$.
- Suppose by way of contradiction that $\mathbf{M}[G]$ contains a bijection $f : \omega \to \omega_1^{\mathbf{M}}$. Let τ be a $\mathcal{P}$-name such that $\tau_G = f$.
- there must be some $p \in \mathcal{P}$ (hence some $p \in \mathcal{P}'$) which forces this, that is, $p \Vdash$ "τ is a bijection from $\check{\omega}$ to $\omega_1{}^{\mathbf{M}}$."
- Inductively define in $\mathbf{M}$ a sequence $\{A_\alpha = (q, n, g) : \alpha < \aleph_1\}$ of partial functions which approximate this bijection (up to n), and elements of $\mathcal{P}'$ which force this approximation. Also define a corresponding (countable) ordinal $h_\alpha = \sup\{h(q) : (q, n, g) \in A_\alpha\}$.

Proof (continued):

- The set $C = \{h_\alpha : \alpha < \aleph_1\}$ is a club in $\aleph_1$, as is the set $C^* = \{h_\alpha : \alpha < \aleph_1, \alpha$ is a limit ordinal$\}$.
- Choose $h_{\alpha*} \in (\aleph_1 - E) \cap C^*$ (by stationarity). We have $h_{\alpha*} = \sup_{\beta<\alpha*} h_\beta = \sup_{n\in\omega} h_{\alpha_n}$.
- Construct in $\mathbf{M}$ a sequence $\{(q_n, n, g_n) : n \in \omega\}$ such that $(q_{n+1}, n+1, g_{n+1}) \in A_{\alpha_{n+1}} - A_{\alpha_n}$. The construction of the A_α's assures that $q_{n+1} \leqslant q_n$ and $g_n \subseteq g_{n+1}$.
- Then $g = \bigcup_{n\in\omega} g_n$ defines a function from ω to ω_1 in $\mathbf{M}$.
- We have $\sup_{n\in\omega} h(q_n) = h_{\alpha*}$. Recall $q_n \in \mathcal{P}'$ is a basis for $H_{h(q_n)}$. Then $q^* := \bigcup_{n\in\omega} q_n$ is a basis for $H_{h_{\alpha*}}$. As $h_{\alpha*} \in \aleph_1 - E$, $H/\langle q^*\rangle$ is $\aleph_1$-free.
- Thus $q^* \in \mathcal{P}'$ with $q^* \leqslant q_n$ for all $n \in \omega$. Thus $q^* \Vdash$ "τ is a bijection from $\check{\omega}$ to $\underbrace{\omega_1}^{\mathbf{M}}$ with $\tau = \check{g}$." But this implies that we have a bijection $g : \omega \to \omega_1$ in $\mathbf{M}$. Contradiction.

Further Work

- Use iterated forcing to make multiple $\aleph_1$-free groups free simultaneously, and explore applications to homological algebra.
- Force isomorphisms between $\aleph_1$-free groups using partial isomorphism poset.
- Develop set-theoretical tools and principles for algebraic constructions with $\aleph_1$-free groups.

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