The Subpower Membership Problem for Finite Algebras with Cube Terms

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Joint work with A. Bulatov and P. Mayr

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Main Result

$\boxed{\text{Algebras}}$ $\boxed{\text{Algorithms}}$

$\boxed{\text{Structure}}$ $\boxed{\text{Complexity}}$

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- INPUT: $b_1, \ldots, b_k, c \in \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ with $\mathbf{A}_1, \ldots, \mathbf{A}_n \in \mathcal{K}$.
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Theorem

If $\mathcal{V}$ is a residually small variety with a cube term, then

$$\mathrm{SMP}(\mathcal{K}) \in \mathsf{P} \qquad \textit{for every finite } \mathcal{K} \subseteq \mathcal{V}_{\mathrm{fin}}.$$

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Definition. A *d-cube term* ($d \geq 2$) for a class $\mathcal{K}$ of algebras is a term C s.t.

$$\mathcal{K} \models C\Bigg(\underbrace{\begin{bmatrix} x \\ y \\ \vdots \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \\ \vdots \\ y \end{bmatrix}, \ldots, \begin{bmatrix} y \\ y \\ \vdots \\ x \end{bmatrix}, \begin{bmatrix} x \\ x \\ \vdots \\ y \end{bmatrix}, \ldots}_{d\text{-tuples in } x,\, y, \text{ with at least one } x}\Bigg) = \begin{bmatrix} y \\ y \\ \vdots \\ y \end{bmatrix}.$$

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For a finite algebra **A**,

- **A** has a cube term $\Leftrightarrow$ **A** has *few subpowers*, i.e.
  $\diamond$ $\log_2 |\mathrm{Sub}(\mathbf{A}^n)| \leq \mathrm{const} \cdot n^k$ for some k
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- $\mathbf{A}$ finitely related & $\mathcal{V}(\mathbf{A})$ CM $\Rightarrow$ $\mathbf{A}$ has a cube term [Barto, 2016?]

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Problem. Is $\mathrm{SMP}(\mathbf{A}) \in \mathsf{P}$ whenever $\mathcal{V}(\mathbf{A})$ has a Mal'tsev/cube term? [Willard, 2007]/[IMMVW, 2010]

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- $\mathrm{SMP}(\mathbf{A}) \in \mathsf{P}$ would yield a simpler proof.

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  - $\exists$ 10-element semigroup $\mathbf{S}$ and a 9-element homomorphic image $\bar{\mathbf{S}}$ of $\mathbf{S}$ such that $\mathrm{SMP}(\mathbf{S}) \in \mathsf{P}$ while $\mathrm{SMP}(\bar{\mathbf{S}})$ is NP-complete [Steindl, 2017?]

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However:

Theorem

If $\mathcal{V}$ has a cube term, then for every finite $\mathcal{K} \subseteq \mathcal{V}_{\text{fin}}$ we have that

$$\mathrm{SMP}(\mathcal{K}) \quad \overset{\text{poly time}}{\Longleftrightarrow} \quad \mathrm{SMP}(\mathbb{H}\mathcal{K}).$$

Critical Algebras in Varieties with a Cube Term: Reduction

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  - $\mathbf{R}$ is completely $\cap$-irreducible in $\mathrm{Sub}(\mathbf{R}_1 \times \cdots \times \mathbf{R}_n)$, and
  - $\mathbf{R}$ is *directly indecomposable*, i.e., $[n]$ has no partition $\{I, J\}$ such that $\mathbf{R}$ and $\mathbf{R}|_I \times \mathbf{R}|_J$ differ only by a permutation of coordinates.

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- Let $\theta = \theta_1 \times \cdots \times \theta_n$ ($\theta_i \in \mathrm{Con}(\mathbf{R}_i)$) be the largest product congruence of $\mathbf{R}_1 \times \cdots \times \mathbf{R}_n$ such that $\mathbf{R}$ is *θ-saturated*, i.e.,

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 & \mathbf{R} & \hookrightarrow & \mathbf{R}_1 \times \cdots \times \mathbf{R}_n & \\
 & \downarrow{\scriptstyle \nu_\theta} & & \downarrow{\scriptstyle \nu_{\theta_1}} \qquad\quad \downarrow{\scriptstyle \nu_{\theta_n}} & \\
\mathbf{R}/\theta|_{\mathbf{R}} = & \overline{\mathbf{R}} & \hookrightarrow & \overline{\mathbf{R}}_1 \times \cdots \times \overline{\mathbf{R}}_n & \cong (\mathbf{R}_1 \times \cdots \times \mathbf{R}_n)/\theta
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$$\begin{array}{ccccc} (\nu_{\theta_1} \times \ldots \times \nu_{\theta_n})^{-1}[\overline{\mathbf{R}}] = & \mathbf{R} & \hookrightarrow & \mathbf{R}_1 \times \cdots \times \mathbf{R}_n & \\ & \big\downarrow^{\nu_\theta} & & \big\downarrow^{\nu_{\theta_1}} \qquad \big\downarrow^{\nu_{\theta_n}} & \\ \mathbf{R}/\theta|_{\mathbf{R}} = & \overline{\mathbf{R}} & \hookrightarrow & \overline{\mathbf{R}}_1 \times \cdots \times \overline{\mathbf{R}}_n & \cong (\mathbf{R}_1 \times \cdots \times \mathbf{R}_n)/\theta \end{array}$$

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$$\mathbf{R}[\theta] := (\nu_{\theta_1} \times \ldots \times \nu_{\theta_n})^{-1}[\overline{\mathbf{R}}] = \mathbf{R}$$

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Let $\mathcal{V}$ be a variety with a d-cube term.
Let $\mathbf{R} \leq_{\mathrm{sd}} \mathbf{R}_1 \times \cdots \times \mathbf{R}_n$ with $\mathbf{R}_1, \ldots, \mathbf{R}_n \in \mathcal{V}$.

- Assume $\mathbf{R}$ is a *critical subalgebra* of $\mathbf{R}_1 \times \cdots \times \mathbf{R}_n$, that is,
  - $\mathbf{R}$ is completely $\cap$-irreducible in $\mathrm{Sub}(\mathbf{R}_1 \times \cdots \times \mathbf{R}_n)$, and
  - $\mathbf{R}$ is *directly indecomposable*, i.e., $[n]$ has no partition $\{I, J\}$ such that $\mathbf{R}$ and $\mathbf{R}|_I \times \mathbf{R}|_J$ differ only by a permutation of coordinates.
- Let $\theta = \theta_1 \times \cdots \times \theta_n$ ($\theta_i \in \mathrm{Con}(\mathbf{R}_i)$) be the largest product congruence of $\mathbf{R}_1 \times \cdots \times \mathbf{R}_n$ such that $\mathbf{R}$ is *θ-saturated*, i.e.,

$$
\begin{array}{rcccl}
\mathbf{R}[\theta] := (\nu_{\theta_1} \times \ldots \times \nu_{\theta_n})^{-1}[\overline{\mathbf{R}}] = & \mathbf{R} & \hookrightarrow & \mathbf{R}_1 \times \cdots \times \mathbf{R}_n & \\
 & \big\downarrow{\scriptstyle \nu_\theta} & & \big\downarrow{\scriptstyle \nu_{\theta_1}} \qquad\quad \big\downarrow{\scriptstyle \nu_{\theta_n}} & \\
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- **Fact.** $a\, \theta_1\, b$ iff $a\mathbf{u}, b\mathbf{u} \in \mathbf{R}$ for some $\mathbf{u} \in \mathbf{R}_2 \times \cdots \times \mathbf{R}_n$.

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$\overline{\mathbf{R}}$ is *the reduced representation* of $\mathbf{R}$.

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Structure Theorem (Kearnes–Sz, 2012)

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is the graph of an isomorphism $\overline{\mathbf{R}}_i/\rho_i \to \overline{\mathbf{R}}_j/\rho_j$ for any $i, j \in [n]$.

Applying the Structure Theorem to SMP($\mathcal{K}$): Prelims

INPUT: $b_1, \ldots, b_k, c \in \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ $(\mathbf{A}_1, \ldots, \mathbf{A}_n \in \mathcal{K} \subseteq \mathcal{V}_{\text{fin}})$

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Will assume (†) from now on.

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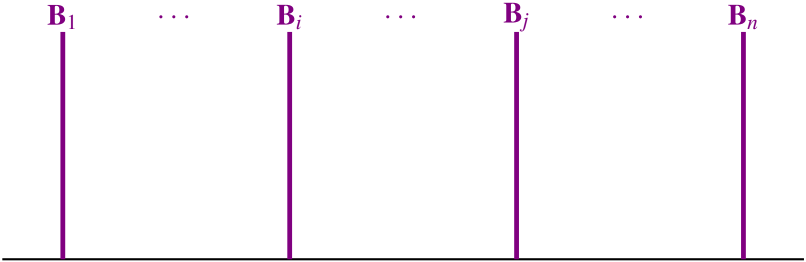

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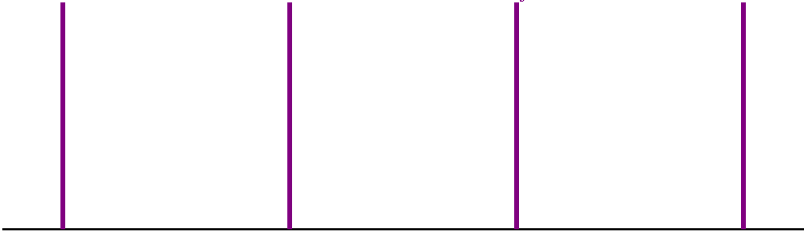

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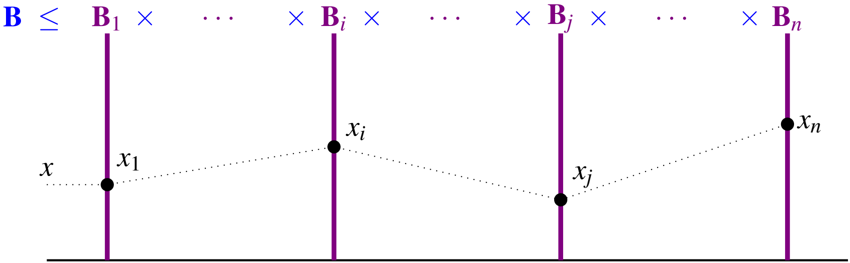

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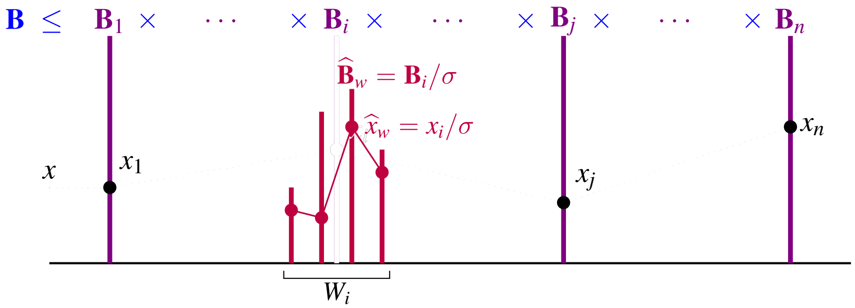

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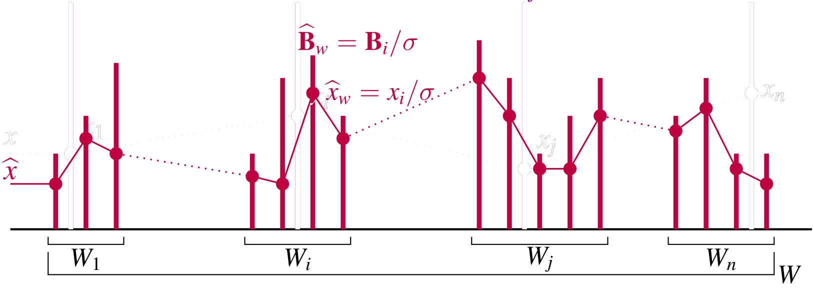

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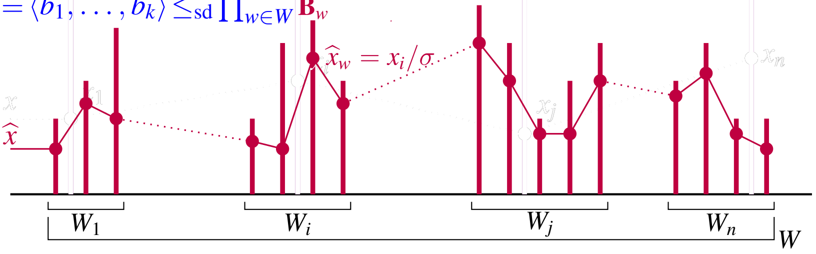

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$\boxed{c \in \mathbf{B} \text{ iff } \widehat{c} \in \widehat{\mathbf{B}}}$

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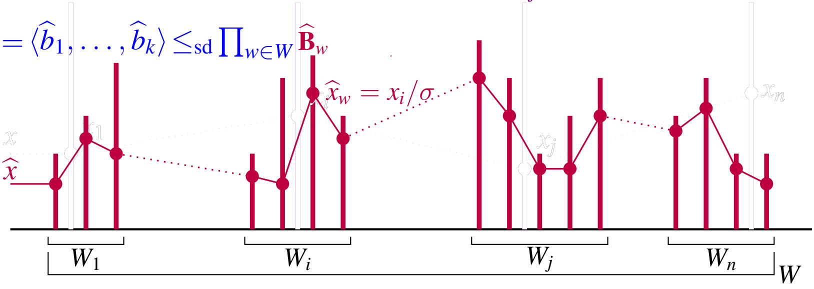

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Have: $\widehat{b}_1, \ldots, \widehat{b}_k, \widehat{c} \in \prod_{w \in W} \widehat{\mathbf{B}}_w$ ($\widehat{\mathbf{B}}_w \in \mathbb{HS}\mathcal{K} \subseteq \mathcal{V}_{\text{fin}}$)

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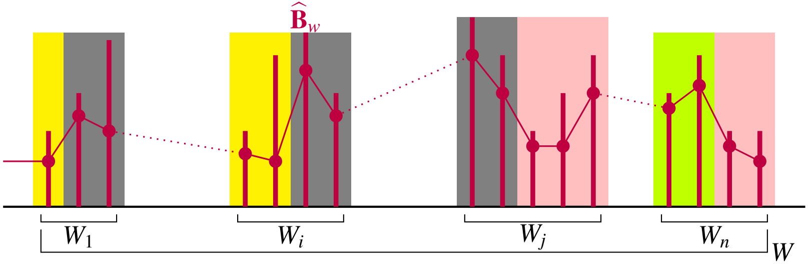

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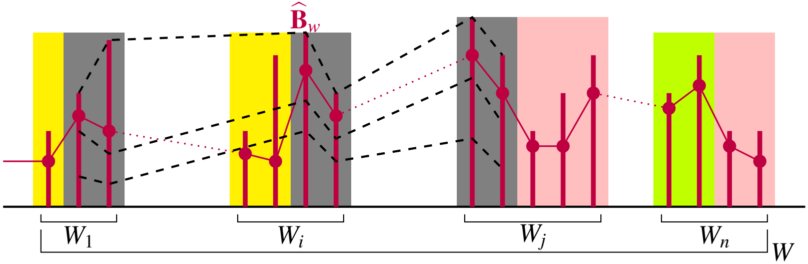

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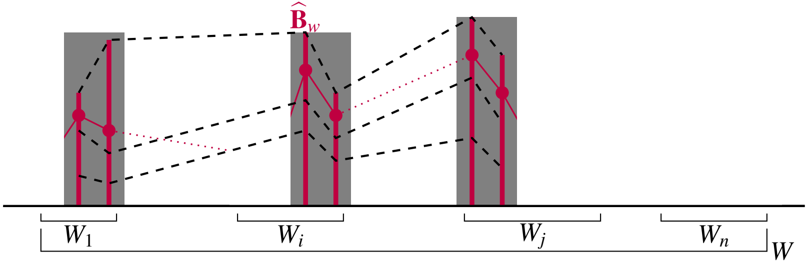

Applying the Structure Theorem to $\mathrm{SMP}(\mathcal{K})$: A Criterion

Theorem

Let $\mathcal{V}$ be a variety with a d-cube term, and let $b_1, \ldots, b_k, c \in \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ be an input for $\mathrm{SMP}(\mathcal{K})$ with $n \geq d$ for some finite $\mathcal{K} \subseteq \mathcal{V}_{\mathrm{fin}}$.

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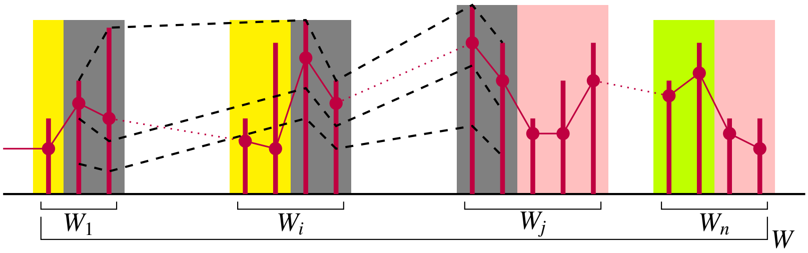

Applying the Structure Theorem to SMP($\mathcal{K}$): A Criterion

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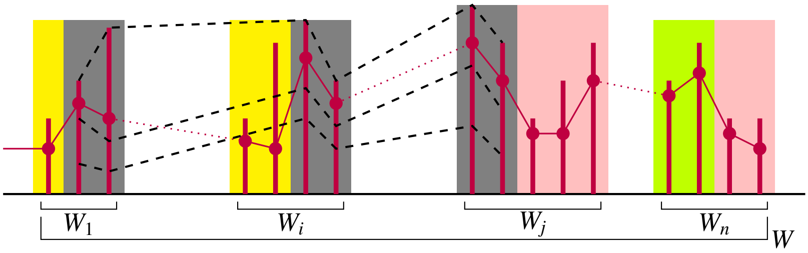

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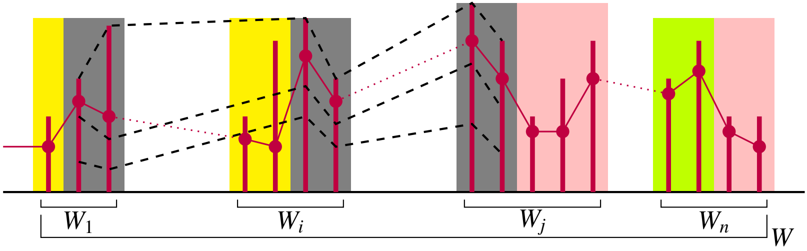

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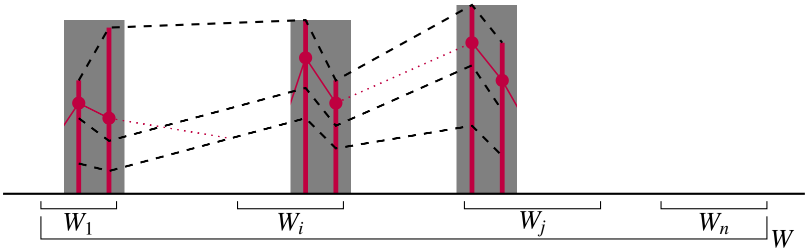

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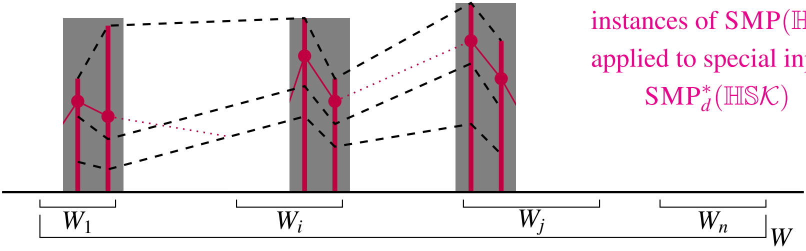

instances of $\mathrm{SMP}(\mathbb{HS}\mathcal{K})$ applied to special inputs: $\mathrm{SMP}^*_d(\mathbb{HS}\mathcal{K})$

Applying the Structure Theorem to $\mathrm{SMP}(\mathcal{K})$: A Corollary

Corollary

Let $\mathcal{V}$ be a variety with a d-cube term.
For every finite $\mathcal{K} \subseteq \mathcal{V}_{\mathrm{fin}}$,

$$\mathrm{SMP}(\mathcal{K}) \quad \overset{\text{poly time}}{\Longleftrightarrow} \quad \mathrm{SMP}_d^*(\mathbb{H}\mathbb{S}\mathcal{K}).$$

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Assume:

(∗) $\mathcal{V}$ is a RS variety with a d-cube term.

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Enough to show:

Claim. $(*) \;\Rightarrow\; \mathrm{SMP}_d^*(\mathcal{K}) \in \mathsf{P}$ *for all finite* $\mathcal{K} \subseteq \mathcal{V}_{\mathrm{fin}}$.

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Important Facts.

- $\mathcal{V}$ has a cube term $\Rightarrow$ $\mathcal{V}$ CM.
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- $\mathcal{V}$ has a cube term $\;\Rightarrow\;$ $\mathcal{V}$ CM.
  (BIMMVW, 2010)
- $\mathcal{V}$ CM & RS $\;\Rightarrow\;$ for every SI $\mathbf{S} \in \mathcal{V}$ with abelian monolith μ,
  the centralizer $\rho = (0 : \mu)$ is abelian.
  (Freese, McKenzie, 1981)

The RS Case: Idea of Proof

INPUT: $b_1, \ldots, b_k, c \in \mathbf{B}_1 \times \cdots \times \mathbf{B}_n$ ($\mathbf{B}_1, \ldots, \mathbf{B}_n \in \mathcal{K} \subseteq \mathcal{V}_{\text{fin}},\ n \geq d, 3$) s.t.

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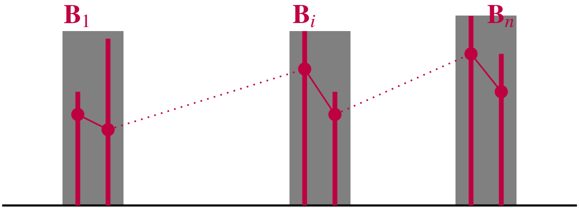

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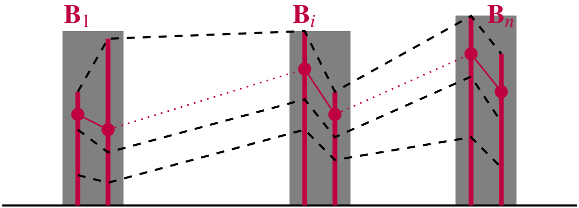

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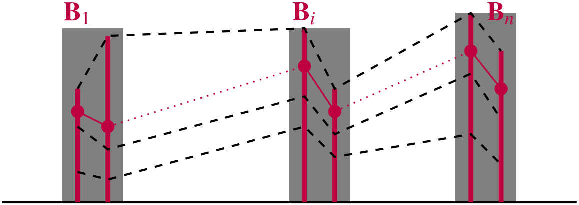

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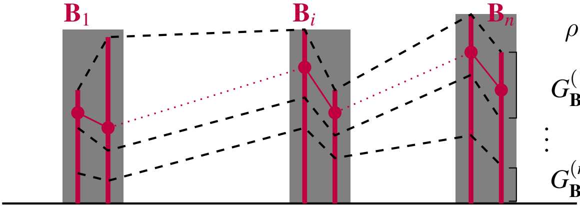

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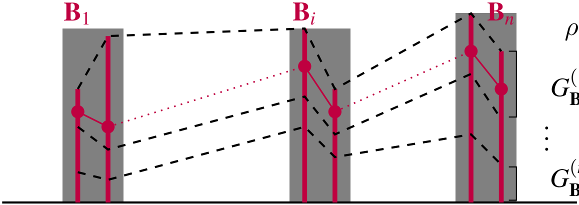

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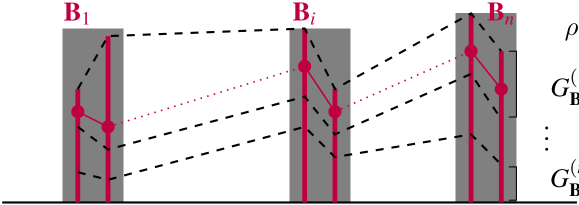

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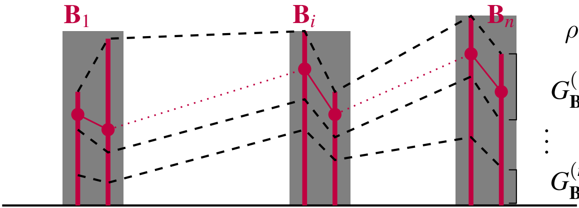

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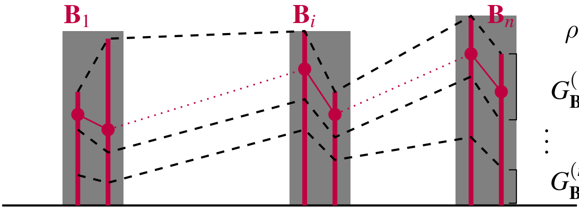

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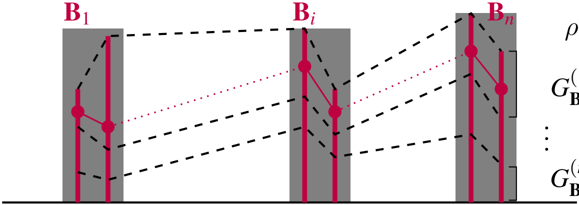

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Hence, a modified group algorithm decides $c \in G^{(r)}_{\mathbf{B}}$ ($\Leftrightarrow c \in \mathbf{B}$)