A CSP Algorithm for some Subvarieties of Maltsev Products

Ian Payne

University of Waterloo

May 20, 2016

$(2,3)$-consistency

Definition

We call an input, $\mathcal{I} = (X, \mathbf{D}, \mathcal{C})$ to $\mathrm{CSP}(\mathbf{D}, 2)$ a standard $(2,3)$-instance if $\mathcal{C}$ takes the form

$$\{(x, P_x) : x \in X\} \cup \{((x,y), R_{xy}) : x, y \in X\}$$

(P1) For each $x \in X$, $R_{x,x} = 0_{P_x}$,

(P2) For $x, y, z \in X$ and any $(a,b) \in R_{xy}$, there is a $c \in P_z$ such that $(a,c) \in R_{x,z}$ and $(b,c) \in R_{y,z}$,

(P3) For each $x, y \in X$, $R_{xy} \leq_{\mathrm{sd}} \mathbf{P}_x \times \mathbf{P}_y$ if P_x and P_y are both non-empty.

(P4) $R_{y,x} = R_{xy}^{-1}$ for each $x, y \in X$.

Definition

We call an input, $\mathcal{I} = (X, \mathbf{D}, \mathcal{C})$ to $\mathrm{CSP}(\mathbf{D}, 2)$ a standard $(2,3)$-instance if $\mathcal{C}$ takes the form

$$\{(x, P_x) : x \in X\} \cup \{((x,y), R_{xy}) : x, y \in X\}$$

(P1) For each $x \in X$, $R_{x,x} = 0_{P_x}$,

(P2) For $x, y, z \in X$ and any $(a,b) \in R_{xy}$, there is a $c \in P_z$ such that $(a,c) \in R_{x,z}$ and $(b,c) \in R_{y,z}$,

(P3) For each $x, y \in X$, $R_{xy} \leq_{\mathrm{sd}} \mathbf{P}_x \times \mathbf{P}_y$ if P_x and P_y are both non-empty.

(P4) $R_{y,x} = R_{xy}^{-1}$ for each $x, y \in X$.

Proposition (Folklore)

There is a polynomial time algorithm which transforms an input, $\mathcal{I}$ of $\mathrm{CSP}(\mathbf{D}, 2)$, *into a standard $(2,3)$-instance with exactly the same set of solutions as $\mathcal{I}$.*

Bulatov's Result on 2-semilattices

Definition

A 2-semilattice operation, $\cdot$, is a binary operation satisfying

- $x \cdot x \approx x$
- $x \cdot y \approx y \cdot x$
- $x \cdot y \approx x \cdot (x \cdot y)$

Definition

A 2-semilattice is an algebra $(A, \cdot)$ where $\cdot$ is a 2-semilattice operation.

Definition

The variety of 2-semilattices will be denoted by $\mathcal{S}$.

Definition

A 2-semilattice operation, $\cdot$, is a binary operation satisfying

- $x \cdot x \approx x$
- $x \cdot y \approx y \cdot x$
- $x \cdot y \approx x \cdot (x \cdot y)$

Definition

A 2-semilattice is an algebra $(A, \cdot)$ where $\cdot$ is a 2-semilattice operation.

Definition

The variety of 2-semilattices will be denoted by $\mathcal{S}$.

Theorem (Bulatov '06)

If $\mathbf{D}$ *has a "2-semilattice" operation, then every "nonempty" standard* $(2, 3)$*-instance of* $\mathrm{CSP}(\mathbf{D}, 2)$ *has a solution.*

Definition

For a finite 2-semilattice, $\mathbf{A}$. Define a digraph relation on A by $a \longrightarrow b$ iff $a \cdot b = b$.

Definition

For a finite 2-semilattice, $\mathbf{A}$. Define a digraph relation on A by $a \longrightarrow b$ iff $a \cdot b = b$.

Lemma

1. *$a \longrightarrow a \cdot b$ and $b \longrightarrow a \cdot b$ for all $a, b \in A$.*
2. *If $\mathbf{A}$ is strongly connected, so are all of its quotients.*
3. *If we quasi-order the strongly connected components by $U \geq V$ if $u \longrightarrow v$ for some $u \in U, v \in V$, there is a unique minimal component, called A'.*
4. *A' is a minimal absorbing subuniverse of $\mathbf{A}$.*

Reduction 1

Fix a standard $(2,3)$-instance, $\mathcal{I}$ of $\mathrm{CSP}(\mathbf{D},2)$ for some finite $\mathbf{D} \in \mathcal{S}$. Reduce $\mathcal{I}$ to $\mathcal{I}'$ as follows:

Reduction 1

Fix a standard $(2,3)$-instance, $\mathcal{I}$ of CSP$(\mathbf{D}, 2)$ for some finite $\mathbf{D} \in \mathcal{S}$. Reduce $\mathcal{I}$ to $\mathcal{I}'$ as follows:

- For each $x \in X$, replace P_x by P_x', its smallest strongly connected component.

Reduction 1

Fix a standard $(2,3)$-instance, $\mathcal{I}$ of CSP$(\mathbf{D}, 2)$ for some finite $\mathbf{D} \in \mathcal{S}$. Reduce $\mathcal{I}$ to $\mathcal{I}'$ as follows:

- For each $x \in X$, replace P_x by P'_x, its smallest strongly connected component.
- For each $(x, y) \in X^2$, replace R_{xy} by R'_{xy}.

More precisely, the constraint (x, P_x) gets replaced by (x, P'_x), and $((x, y), R_{xy})$ by $((x, y), R'_{xy})$.

Reduction 1

Fix a standard $(2,3)$-instance, $\mathcal{I}$ of $\mathrm{CSP}(\mathbf{D},2)$ for some finite $\mathbf{D} \in \mathcal{S}$. Reduce $\mathcal{I}$ to $\mathcal{I}'$ as follows:

- For each $x \in X$, replace P_x by P'_x, its smallest strongly connected component.
- For each $(x,y) \in X^2$, replace R_{xy} by R'_{xy}.

More precisely, the constraint (x, P_x) gets replaced by (x, P'_x), and $((x,y), R_{xy})$ by $((x,y), R'_{xy})$.

Lemma (Reduction 1 Lemma)

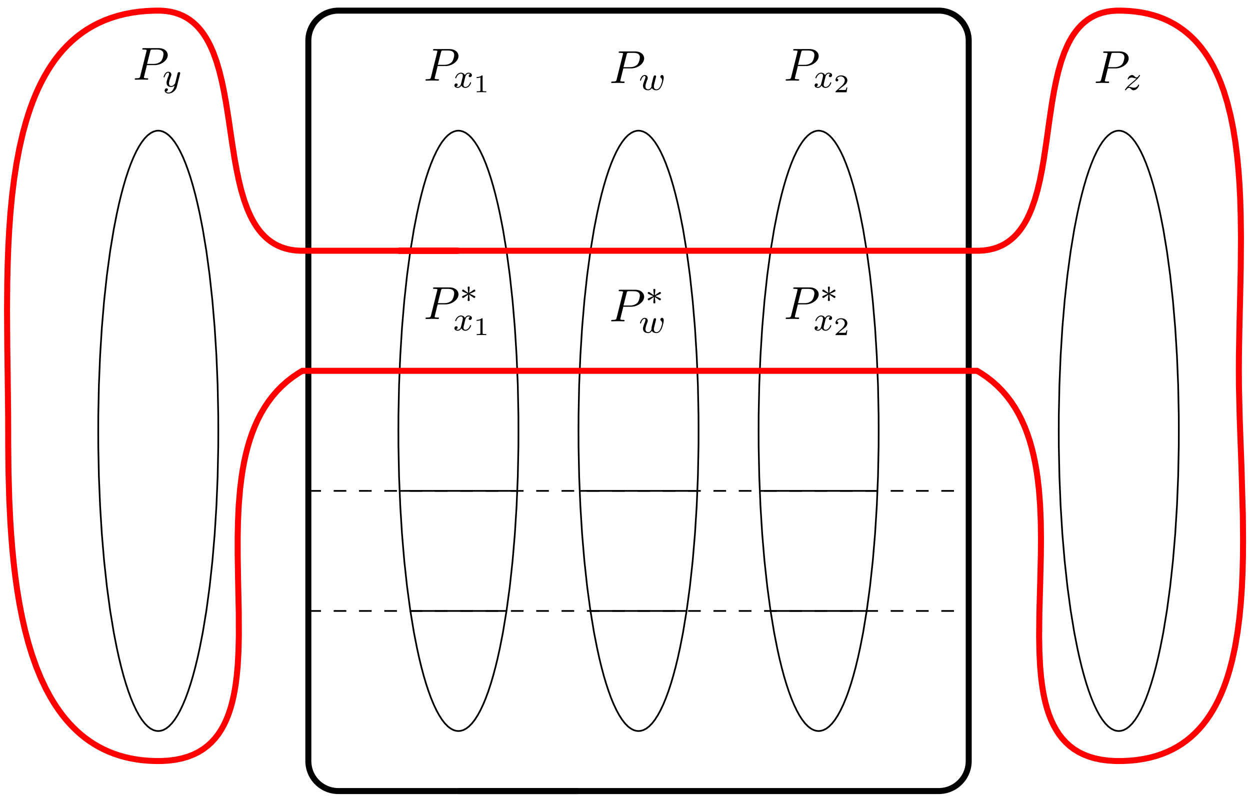
If $\mathcal{I}$ is a nonempty standard $(2,3)$-instance, so is $\mathcal{I}'$.

A Picture of Reduction 2

In the situation where $\mathcal{J} = \mathcal{J}'$ and some potato, $\mathbf{P}_w$ is not simple, there is another way to shrink.

A Picture of Reduction 2

In the situation where $\mathcal{J} = \mathcal{J}'$ and some potato, $\mathbf{P}_w$ is not simple, there is another way to shrink.



Lemma (Reduction 2 Lemma)

If $\mathcal{J}$ is a standard $(2,3)$-instance, so is $\mathcal{I}^*$, where $\mathcal{I}^*$ is the result of applying reduction 2 to $\mathcal{I}$.

Lemma

Let $\mathbf{A}, \mathbf{B} \in \mathcal{S}$ *be simple and strongly connected. If* $R \leq_{\mathrm{sd}} \mathbf{A} \times \mathbf{B}$, *then Either* R *is the graph of a bijection, or it is* $A \times B$.

Lemma

Let $\mathbf{A}, \mathbf{B} \in \mathcal{S}$ *be simple and strongly connected. If* $R \leq_{\text{sd}} \mathbf{A} \times \mathbf{B}$, *then Either* R *is the graph of a bijection, or it is* $A \times B$.

Proof/"Algorithm" for Bulatov's Theorem.

Let $\mathcal{I}$ be a standard $(2,3)$-instance of $\mathrm{CSP}(\mathbf{D}, 2)$ for some $\mathbf{D} \in \mathcal{S}$.

1. Apply Reduction 1 ($\mathcal{I} \leftarrow \mathcal{I}'$).
2. If $\mathcal{I}$ has a non-simple potato, apply Reduction 2 ($\mathcal{I} \leftarrow \mathcal{I}^*$). Go to 1 .
3. Get to a situation where R_{xy} is a bijection or direct product for each $x, y \in X$. Choose a solution on each "bijection"-class to find a global solution.

$\square$

Bulatov Solutions

What is a Bulatov Solution?

Definition

A Bulatov solution to a standard $(2, 3)$-instance is any solution obtained from the proof on the previous slide.

What is a Bulatov Solution?

Definition

A Bulatov solution to a standard $(2,3)$-instance is any solution obtained from the proof on the previous slide.

Remark

Recall that $\mathrm{Sol}(\mathcal{J}) \leq \mathbf{D}^{|X|}$. This means $\mathrm{Sol}(\mathcal{J})$ is in $\mathcal{S}$, so it has a digraph structure.

What is a Bulatov Solution?

Definition

A Bulatov solution to a standard $(2,3)$-instance is any solution obtained from the proof on the previous slide.

Remark

Recall that $\mathrm{Sol}(\mathcal{I}) \leq \mathbf{D}^{|X|}$. This means $\mathrm{Sol}(\mathcal{I})$ is in $\mathcal{S}$, so it has a digraph structure.

Lemma (Bulatov Solution Walk Lemma)

Let $\mathcal{I}$ be a standard $(2,3)$-instance of $\mathrm{CSP}(\mathbf{D},2)$ for some $\mathbf{D} \in \mathcal{S}$. Suppose φ is a solution to $\mathcal{I}$, and ψ is a Bulatov solution to $\mathcal{I}$. Then there is a directed walk from φ to ψ.

Proof Sketch

- "Prove the Lemma" for each reduction type.
- Notice that if no reduction is possible, the solution space is strongly connected.
- Concatenate the walks.

Results

Theorem

Let $\mathcal{W}$ be an idempotent variety so that for every finite $\mathbf{A} \in \mathcal{W}$, $\mathrm{CSP}(\mathbf{A}, 2)$ has a polynomial time algorithm. Let $\mathbf{D}$ be finite, idempotent, and similar to $\mathcal{W}$. Suppose $\mathbf{D}$ has a binary term, $\cdot$, and a congruence, θ such that $\cdot$ is a 2-semilattice operation on $\mathbf{D}/\theta$, and each θ-class as a subalgebra of $\mathbf{D}$ is in $\mathcal{W}$. Also suppose the following hold:

1. $\mathcal{W} \models x \cdot y \approx x$
2. $\mathbf{D} \models x \cdot (y \cdot z) \approx x \cdot (z \cdot y)$.

The $\mathrm{CSP}(\mathbf{D}, 2)$ has a polynomial time algorithm.

The Algorithm (roughly)

- Let $\mathbf{D}$ be as in the statement of the Theorem, and fix $\mathcal{I}$, a standard $(2, 3)$-instance of $\mathrm{CSP}(\mathbf{D}, 2)$.

The Algorithm (roughly)

- Let $\mathbf{D}$ be as in the statement of the Theorem, and fix $\mathcal{I}$, a standard $(2,3)$-instance of $\mathrm{CSP}(\mathbf{D},2)$.
- Construct a “quotient instance” by taking the quotient in each potato by its “special” congruence. The binary relations are constructed in the natural way.

The Algorithm (roughly)

- Let $\mathbf{D}$ be as in the statement of the Theorem, and fix $\mathcal{I}$, a standard $(2,3)$-instance of $\mathrm{CSP}(\mathbf{D},2)$.
- Construct a "quotient instance" by taking the quotient in each potato by its "special" congruence. The binary relations are constructed in the natural way.
- This quotient instance is a standard $(2,3)$-instance, and (modulo some isomorphism trickery) it is an instance of $\mathrm{CSP}(\mathbf{E},2)$ for some $\mathbf{E}$ with a 2-semilattice operation.

The Algorithm (roughly)

- Let $\mathbf{D}$ be as in the statement of the Theorem, and fix $\mathcal{I}$, a standard $(2, 3)$-instance of $\mathrm{CSP}(\mathbf{D}, 2)$.
- Construct a "quotient instance" by taking the quotient in each potato by its "special" congruence. The binary relations are constructed in the natural way.
- This quotient instance is a standard $(2, 3)$-instance, and (modulo some isomorphism trickery) it is an instance of $\mathrm{CSP}(\mathbf{E}, 2)$ for some $\mathbf{E}$ with a 2-semilattice operation.
- Replace every potato and relation by the reduct to this operation; find a Bulatov solution.

The Algorithm (roughly)

- Let $\mathbf{D}$ be as in the statement of the Theorem, and fix $\mathcal{I}$, a standard $(2,3)$-instance of CSP$(\mathbf{D}, 2)$.
- Construct a "quotient instance" by taking the quotient in each potato by its "special" congruence. The binary relations are constructed in the natural way.
- This quotient instance is a standard $(2,3)$-instance, and (modulo some isomorphism trickery) it is an instance of CSP$(\mathbf{E}, 2)$ for some $\mathbf{E}$ with a 2-semilattice operation.
- Replace every potato and relation by the reduct to this operation; find a Bulatov solution.
- Check this Bulatov solution, as a subinstance of $\mathcal{I}$, for a solution using the algorithm from $\mathcal{W}$.

The Algorithm (roughly)

- Let $\mathbf{D}$ be as in the statement of the Theorem, and fix $\mathcal{I}$, a standard $(2,3)$-instance of $\mathrm{CSP}(\mathbf{D},2)$.
- Construct a "quotient instance" by taking the quotient in each potato by its "special" congruence. The binary relations are constructed in the natural way.
- This quotient instance is a standard $(2,3)$-instance, and (modulo some isomorphism trickery) it is an instance of $\mathrm{CSP}(\mathbf{E},2)$ for some $\mathbf{E}$ with a 2-semilattice operation.
- Replace every potato and relation by the reduct to this operation; find a Bulatov solution.
- Check this Bulatov solution, as a subinstance of $\mathcal{I}$, for a solution using the algorithm from $\mathcal{W}$.
- The original instance has a solution if and only if this subinstance does.

Why the Algorithm Works

Fix $\mathcal{I}$, and let $\mathcal{I}/\theta$ be the "quotient instance" from the previous slide.

- Every solution, φ, to $\mathcal{I}$ naturally gives rise to a solution, Φ to $\mathcal{I}/\theta$. We say that φ passes through Φ in this situation.

Why the Algorithm Works

Fix $\mathcal{I}$, and let $\mathcal{I}/\theta$ be the "quotient instance" from the previous slide.

- Every solution, φ, to $\mathcal{I}$ naturally gives rise to a solution, Φ to $\mathcal{I}/\theta$. We say that φ passes through Φ in this situation.

Lemma

If φ passes through Φ, and Ψ is a solution to $\mathcal{I}/\theta$ so that $\Phi \longrightarrow \Psi$, then $\mathcal{I}$ has a solution which passes through Ψ.

- This Lemma with the Bulatov Solution Walk Lemma implies that if $\mathcal{I}$ has a solution, then it has a solution passing through **every** Bulatov solution.

Corollary

Corollary

Suppose $\mathcal{W}$ *and* $\mathcal{T}$ *are similar idempotent varieties. If* $\mathcal{W}$ *has an edge term and* $\mathcal{T}$ *is term equivalent to the variety of 2-semilattices, then* $\mathrm{CSP}(\mathbf{D}, 2)$ *has a polynomial time algorithm for every finite* $\mathbf{D} \in \mathcal{W} \vee \mathcal{T}$.

Corollary

Corollary

Suppose $\mathcal{W}$ and $\mathcal{T}$ are similar idempotent varieties. If $\mathcal{W}$ has an edge term and $\mathcal{T}$ is term equivalent to the variety of 2-semilattices, then $\mathrm{CSP}(\mathbf{D}, 2)$ has a polynomial time algorithm for every finite $\mathbf{D} \in \mathcal{W} \vee \mathcal{T}$.

- Let $\mathcal{W}^{(2)}$ be the variety axiomatized by all at most 2-variable identities which hold in $\mathcal{W}$.
- Since $\mathcal{W} \leq \mathcal{W}^{(2)}$ and $\mathcal{W}^{(2)}$ retains the edge term, it suffices to prove the Corollary for $\mathcal{W}^{(2)} \vee \mathcal{T}$.

"Proof" of Corollary

Fact

I know how to construct a binary term, $\cdot$, which is the first projection in $\mathcal{W}^{(2)}$, and a 2-semilattice operation in $\mathcal{T}$.

Fact

Suppose $\mathcal{A}$ and $\mathcal{B}$ are similar idempotent varieties. Further suppose there is a binary term, $*$, in their signature which is the first projection in $\mathcal{A}$ and commutative in $\mathcal{B}$. If $\mathcal{A}$ exhibits an axiomatization of at most two variable identities, then $\mathcal{A} \circ \mathcal{B}$ is a variety.

"Proof" of Corollary

Apply the Theorem.

- We already have $\cdot$ in the signature of $\mathcal{W}^{(2)}$ and $\mathcal{T}$ which has the desired properties with respect to those varieties.
- The Theorem, therefore, will apply to any (idempotent) $\mathbf{D}$ in $\mathcal{W}^{(2)} \circ \mathcal{T}$ satisfying $x \cdot (y \cdot z) \approx x \cdot (z \cdot y)$.
- The result then follows from compiling the following facts:
    - $\mathcal{U} := (\mathcal{W}^{(2)} \circ \mathcal{T}) \cap \big[x \cdot (y \cdot z) \approx x \cdot (z \cdot y)\big]$ is a variety
    - $\mathcal{W}^{(2)}, \mathcal{T} \leq \mathcal{W}^{(2)} \circ \mathcal{T}$,
    - $\mathcal{W}^{(2)}, \mathcal{T} \models x \cdot (y \cdot z) \approx x \cdot (z \cdot y)$

  $\Rightarrow$ $\mathcal{W}^{(2)} \vee \mathcal{T} \leq \mathcal{U}$.
    - $\mathrm{CSP}(\mathbf{D}, 2)$ has a polynomial time algorithm for every $\mathbf{D} \in \mathcal{U}$,

  $\Rightarrow$ $\mathrm{CSP}(\mathbf{D}, 2)$ has a polynomial time algorithm for every $\mathbf{D} \in \mathcal{W}^{(2)} \vee \mathcal{T}$.

Thank You!