Constraint Satisfaction Problems
A Survey

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CSP = specifications of subpowers of a finite algebra

Fix a finite algebra \( A \).

**Definition**
A constraint network over \( A \) is a pair \((n, \varphi)\) where
- \( n \geq 1 \)
- \( \varphi \) is a quantifier-free formula of the form \( \bigwedge_{i \in I} R_i(x_i) \),
where for each \( i \in I \),
  - \( x_i \) is a \( d \)-tuple of variables from \( \{x_1, \ldots, x_n\} \) (for some \( d \))
  - \( R_i \) is a subuniverse of \( A^d \).

The relation defined by \((n, \varphi)\) is
\[
\text{Rel}_A(n, \varphi) = \{ \mathbf{a} \in A^n : \varphi(\mathbf{a}) \}.
\]
Example

Let
\[ A = \{0, 1\}; \ x+y+z \]
\[ R_0 = \{(0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0)\} \]
\[ R_1 = \{(0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 1, 1)\}. \]

\( R_0, R_1 \leq A^3 \). Thus the following is a constraint network over \( A \):

\[ (6, R_0(x_1, x_2, x_3) \land R_1(x_1, x_4, x_5) \land R_0(x_2, x_4, x_6) \land R_1(x_3, x_5, x_6)). \]

We can view \( \varphi \) as asserting (over \( \mathbb{Z}_2 \))

\[
\begin{align*}
  x_1 + x_2 + x_3 & = 0 \\
  x_1 + x_4 + x_5 & = 1 \\
  x_2 + x_4 + x_6 & = 0 \\
  + x_3 + x_5 + x_6 & = 1.
\end{align*}
\]

\( \text{Rel}_A(6, \varphi) \) is the solution-set to this linear system.
Variant notations

A constraint network over $A$ is a pair $(n, \varphi)$, $\varphi = \bigwedge_i R_i(x_i)$ . . .

... may be written as...

<table>
<thead>
<tr>
<th>$n$</th>
<th>${x_1, \ldots, x_n}$ ($= V$, the set of variables)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi$</td>
<td>${(x_i, R_i) : i \in I}$ ($= C$)</td>
</tr>
<tr>
<td></td>
<td>$\cdot$ $(x_i, R_i)$ is called a constraint</td>
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<tr>
<td></td>
<td>$\cdot$ $x_i$ is its scope</td>
</tr>
<tr>
<td></td>
<td>$\cdot$ $R_i$ is its constraint relation</td>
</tr>
<tr>
<td>$(n, \varphi)$</td>
<td>$(V, C)$ or $(V, A, C)$</td>
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</tbody>
</table>
Decision Problems

Definition

$(n, \varphi)$ is $k$-ary if each scope has length $\leq k$.

Definition

CSP($A$, $k$)

Input: A $k$-ary constraint network $(n, \varphi)$ over $A$.

Question: Is $\text{Rel}_A(n, \varphi) \neq \emptyset$?

Dichotomy Conjecture (Feder & Vardi)

For all $A$ and $k$, CSP($A$, $k$) is in P or is NP-hard.

Algebraic Dichotomy Conjecture (Bulatov, Krokhin & Jeavons)

If $A$ has a Taylor operation, then CSP($A$, $k$) is in P for every $k$. $A$ is tractable
Taylor operations

Definition
An operation \( t : A^n \to A \) is a **Taylor operation** if

1. \( t \) is idempotent \( (t(x, x, \ldots, x) \approx x) \);
2. For each \( i = 1, \ldots, n \), \( t \) satisfies an identity of the form \( t(x) \approx t(y) \) with \( x_i \neq y_i \).

Theorem (Taylor; Barto & Kozik; Hobby & McKenzie)
For a finite algebra \( A \), the following are equivalent:

1. \( A \) has a Taylor (term) operation.
2. \( A \) satisfies some idempotent Maltsev condition not satisfied by \( \text{SETS} \).
3. \( A \) has an idempotent **cyclic** term \( t(x_1, \ldots, x_n) \), i.e.,

   \[
   t(x_1, x_2, \ldots, x_n) \approx t(x_2, \ldots, x_n, x_1).
   \]

4. \( \text{V}(A) \) omits type 1.
Progress

**Algebraic Dichotomy Conjecture**
If $A$ has a Taylor operation, then $\text{CSP}(A, k)$ is in P for every $k$. $A$ is tractable.

**Theorem**
$A$ is known to be tractable if:

1. $V(A)$ is CM. (Dalmau ‘05 + IMMVM ‘07, using Barto ‘16?)
2. $V(A)$ is SD($\land$). (Barto & Kozik ‘09; Bulatov ‘09)
3. $A$ is Taylor + conservative, i.e. $Su(A) = P(A)$. (Bulatov ‘03)
4. $A$ is Taylor and $|A| = 2$ or $3$. (Schaefer ‘78, Bulatov ‘02)
Definition
Let \( A \) be a finite algebra, \( \mathcal{A} \) a set of finite algebras.

1. \( \text{CSP}(A) = \bigcup_k \text{CSP}(A, k) \). “Global”
2. \( \text{CSP}(\mathcal{A}, k) = \bigcup_{A \in \mathcal{A}} \text{CSP}(A, k) \). “Uniform”

Can’t ask these problems to be in P. (Set of inputs is problematic.)

Definition
Say \( \text{CSP}(A) \) [\( \text{CSP}(\mathcal{A}, k) \)] is “in” P if there is a poly-time algorithm which correctly decides all inputs to \( \text{CSP}(A) \) [\( \text{CSP}(\mathcal{A}, k) \)].

Global Tractability Problem
If \( A \) is tractable, does it follow that \( \text{CSP}(A) \) is “in” P?

\[ \text{A is globally tractable} \]

Uniform Tractability Question
(For a given Taylor class \( \mathcal{A} \)): Is \( \text{CSP}(\mathcal{A}, k) \) “in” P for all \( k \)?

\[ \text{\( \mathcal{A} \) is uniformly tractable} \]
Theorem

A is known to be globally tractable if:

1. A has a **cube term**. (Dalmau ‘05 + IMMVW ‘07)
2. \( V(A) \) is SD(\( \wedge \)). (Bulatov ‘09; Barto ‘14)
3. A is Taylor + conservative. (Bulatov ‘03)
4. A is Taylor and \(|A| = 2 \text{ or } 3\). (Schaefer ‘78, Bulatov ‘02)

Theorem (Bulatov ‘09; Barto ‘14)

The class \( SD_\wedge \) of all finite algebras generating an SD(\( \wedge \)) variety is uniformly globally tractable.
Open problems

1. If $V(A)$ is congruence modular, is $A$ globally tractable?

2. Is the class $\mathcal{M}$ of finite Maltsev algebras uniformly tractable?

3. If $A$ has a difference term, is $A$ tractable?

4. Suppose $A$ is idempotent and has a congruence $\theta$ such that
   - $A/\theta \in SD_\wedge$, and
   - Each $\theta$-block is in $\mathcal{M}$.
   
   ("SD(\wedge) over Maltsev.") Is $A$ tractable?
CSP(\(A, k\)) reduces to:

1. CSP(\(A\|_U, k\)), where \(U\) is a minimal range of a unary idempotent term, and \(A\|_U\) is the induced term-minimal algebra defined on \(U\).

2. CSP(((\(A\|_U\))^id, k)) where (\(B\))^id is the idempotent reduct of \(B\). (This is the “reduction to the idempotent case.”)

3. CSP(\(A^{\lceil k/2 \rceil}, 2\))

4. multi-CSP(\(H(A)_{si}, kd\)), where \(A\) is a subdirect product of \(d\) subdirectly irreducible homomorphic images.

5. CSP(\(A^+, k\)) where \(A^+ = (A; Pol(Su(A^k)))\).
Conditioning the input – local consistency

Let \((n, \varphi)\) be a 2-ary constraint network over \(A\).

At essentially no cost, one can assume that \((n, \varphi)\) is “determined” by a “(2,3)-minimal” constraint network.

Definition
A 2-ary constraint network \((n, \varphi)\) is a \((2,3)\)-system\(^1\) provided for all \(i, j \in \{1, 2, \ldots, n\}\):

1. \(\varphi\) has exactly one constraint \(R_{i,j}(x_i, x_j)\) with scope \((x_i, x_j)\).
2. \(R_{j,i} = (R_{i,j})^{-1}\).
3. For all \(k\), \(R_{i,j} \subseteq R_{i,k} \circ R_{k,j}\).

The “associated potatoes” are \(A_i := \text{proj}_1(R_{i,j}),\ i = 1, \ldots, n\).

Fact
There is a poly-time algorithm which, given a 2-ary constraint network over \(A\), outputs an equivalent (2,3)-system over \(A\).

\(^1\)There is no standard terminology.
Conditioning the input – absorption

Definition
Suppose $A$ is a finite idempotent algebra and $B \leq A$.

1. $B$ is an **absorbing subalgebra** if there exists a term operation $t(x_1, \ldots, x_m)$ of $A$ such that
   \[ t(B, \ldots, B, A, B, \ldots, B) \subseteq B \]
   for all possible positions of $A$.

2. $A$ is **absorption-free** if it has no proper absorbing subalgebra.

Given a $(2,3)$-system $(n, \varphi)$ over an idempotent $A$, Barto & Kozik show how to “shrink” the associated potatoes to absorption-free algebras, though losing $(2,3)$-systemhood and equivalency.

In some situations this has proven to be useful.
Miklós magic

Lemma (Maróti ‘09)

Suppose $A$ is idempotent and has a term operation $t(x, y)$ such that:

1. $A \models t(x, t(x, y)) \approx t(x, y)$.
2. $t(a, x)$ is non-surjective, for all $a \in A$.
3. There exists a proper subalgebra $C < A$ such that if $t(x, a)$ is surjective then $a \in C$.

Then $\text{CSP}(A, k)$ can be reduced to $\text{multi-CSP}(B \setminus \{A\}, \ell)$, where

- $B$ is the closure of $\{A\}$ under $H$, $S$, and “idempotent unary polynomial retracts.”
- $\ell = \max(k, |A|)$.

This may seem random, but it is useful (and the proof is beautiful).
Moving forward

Suppose \((n, \varphi)\) is a \(k\)-ary constraint network over \(A\), and \(R = \text{Rel}_A(n, \varphi) \leq A^n\).

**Definition**

A **compact** \(k\)-frame for \(R\) is a subset \(F \subseteq R\) such that

1. \(\text{proj}_J(F) = \text{proj}_J(R)\) for all \(J \subseteq \{1, \ldots, n\}\) with \(|J| \leq k\).
2. \(|F| \leq |A|^{k(n)}\).

Every relation definable by a \(k\)-ary constraint network over \(A\) has a compact \(k\)-frame, and is determined by any one of its \(k\)-frames.

**Speculation**: Is it possible to mimic the few subpowers algorithm without having few subpowers?
To carry this out, we would need a notion of “compact $k$-representation” extending compact $k$-frames with more data.

The following problem seems central:

**Functional Dependency Problem**

Suppose

- $\mathbf{A}$ is finite, idempotent, Taylor.
- $F$ is a compact $k$-frame for a relation $R \leq \mathbf{A}^n$ defined by some $k$-ary constraint network over $\mathbf{A}$.
- $X \subseteq \{1, \ldots, n\}$ and $\ell \in \{1, \ldots, n\} \setminus X$.

What additional data would enable us to efficiently decide whether $\text{proj}_{X \cup \{\ell\}}(R)$ is the graph of a function $f : \text{proj}_X(R) \to \text{proj}_\ell(R)$?
Barto ‘14: The collapse of the bounded width hierarchy, *J. Logic Comput.* (online)

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Bulatov ‘02: A dichotomy theorem for constraints on a 3-element set, *FOCS 2002*; see also *J. ACM* 2006.


Bulatov ‘09: Bounded relational width (unpublished; available on Bulatov’s website).


Hobby & McKenzie ‘88: *The Structure of Finite Algebras*.


Maróti ‘09: Tree on top of Maltsev (unpublished; available from Maróti’s website).

Schaefer ‘78: The complexity of satisfiability problems, *STOC ‘78*.