Higher Commutator Theory for Congruence Modular Varieties

Andrew Moorhead

CU Boulder

May 21, 2016

Some History

Some History

Smith developed a general commutator theory for congruence permutable varieties (1976).

Some History

Smith developed a general commutator theory for congruence permutable varieties (1976).

Hagemann and Hermann extend this theory to congruence modular varieties (1979).

Some History

Smith developed a general commutator theory for congruence permutable varieties (1976).

Hagemann and Hermann extend this theory to congruence modular varieties (1979).

Gumm presented this material from a geometric perspective (1983).

Some History

Smith developed a general commutator theory for congruence permutable varieties (1976).

Hagemann and Hermann extend this theory to congruence modular varieties (1979).

Gumm presented this material from a geometric perspective (1983).

Freese and McKenzie write *Commutator theory for Congruence Modular Varieties*. In it they define the commutator with a term condition and prove its equivalence to other definitions (1987).

Some History

Smith developed a general commutator theory for congruence permutable varieties (1976).

Hagemann and Hermann extend this theory to congruence modular varieties (1979).

Gumm presented this material from a geometric perspective (1983).

Freese and McKenzie write *Commutator theory for Congruence Modular Varieties*. In it they define the commutator with a term condition and prove its equivalence to other definitions (1987).

Bulatov generalizes the term condition and defines higher commutators of arbitrary arity to distinguish polynomially inequivalent Mal'cev algebras (2000)

Some History

Smith developed a general commutator theory for congruence permutable varieties (1976).

Hagemann and Hermann extend this theory to congruence modular varieties (1979).

Gumm presented this material from a geometric perspective (1983).

Freese and McKenzie write *Commutator theory for Congruence Modular Varieties*. In it they define the commutator with a term condition and prove its equivalence to other definitions (1987).

Bulatov generalizes the term condition and defines higher commutators of arbitrary arity to distinguish polynomially inequivalent Mal'cev algebras (2000)

Aichinger and Mudrinski develop the basic properties of the higher commutator for congruence permutable varieties (2010)

Centralization

Definition

Centralization

Definition

Let $\mathbb{A}$ be an algebra, $k \in \mathbb{N}_{\geq 2}$, and choose $\alpha_0, \ldots, \alpha_{k-1}, \delta \in \mathrm{Con}(\mathbb{A})$.

Centralization

Definition

Let $\mathbb{A}$ be an algebra, $k \in \mathbb{N}_{\geq 2}$, and choose $\alpha_0, \ldots, \alpha_{k-1}, \delta \in \mathrm{Con}(\mathbb{A})$. We say that $\alpha_0, \ldots, \alpha_{k-2}$ **centralize** α_{k-1} **modulo** δ if for all $f \in \mathrm{Pol}(\mathbb{A})$ and tuples $\mathbf{a}_0, \mathbf{b}_0, \ldots, \mathbf{a}_{k-1}, \mathbf{b}_{k-1}$ from $\mathbb{A}$ such that

Centralization

Definition

Let $\mathbb{A}$ be an algebra, $k \in \mathbb{N}_{\geq 2}$, and choose $\alpha_0, \dots, \alpha_{k-1}, \delta \in \mathsf{Con}(\mathbb{A})$. We say that $\alpha_0, \dots, \alpha_{k-2}$ **centralize** α_{k-1} **modulo** δ if for all $f \in \mathsf{Pol}(\mathbb{A})$ and tuples $\mathbf{a}_0, \mathbf{b}_0, \dots, \mathbf{a}_{k-1}, \mathbf{b}_{k-1}$ from $\mathbb{A}$ such that

1. $\mathbf{a}_i \equiv_{\alpha_i} \mathbf{b}_i$ for each $0 \leq i \leq k-1$

Centralization

Definition

Let $\mathbb{A}$ be an algebra, $k \in \mathbb{N}_{\geq 2}$, and choose $\alpha_0, \ldots, \alpha_{k-1}, \delta \in \mathrm{Con}(\mathbb{A})$. We say that $\alpha_0, \ldots, \alpha_{k-2}$ **centralize** α_{k-1} **modulo** δ if for all $f \in \mathrm{Pol}(\mathbb{A})$ and tuples $\mathbf{a}_0, \mathbf{b}_0, \ldots, \mathbf{a}_{k-1}, \mathbf{b}_{k-1}$ from $\mathbb{A}$ such that

1. $\mathbf{a}_i \equiv_{\alpha_i} \mathbf{b}_i$ for each $0 \leq i \leq k-1$
2. If $f(\mathbf{z}_0, \ldots, \mathbf{z}_{k-2}, \mathbf{a}_{k-1}) \equiv_\delta f(\mathbf{z}_0, \ldots, \mathbf{z}_{k-2}, \mathbf{b}_{k-1})$ for all $(\mathbf{z}_0, \ldots, \mathbf{z}_{k-2}) \in \{\mathbf{a}_0, \mathbf{b}_0\} \times \cdots \times \{\mathbf{a}_{k-2}, \mathbf{b}_{k-2}\} \setminus \{(\mathbf{b}_0, \ldots, \mathbf{b}_{k-2})\}$

Centralization

Definition

Let $\mathbb{A}$ be an algebra, $k \in \mathbb{N}_{\geq 2}$, and choose $\alpha_0, \ldots, \alpha_{k-1}, \delta \in \mathrm{Con}(\mathbb{A})$. We say that $\alpha_0, \ldots, \alpha_{k-2}$ **centralize** α_{k-1} **modulo** δ if for all $f \in \mathrm{Pol}(\mathbb{A})$ and tuples $\mathbf{a}_0, \mathbf{b}_0, \ldots, \mathbf{a}_{k-1}, \mathbf{b}_{k-1}$ from $\mathbb{A}$ such that

1. $\mathbf{a}_i \equiv_{\alpha_i} \mathbf{b}_i$ for each $0 \leq i \leq k-1$
2. If $f(\mathbf{z}_0, \ldots, \mathbf{z}_{k-2}, \mathbf{a}_{k-1}) \equiv_\delta f(\mathbf{z}_0, \ldots, \mathbf{z}_{k-2}, \mathbf{b}_{k-1})$ for all $(\mathbf{z}_0, \ldots, \mathbf{z}_{k-2}) \in \{\mathbf{a}_0, \mathbf{b}_0\} \times \cdots \times \{\mathbf{a}_{k-2}, \mathbf{b}_{k-2}\} \setminus \{(\mathbf{b}_0, \ldots, \mathbf{b}_{k-2})\}$

we have that

$$f(\mathbf{b}_0, \ldots, \mathbf{b}_{k-2}, \mathbf{a}_{k-1}) \equiv_\delta f(\mathbf{b}_0, \ldots, \mathbf{b}_{k-2}, \mathbf{b}_{k-1})$$

Centralization

Definition

Let $\mathbb{A}$ be an algebra, $k \in \mathbb{N}_{\geq 2}$, and choose $\alpha_0, \ldots, \alpha_{k-1}, \delta \in \mathrm{Con}(\mathbb{A})$. We say that $\alpha_0, \ldots, \alpha_{k-2}$ **centralize** α_{k-1} **modulo** δ if for all $f \in \mathrm{Pol}(\mathbb{A})$ and tuples $\mathbf{a}_0, \mathbf{b}_0, \ldots, \mathbf{a}_{k-1}, \mathbf{b}_{k-1}$ from $\mathbb{A}$ such that

1. $\mathbf{a}_i \equiv_{\alpha_i} \mathbf{b}_i$ for each $0 \leq i \leq k-1$
2. If $f(\mathbf{z}_0, \ldots, \mathbf{z}_{k-2}, \mathbf{a}_{k-1}) \equiv_\delta f(\mathbf{z}_0, \ldots, \mathbf{z}_{k-2}, \mathbf{b}_{k-1})$ for all $(\mathbf{z}_0, \ldots, \mathbf{z}_{k-2}) \in \{\mathbf{a}_0, \mathbf{b}_0\} \times \cdots \times \{\mathbf{a}_{k-2}, \mathbf{b}_{k-2}\} \setminus \{(\mathbf{b}_0, \ldots, \mathbf{b}_{k-2})\}$

we have that

$$f(\mathbf{b}_0, \ldots, \mathbf{b}_{k-2}, \mathbf{a}_{k-1}) \equiv_\delta f(\mathbf{b}_0, \ldots, \mathbf{b}_{k-2}, \mathbf{b}_{k-1})$$

This condition is abbreviated as $C(\alpha_0, \ldots, \alpha_{k-1}; \delta)$.

Matrices

Matrices

Let's look at the binary case.

Matrices

Let's look at the binary case. Let $\mathcal{V}$ be a congruence modular variety, and take $\mathbb{A} \in \mathcal{V}$. For $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$ set

$$M(\alpha, \beta) = \left\{ \begin{bmatrix} t(\mathbf{a}_0, \mathbf{a}_1) & t(\mathbf{a}_0, \mathbf{b}_1) \\ t(\mathbf{b}_0, \mathbf{a}_1) & t(\mathbf{b}_0, \mathbf{b}_1) \end{bmatrix} : t \in \mathrm{Pol}(\mathbb{A}), \mathbf{a}_0 \equiv_\alpha \mathbf{b}_0, \mathbf{a}_1 \equiv_\beta \mathbf{b}_1 \right\}$$

Matrices

Let's look at the binary case. Let $\mathcal{V}$ be a congruence modular variety, and take $\mathbb{A} \in \mathcal{V}$. For $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$ set

$$M(\alpha, \beta) = \left\{ \begin{bmatrix} t(\mathbf{a}_0, \mathbf{a}_1) & t(\mathbf{a}_0, \mathbf{b}_1) \\ t(\mathbf{b}_0, \mathbf{a}_1) & t(\mathbf{b}_0, \mathbf{b}_1) \end{bmatrix} : t \in \mathrm{Pol}(\mathbb{A}), \mathbf{a}_0 \equiv_\alpha \mathbf{b}_0, \mathbf{a}_1 \equiv_\beta \mathbf{b}_1 \right\}$$

This is the algebra of (α, β)-matrices.

Matrices

Let's look at the binary case. Let $\mathcal{V}$ be a congruence modular variety, and take $\mathbb{A} \in \mathcal{V}$. For $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$ set

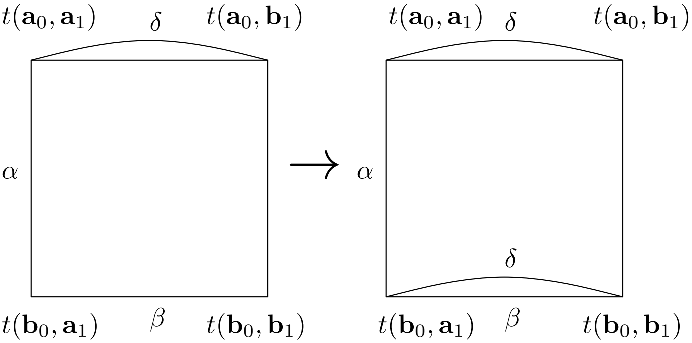
$$M(\alpha,\beta) = \left\{ \begin{bmatrix} t(\mathbf{a}_0, \mathbf{a}_1) & t(\mathbf{a}_0, \mathbf{b}_1) \\ t(\mathbf{b}_0, \mathbf{a}_1) & t(\mathbf{b}_0, \mathbf{b}_1) \end{bmatrix} : t \in \mathrm{Pol}(\mathbb{A}), \mathbf{a}_0 \equiv_\alpha \mathbf{b}_0, \mathbf{a}_1 \equiv_\beta \mathbf{b}_1 \right\}$$

This is the algebra of (α, β)-matrices.
$M(\alpha, \beta)$ is a subalgebra of $\mathbb{A}^4$ with generators

$$\left\{ \begin{bmatrix} x & x \\ y & y \end{bmatrix} : x \equiv_\alpha y \right\} \bigcup \left\{ \begin{bmatrix} x & y \\ x & y \end{bmatrix} : x \equiv_\beta y \right\}$$

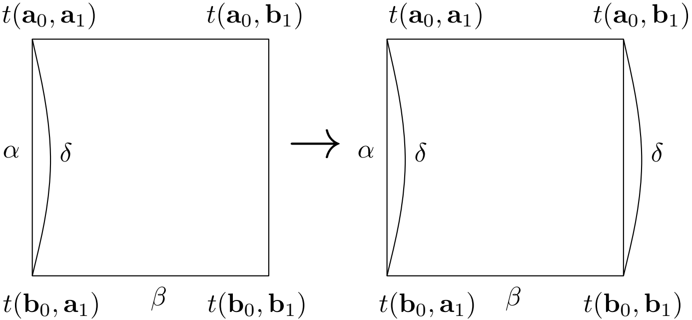
Matrices

For $\delta \in \mathrm{Con}(\mathbb{A})$ we have that α **centralizes** β **modulo** δ if the implication



holds for all (α, β)-matrices. This condition is abbreviated $C(\alpha, \beta; \delta)$.

Similarly, we have that β **centralizes** α **modulo** δ if the implication



holds for all (α, β)-matrices. This condition is abbreviated $C(\beta, \alpha; \delta)$.

Matrices

The binary commutator is defined to be

$$[\alpha, \beta] = \bigwedge\{\delta : C(\alpha, \beta; \delta)\}$$

Matrices

The binary commutator is defined to be

$$[\alpha, \beta] = \bigwedge\{\delta : C(\alpha, \beta; \delta)\}$$

We do the same thing for the higher commutator.

Matrices

The binary commutator is defined to be

$$[\alpha, \beta] = \bigwedge\{\delta : C(\alpha, \beta; \delta)\}$$

We do the same thing for the higher commutator. For congruences $\theta_0, \theta_1, \theta_2$ of $\mathbb{A}$ set $M(\theta_0, \theta_1, \theta_2)$ to be the collection of cubes

$$\begin{tikzcd}
t(\mathbf{a}_0,\mathbf{a}_1,\mathbf{a}_2) \arrow[rr, dash, "\theta_1"] \arrow[dr, dash] \arrow[dd, dash, "\theta_0"'] & & t(\mathbf{a}_0,\mathbf{b}_1,\mathbf{a}_2) \arrow[dr, dash] \arrow[dd, dash] & \\
& t(\mathbf{a}_0,\mathbf{a}_1,\mathbf{b}_2) \arrow[rr, dash] \arrow[dd, dash] & & t(\mathbf{a}_0,\mathbf{b}_1,\mathbf{b}_2) \arrow[dd, dash] \\
t(\mathbf{b}_0,\mathbf{a}_1,\mathbf{a}_2) \arrow[rr, dash] \arrow[dr, dash, "\theta_2"'] & & t(\mathbf{b}_0,\mathbf{b}_1,\mathbf{a}_2) \arrow[dr, dash] & \\
& t(\mathbf{b}_0,\mathbf{a}_1,\mathbf{b}_2) \arrow[rr, dash] & & t(\mathbf{b}_0,\mathbf{b}_1,\mathbf{b}_2)
\end{tikzcd}$$

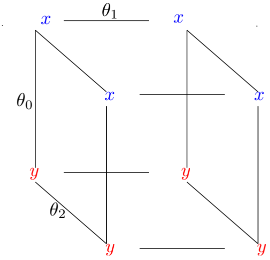
for $t \in \mathrm{Pol}(\mathbb{A})$

Matrices

$M(\theta_0, \theta_1, \theta_2)$ is the subalgebra of $\mathbb{A}^8$ generated by cubes of the form

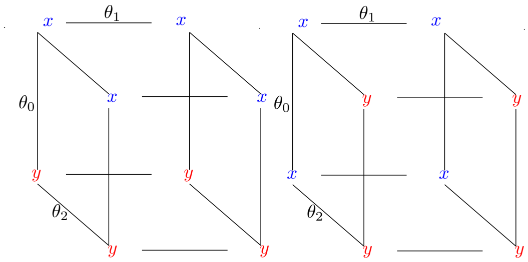
Matrices

$M(\theta_0, \theta_1, \theta_2)$ is the subalgebra of $\mathbb{A}^8$ generated by cubes of the form



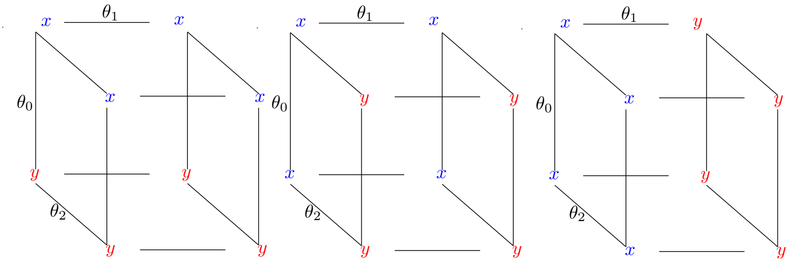
Matrices

$M(\theta_0, \theta_1, \theta_2)$ is the subalgebra of $\mathbb{A}^8$ generated by cubes of the form



Matrices

$M(\theta_0, \theta_1, \theta_2)$ is the subalgebra of $\mathbb{A}^8$ generated by cubes of the form

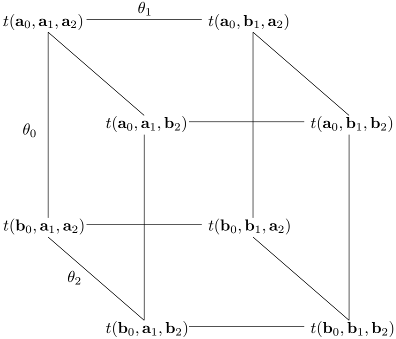


Matrices

For $\delta \in \mathrm{Con}(\mathbb{A})$, we say that θ_0, θ_1 **centralize** θ_2 **modulo** δ if the following implication holds for all $(\theta_0, \theta_1, \theta_2)$-matrices:

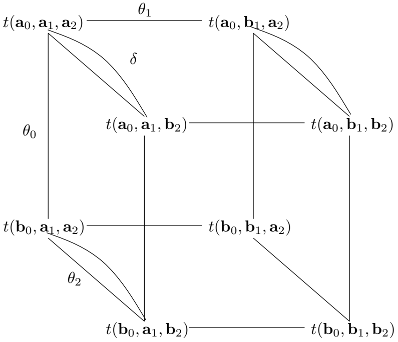
Matrices

For $\delta \in \mathrm{Con}(\mathbb{A})$, we say that θ_0, θ_1 **centralize** θ_2 **modulo** δ if the following implication holds for all $(\theta_0, \theta_1, \theta_2)$-matrices:



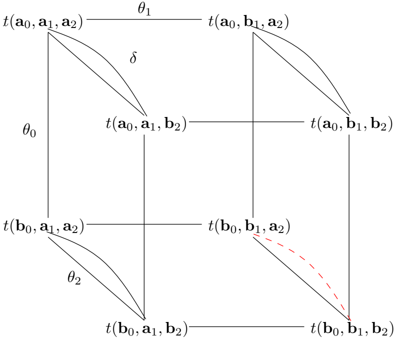
Matrices

For $\delta \in \mathrm{Con}(\mathbb{A})$, we say that θ_0, θ_1 **centralize** θ_2 **modulo** δ if the following implication holds for all $(\theta_0, \theta_1, \theta_2)$-matrices:



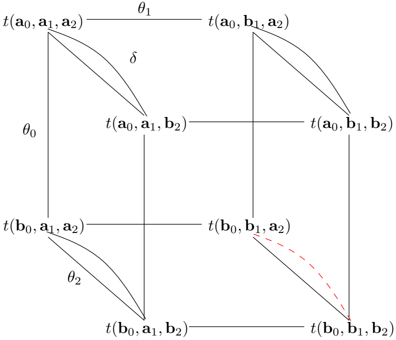
Matrices

For $\delta \in \mathrm{Con}(\mathbb{A})$, we say that θ_0, θ_1 **centralize** θ_2 **modulo** δ if the following implication holds for all $(\theta_0, \theta_1, \theta_2)$-matrices:



Matrices

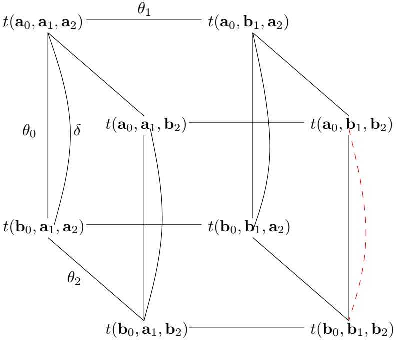
For $\delta \in \mathrm{Con}(\mathbb{A})$, we say that θ_0, θ_1 **centralize** θ_2 **modulo** δ if the following implication holds for all $(\theta_0, \theta_1, \theta_2)$-matrices:



This condition is abbreviated $C(\theta_0, \theta_1, \theta_2; \delta)$.

Matrices

Here is a picture of $C(\theta_1, \theta_2, \theta_0; \delta)$:



Matrices

For congruences $\theta_0, \theta_1, \theta_2$ we set

$$[\theta_0, \theta_1, \theta_2] = \bigwedge \{\delta : C(\theta_0, \theta_1, \theta_2; \delta)\}$$

Matrices

For congruences $\theta_0, \theta_1, \theta_2$ we set

$$[\theta_0, \theta_1, \theta_2] = \bigwedge\{\delta : C(\theta_0, \theta_1, \theta_2; \delta)\}$$

For a sequence of congruences $(\theta_0, \ldots, \theta_{k-1})$ we analogously define $M(\theta_0, \ldots, \theta_{k-1})$. The condition $C(\theta_0, \ldots, \theta_{k-1}, \delta)$ can be defined in terms of these matrices.

Definition of Commutator

Definition

Let $\mathbb{A}$ be an algebra, and let $\alpha_0, ..., \alpha_{k-1} \in \text{Con}(\mathbb{A})$ for $k \geq 2$. The k**-ary commutator of** $\alpha_1, ..., \alpha_k$ is defined to be

$$[\alpha_0, ..., \alpha_{k-1}] = \bigwedge \{\delta : C(\alpha_0, ..., \alpha_{k-1}; \delta)\}$$

Properties

The following properties are consequences of the definition:

Properties

The following properties are consequences of the definition:

(1) $[\alpha_0, \ldots, \alpha_{k-1}] \leq \bigwedge_{0 \leq i \leq k-1} \alpha_i$

Properties

The following properties are consequences of the definition:

(1) $[\alpha_0, ..., \alpha_{k-1}] \leq \bigwedge_{0 \leq i \leq k-1} \alpha_i$

(2) For $\alpha_0 \leq \beta_0, ..., \alpha_{k-1} \leq \beta_{k-1}$ in $\mathrm{Con}(\mathbb{A})$, we have $[\alpha_0, ..., \alpha_{k-1}] \leq [\beta_0, ..., \beta_{k-1}]$ (Monotonicity)

Properties

The following properties are consequences of the definition:

(1) $[\alpha_0, \ldots, \alpha_{k-1}] \leq \bigwedge_{0 \leq i \leq k-1} \alpha_i$

(2) For $\alpha_0 \leq \beta_0, \ldots, \alpha_{k-1} \leq \beta_{k-1}$ in $\mathrm{Con}(\mathbb{A})$, we have $[\alpha_0, \ldots, \alpha_{k-1}] \leq [\beta_0, \ldots, \beta_{k-1}]$ (Monotonicity)

(3) $[\alpha_0, \ldots, \alpha_{k-1}] \leq [\alpha_1, \ldots, \alpha_{k-1}]$

Properties Continued

Properties Continued

The following additional properties hold for the higher commutator in a congruence modular variety $\mathcal{V}$, which are developed for the binary commutator in Freese-McKenzie.

Properties Continued

The following additional properties hold for the higher commutator in a congruence modular variety $\mathcal{V}$, which are developed for the binary commutator in Freese-McKenzie.

(4) $[\alpha_0, ..., \alpha_{k-1}] = [\alpha_{\sigma(0)}, ..., \alpha_{\sigma(k-1)}]$ for any permutation of σ of the congruences $\alpha_0, ..., \alpha_{k-1}$ (Symmetry)

Properties Continued

The following additional properties hold for the higher commutator in a congruence modular variety $\mathcal{V}$, which are developed for the binary commutator in Freese-McKenzie.

(4) $[\alpha_0, ..., \alpha_{k-1}] = [\alpha_{\sigma(0)}, ..., \alpha_{\sigma(k-1)}]$ for any permutation of σ of the congruences $\alpha_0, ..., \alpha_{k-1}$ (Symmetry)

(5) $[\bigvee_{i \in I} \gamma_i, \alpha_1, ..., \alpha_{k-1}] = \bigvee_{i \in I} [\gamma_i, \alpha_1, ..., \alpha_{k-1}]$ (Additivity)

Properties Continued

The following additional properties hold for the higher commutator in a congruence modular variety $\mathcal{V}$, which are developed for the binary commutator in Freese-McKenzie.

(4) $[\alpha_0, ..., \alpha_{k-1}] = [\alpha_{\sigma(0)}, ..., \alpha_{\sigma(k-1)}]$ for any permutation of σ of the congruences $\alpha_0, ..., \alpha_{k-1}$ (Symmetry)

(5) $[\bigvee_{i \in I} \gamma_i, \alpha_1, ..., \alpha_{k-1}] = \bigvee_{i \in I} [\gamma_i, \alpha_1, ..., \alpha_{k-1}]$ (Additivity)

(6) $[\alpha_0, ..., \alpha_{k-1}] \vee \pi = f^{-1}([f(\alpha_0 \vee \pi), ..., f(\alpha_{k-1} \vee \pi))])$, where $f : \mathbb{A} \to \mathbb{B}$ is a surjective homomorphism with kernel π. (Homomorphism property)

Properties Continued

The following additional properties hold for the higher commutator in a congruence modular variety $\mathcal{V}$, which are developed for the binary commutator in Freese-McKenzie.

(4) $[\alpha_0, ..., \alpha_{k-1}] = [\alpha_{\sigma(0)}, ..., \alpha_{\sigma(k-1)}]$ for any permutation of σ of the congruences $\alpha_0, ..., \alpha_{k-1}$ (Symmetry)

(5) $[\bigvee_{i \in I} \gamma_i, \alpha_1, ..., \alpha_{k-1}] = \bigvee_{i \in I}[\gamma_i, \alpha_1, ..., \alpha_{k-1}]$ (Additivity)

(6) $[\alpha_0, ..., \alpha_{k-1}] \vee \pi = f^{-1}([f(\alpha_0 \vee \pi), ..., f(\alpha_{k-1} \vee \pi))])$, where $f : \mathbb{A} \to \mathbb{B}$ is a surjective homomorphism with kernel π. (Homomorphism property)

(7) If $C^{\mathcal{V}} : \mathrm{Con}(\mathbb{A})^k \to \mathrm{Con}(\mathbb{A})$ is defined for all $\mathbb{A} \in \mathcal{V}$ such that (1) and (6) hold, then $C^{\mathcal{V}}(\alpha_0, \ldots, \alpha_{k-1}) \leq [\alpha_0, \ldots, \alpha_{k-1}]$ for all $\mathbb{A} \in \mathcal{V}$ and $\alpha_0, \ldots \alpha_{k-1} \in \mathrm{Con}(\mathbb{A})$. (Greatest global operation)

Properties Continued

The following also hold:

Properties Continued

The following also hold:

(8) $[[\alpha_0, \dots, \alpha_{j-1}], \alpha_j, \dots, \alpha_{k-1}] \leq [\alpha_0, \dots, \alpha_{k-1}]$

Properties Continued

The following also hold:

(8) $[[\alpha_0, \ldots, \alpha_{j-1}], \alpha_j, \ldots, \alpha_{k-1}] \le [\alpha_0, \ldots, \alpha_{k-1}]$
(Follows easily from (7))

Properties Continued

The following also hold:

(8) $[[\alpha_0, \ldots, \alpha_{j-1}], \alpha_j, \ldots, \alpha_{k-1}] \leq [\alpha_0, \ldots, \alpha_{k-1}]$
(Follows easily from (7))

(9) Kiss showed that for congruence modular varieties the binary commutator is equivalent to a binary commutator defined with a two-term condition. This is true for the higher commutator also.

Day Terms

Proposition

A variety $\mathcal{V}$ is congruence modular if and only if there exist term operations $m_e(x, y, z, u)$ for $e \in n + 1$ satisfying the following identities:

Day Terms

Proposition

A variety $\mathcal{V}$ is congruence modular if and only if there exist term operations $m_e(x, y, z, u)$ for $e \in n + 1$ satisfying the following identities:

1. $m_e(x, y, y, x) \approx x$ *for each* $0 \leq e \leq n$*,*

Day Terms

Proposition

A variety $\mathcal{V}$ is congruence modular if and only if there exist term operations $m_e(x, y, z, u)$ for $e \in n + 1$ satisfying the following identities:

1. $m_e(x, y, y, x) \approx x$ *for each* $0 \leq e \leq n$,
2. $m_0(x, y, z, u) \approx x$,

Day Terms

Proposition

A variety $\mathcal{V}$ is congruence modular if and only if there exist term operations $m_e(x, y, z, u)$ for $e \in n + 1$ satisfying the following identities:

1. $m_e(x, y, y, x) \approx x$ *for each* $0 \leq e \leq n$,
2. $m_0(x, y, z, u) \approx x$,
3. $m_n(x, y, z, u) \approx u$,

Day Terms

Proposition

A variety $\mathcal{V}$ is congruence modular if and only if there exist term operations $m_e(x, y, z, u)$ for $e \in n + 1$ satisfying the following identities:

1. $m_e(x, y, y, x) \approx x$ *for each* $0 \leq e \leq n$*,*
2. $m_0(x, y, z, u) \approx x$*,*
3. $m_n(x, y, z, u) \approx u$*,*
4. $m_e(x, x, u, u) \approx m_{e+1}(x, x, u, u)$ *for even* e*, and*

Day Terms

Proposition

A variety $\mathcal{V}$ is congruence modular if and only if there exist term operations $m_e(x, y, z, u)$ for $e \in n + 1$ satisfying the following identities:

1. $m_e(x, y, y, x) \approx x$ *for each* $0 \leq e \leq n$,
2. $m_0(x, y, z, u) \approx x$,
3. $m_n(x, y, z, u) \approx u$,
4. $m_e(x, x, u, u) \approx m_{e+1}(x, x, u, u)$ *for even* e, *and*
5. $m_e(x, y, y, u) \approx m_{e+1}(x, y, y, u)$ *for odd* e

Day Terms and Matrices

$$\boxed{\begin{array}{cc} r & s \\ \\ u & v \end{array}} \in M(\alpha, \beta)$$

Day Terms and Matrices

$$\boxed{\begin{matrix} r & & s \\ & & \\ u & & v \end{matrix}} \in M(\alpha, \beta)$$

Then also

$$\boxed{\begin{matrix} s & & s \\ & & \\ s & & s \end{matrix}} \qquad \in M(\alpha, \beta)$$

Day Terms and Matrices

$$\boxed{\begin{matrix} r & s \\ \\ u & v \end{matrix}} \in M(\alpha, \beta)$$

Then also

$$\boxed{\begin{matrix} s & s \\ \\ s & s \end{matrix}} \in M(\alpha, \beta)$$

Day Terms and Matrices

$$\boxed{\begin{matrix} r & s \\ & \\ u & v \end{matrix}} \in M(\alpha, \beta)$$

Then also

$$\boxed{\begin{matrix} s & s \\ & \\ s & s \end{matrix}}\ ,\ \boxed{\begin{matrix} u & v \\ & \\ r & s \end{matrix}} \qquad \in M(\alpha, \beta)$$

Day Terms and Matrices

$$\begin{array}{|cc|}\hline r & s \\ & \\ u & v \\ \hline\end{array} \in M(\alpha, \beta)$$

Then also

$$\begin{array}{|cc|}\hline s & s \\ & \\ s & s \\ \hline\end{array}\, , \begin{array}{|cc|}\hline u & v \\ & \\ r & s \\ \hline\end{array}\, , \begin{array}{|cc|}\hline u & v \\ & \\ u & v \\ \hline\end{array} \in M(\alpha, \beta)$$

Day Terms and Matrices

$$\boxed{\begin{matrix} r & s \\ u & v \end{matrix}} \in M(\alpha, \beta)$$

Then also

$$\boxed{\begin{matrix} s & s \\ s & s \end{matrix}}\,,\ \boxed{\begin{matrix} u & v \\ r & s \end{matrix}}\,,\ \boxed{\begin{matrix} u & v \\ u & v \end{matrix}}\,,\ \boxed{\begin{matrix} s & s \\ v & v \end{matrix}} \in M(\alpha, \beta)$$

Day Terms and Matrices

$$\boxed{\begin{matrix} r & s \\ & \\ u & v \end{matrix}} \in M(\alpha, \beta)$$

Then also

$$m_e\left(\boxed{\begin{matrix} s & s \\ & \\ s & s \end{matrix}}\ ,\ \boxed{\begin{matrix} u & v \\ & \\ r & s \end{matrix}}\ ,\ \boxed{\begin{matrix} u & v \\ & \\ u & v \end{matrix}}\ ,\ \boxed{\begin{matrix} s & s \\ & \\ v & v \end{matrix}} \right) \in M(\alpha, \beta)$$

Day Terms and Matrices

$$\boxed{\begin{matrix} r & s \\ u & v \end{matrix}} \in M(\alpha, \beta)$$

Then also

$$m_e\left(\boxed{\begin{matrix} s & s \\ s & s \end{matrix}}, \boxed{\begin{matrix} u & v \\ r & s \end{matrix}}, \boxed{\begin{matrix} u & v \\ u & v \end{matrix}}, \boxed{\begin{matrix} s & s \\ v & v \end{matrix}} \right) \in M(\alpha, \beta)$$

Therefore

$$\boxed{\begin{matrix} m_e(s,u,u,s) & m_e(s,v,v,s) \\ m_e(s,r,u,v) & m_e(s,s,v,v) \end{matrix}} \in M(\alpha, \beta)$$

Day Terms and Matrices

$$\left(\begin{array}{cc} r & s \\ \delta & \\ u & v \end{array}\right) \in M(\alpha, \beta)$$

Then also

$$m_e\left(\begin{bmatrix} s & s \\ s & s \end{bmatrix}, \begin{bmatrix} u & v \\ r & s \end{bmatrix}, \begin{bmatrix} u & v \\ u & v \end{bmatrix}, \begin{bmatrix} s & s \\ v & v \end{bmatrix} \right) \in M(\alpha, \beta)$$

Therefore

$$\begin{bmatrix} m_e(s,u,u,s) & m_e(s,v,v,s) \\ m_e(s,r,u,v) & m_e(s,s,v,v) \end{bmatrix} \in M(\alpha, \beta)$$

Day Terms and Matrices

$$\begin{pmatrix} r & s \\ \delta & \\ u & v \end{pmatrix} \in M(\alpha, \beta)$$

Then also

$$m_e\left(\begin{pmatrix} s & s \\ s & s \end{pmatrix}, \begin{pmatrix} u & v \\ r & s \end{pmatrix}, \begin{pmatrix} u & v \\ u & v \end{pmatrix}, \begin{pmatrix} s & s \\ v & v \end{pmatrix} \right) \in M(\alpha, \beta)$$

Therefore

$$\begin{bmatrix} m_e(s,u,u,s) & m_e(s,v,v,s) \\ m_e(s,r,u,v) & m_e(s,s,v,v) \end{bmatrix} \in M(\alpha, \beta)$$

Day Terms and Matrices

$$\begin{pmatrix} r & s \\ \delta & \\ u & v \end{pmatrix} \in M(\alpha, \beta)$$

Then also

$$m_e\left(\begin{pmatrix} s & s \\ s & s \end{pmatrix}, \begin{pmatrix} u & v \\ r & s \end{pmatrix}, \begin{pmatrix} u & v \\ u & v \end{pmatrix}, \begin{pmatrix} s & s \\ v & v \end{pmatrix} \right) \in M(\alpha, \beta)$$

Therefore

$$\begin{pmatrix} m_e(s,u,u,s) & m_e(s,v,v,s) \\ m_e(s,r,u,v) & m_e(s,s,v,v) \end{pmatrix} \in M(\alpha, \beta)$$

Day Terms and Matrices

$$\left(\begin{array}{cc} r & s \\ \delta & \\ u & v \end{array}\right) \in M(\alpha, \beta)$$

Then also

$$m_e\left(\begin{pmatrix} s & s \\ s & s \end{pmatrix}, \begin{pmatrix} u & v \\ r & s \end{pmatrix}, \begin{pmatrix} u & v \\ u & v \end{pmatrix}, \begin{pmatrix} s & s \\ v & v \end{pmatrix} \right) \in M(\alpha, \beta)$$

$s = m_e(s, u, u, s)$ Apply identity (1) $m_e(s, v, v, s) = s$

$$\begin{pmatrix} & \\ m_e(s, r, u, v) & m_e(s, s, v, v) \end{pmatrix} \in M(\alpha, \beta)$$

Day Terms and Matrices

$$\begin{bmatrix} r & & s \\ \delta & & \\ u & & v \end{bmatrix} \in M(\alpha, \beta)$$

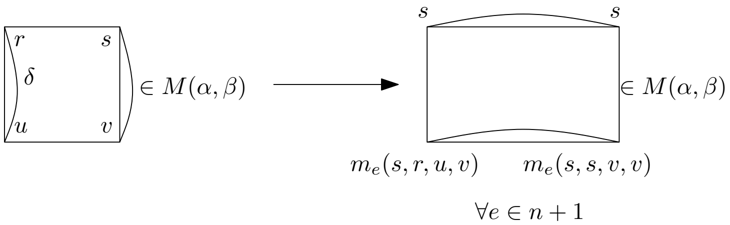
Then also

$$m_e\left(\begin{bmatrix} s & s \\ s & s \end{bmatrix}, \begin{bmatrix} u & v \\ r & s \end{bmatrix}, \begin{bmatrix} u & v \\ u & v \end{bmatrix}, \begin{bmatrix} s & s \\ v & v \end{bmatrix} \right) \in M(\alpha, \beta)$$

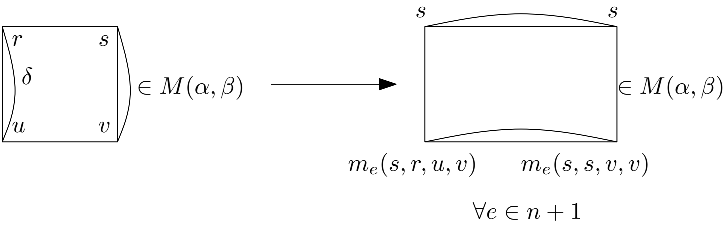
$s = m_e(s, u, u, s)$ Apply identity (1) $m_e(s, v, v, s) = s$

$$\begin{bmatrix} & \delta & \\ \delta & & \delta \\ & \delta & \\ m_e(s, r, u, v) & & m_e(s, s, v, v) \end{bmatrix} \in M(\alpha, \beta)$$

Day Terms and Matrices

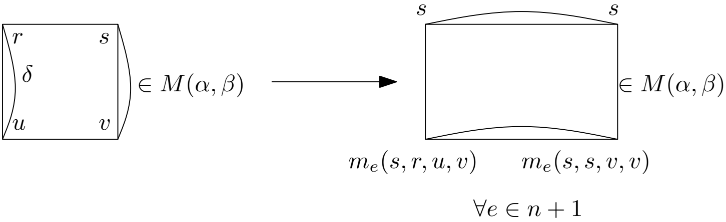


Day Terms and Matrices

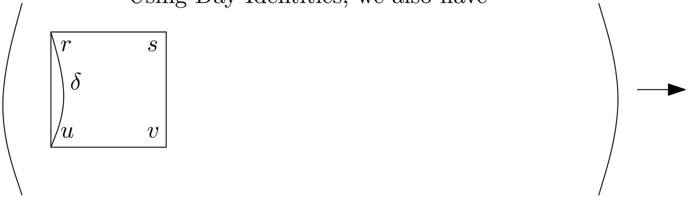


Using Day Identities, we also have

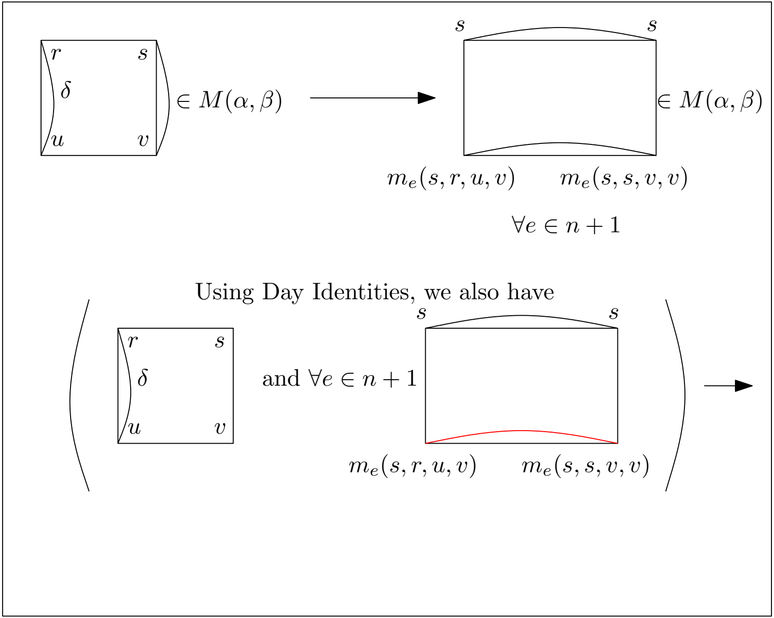
Day Terms and Matrices



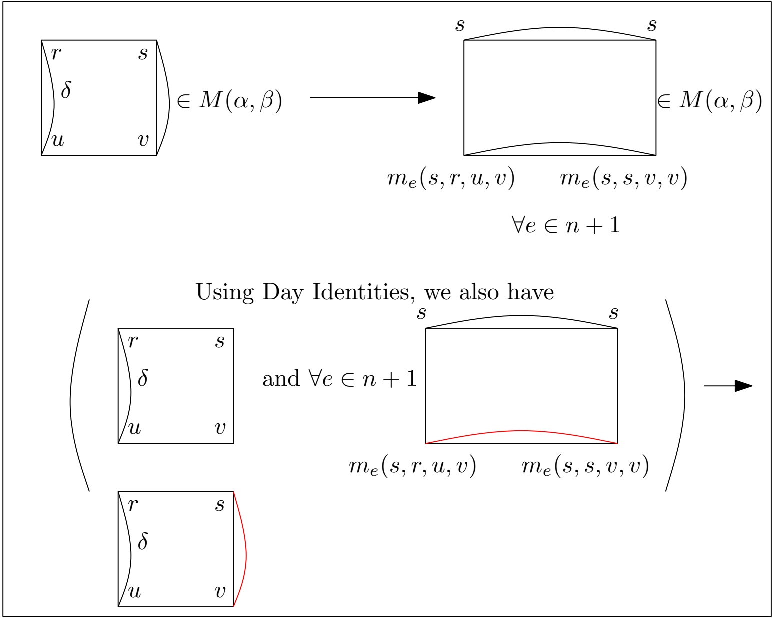
Using Day Identities, we also have



Day Terms and Matrices



Day Terms and Matrices

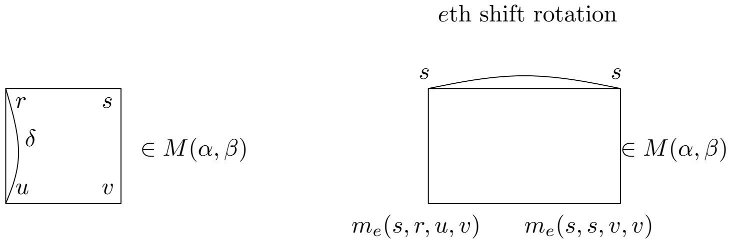


Shift Rotations

To summarize:

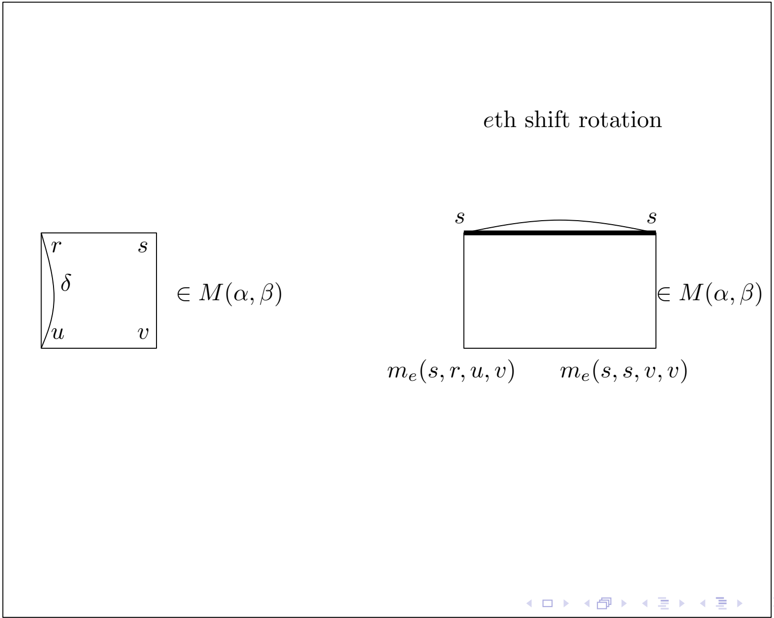
Shift Rotations

To summarize:



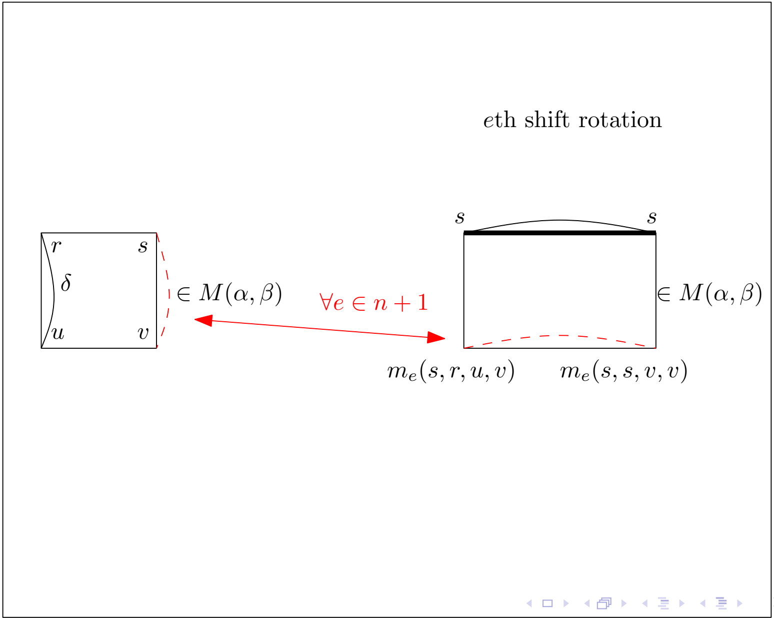
Shift Rotations

To summarize:



Shift Rotations

To summarize:

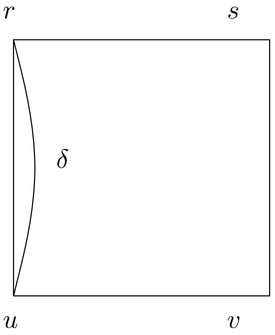


Symmetry and Generators for Binary Commutator

Suppose that $C(\alpha, \beta; \delta)$ holds. We want to show that $C(\beta, \alpha; \delta)$ holds also.

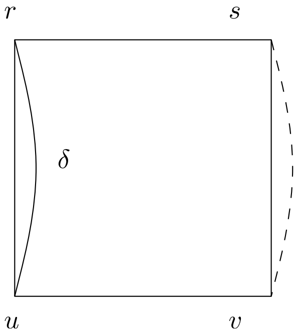
Symmetry and Generators for Binary Commutator

Suppose that $C(\alpha, \beta; \delta)$ holds. We want to show that $C(\beta, \alpha; \delta)$ holds also. So take an (α, β)-matrix with one column a δ-pair:



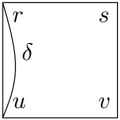
Symmetry and Generators for Binary Commutator

Suppose that $C(\alpha, \beta; \delta)$ holds. We want to show that $C(\beta, \alpha; \delta)$ holds also. So take an (α, β)-matrix with one column a δ-pair:



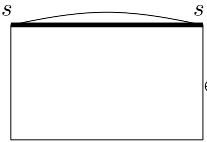
We need to show that the second column is a δ-pair, shown here with a dashed line.

Symmetry and Generators for Binary Commutator



$\in M(\alpha, \beta)$

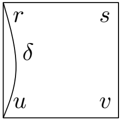
eth shift rotation



$\in M(\alpha, \beta)$

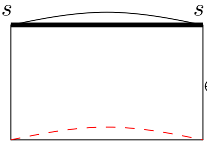
$m_e(s, r, u, v)$ $\quad$ $m_e(s, s, v, v)$

Symmetry and Generators for Binary Commutator



$\in M(\alpha, \beta)$

eth shift rotation

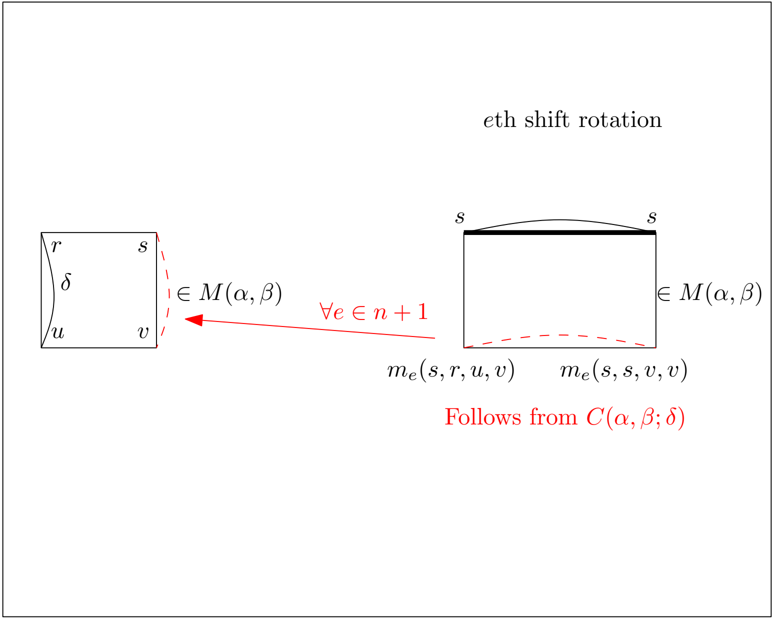


$\in M(\alpha, \beta)$

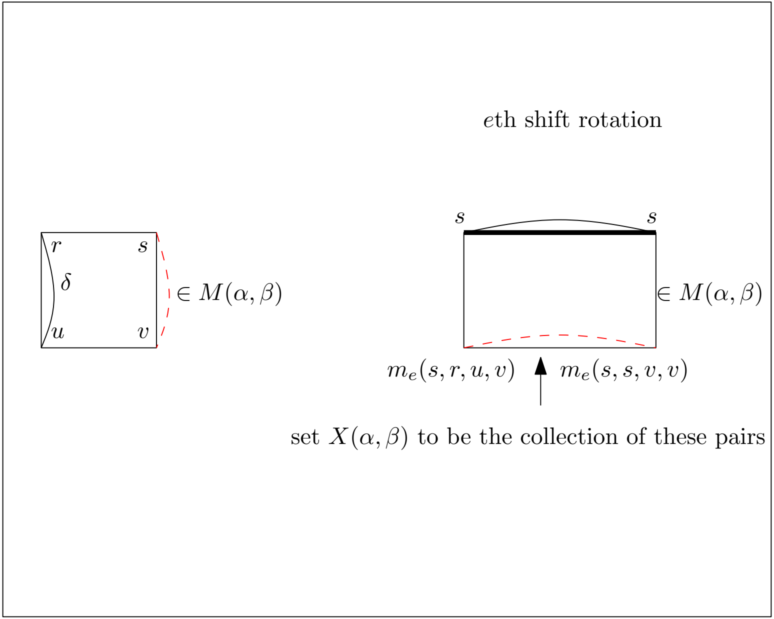
$m_e(s, r, u, v)$ $m_e(s, s, v, v)$

Follows from $C(\alpha, \beta; \delta)$

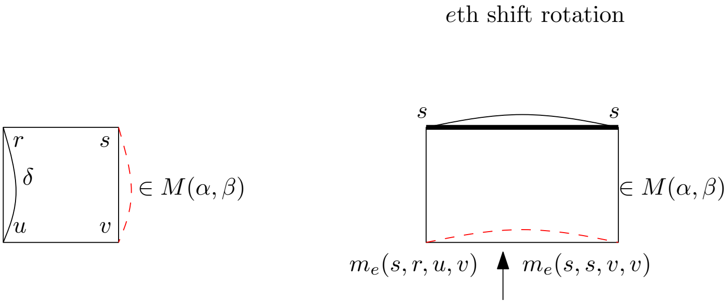
Symmetry and Generators for Binary Commutator



Symmetry and Generators for Binary Commutator



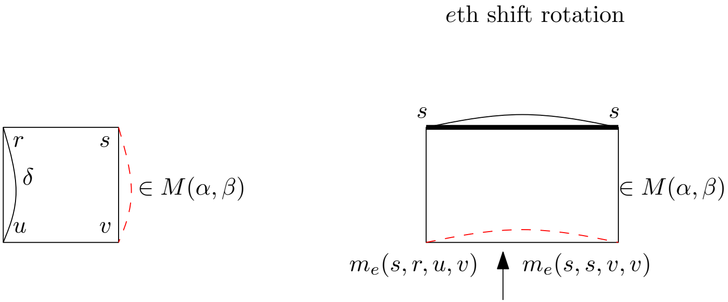
Symmetry and Generators for Binary Commutator



Set $X(\alpha, \beta)$ to be the collection of these pairs.

$X(\alpha, \beta) \subseteq \delta \iff C(\alpha, \beta; \delta) \iff C(\beta, \alpha; \delta)$

Symmetry and Generators for Binary Commutator

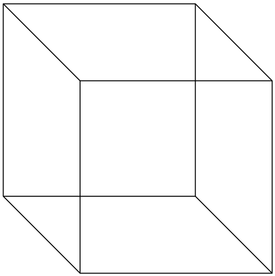


Set $X(\alpha,\beta)$ to be the collection of these pairs.

$X(\alpha,\beta) \subseteq \delta \iff C(\alpha,\beta;\delta) \iff C(\beta,\alpha;\delta)$

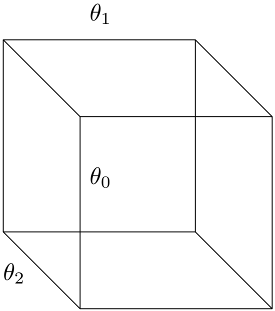
Therefore $[\alpha,\beta] = \mathrm{Cg}(X(\alpha,\beta))$

Symmetry of 3-ary commutator



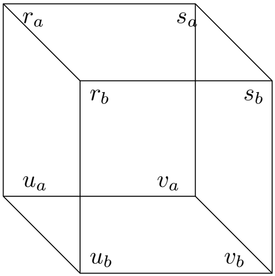
$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2)$

Symmetry of 3-ary commutator



$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2)$

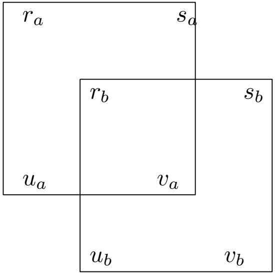
Symmetry of 3-ary commutator



$$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2)$$

Symmetry of 3-ary commutator

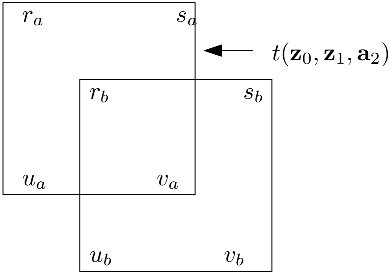
Decomposes into two (θ_0, θ_1)-matrices!



$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2)$

Symmetry of 3-ary commutator

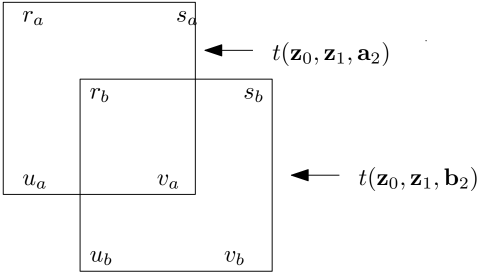
Decomposes into two (θ_0, θ_1)-matrices!



$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2)$

Symmetry of 3-ary commutator

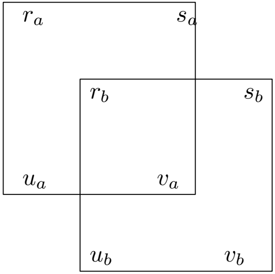
Decomposes into two (θ_0, θ_1)-matrices!



$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2)$

Symmetry of 3-ary commutator

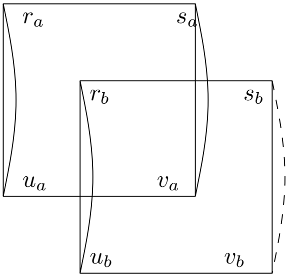
Suppose that $C(\theta_0, \theta_2, \theta_1; \delta)$ holds. Want to show $C(\theta_1, \theta_2, \theta_0; \delta)$ holds.



$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2)$

Symmetry of 3-ary commutator

Suppose that $C(\theta_0, \theta_2, \theta_1; \delta)$ holds. Want to show $C(\theta_1, \theta_2, \theta_0; \delta)$ holds.

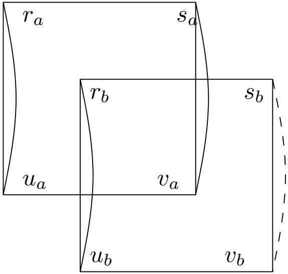


$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2)$

Symmetry of 3-ary commutator

Suppose that $C(\theta_0, \theta_2, \theta_1; \delta)$ holds. Want to show $C(\theta_1, \theta_2, \theta_0; \delta)$ holds.

Take $e \in n+1$. "Rotate" each square.

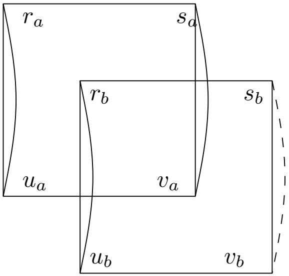


$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2)$

Symmetry of 3-ary commutator

Suppose that $C(\theta_0, \theta_2, \theta_1; \delta)$ holds. Want to show $C(\theta_1, \theta_2, \theta_0; \delta)$ holds.

Take $e \in n+1$. "Rotate" each square.



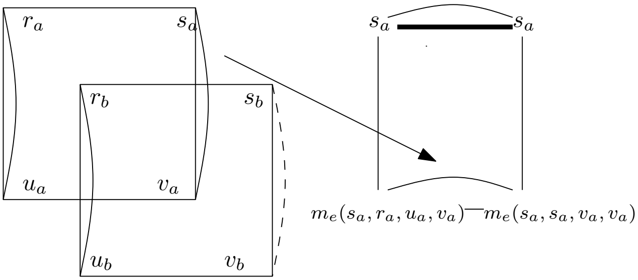
$$
\begin{tikzcd}
s_a \arrow[r, dash] \arrow[d, dash] & s_a \arrow[d, dash] \\
m_e(s_a, r_a, u_a, v_a) \arrow[r, dash] & m_e(s_a, s_a, v_a, v_a)
\end{tikzcd}
$$

$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2)$

Symmetry of 3-ary commutator

Suppose that $C(\theta_0, \theta_2, \theta_1; \delta)$ holds. Want to show $C(\theta_1, \theta_2, \theta_0; \delta)$ holds.

Take $e \in n+1$. "Rotate" each square.

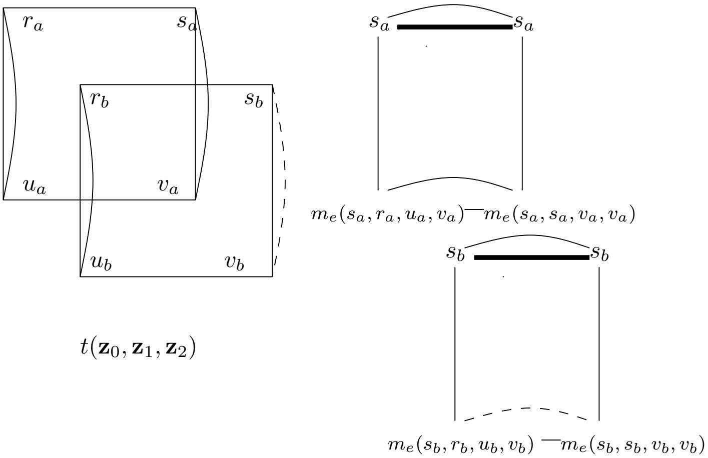


$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2)$

Symmetry of 3-ary commutator

Suppose that $C(\theta_0, \theta_2, \theta_1; \delta)$ holds. Want to show $C(\theta_1, \theta_2, \theta_0; \delta)$ holds.

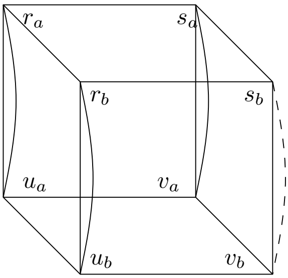
Take $e \in n+1$. "Rotate" each square.



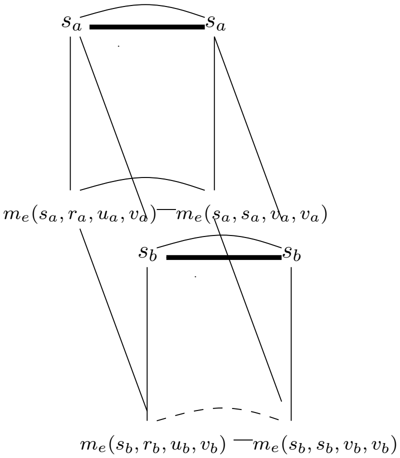
Symmetry of 3-ary commutator

Suppose that $C(\theta_0, \theta_2, \theta_1; \delta)$ holds. Want to show $C(\theta_1, \theta_2, \theta_0; \delta)$ holds.

Take $e \in n+1$. "Rotate" each square.



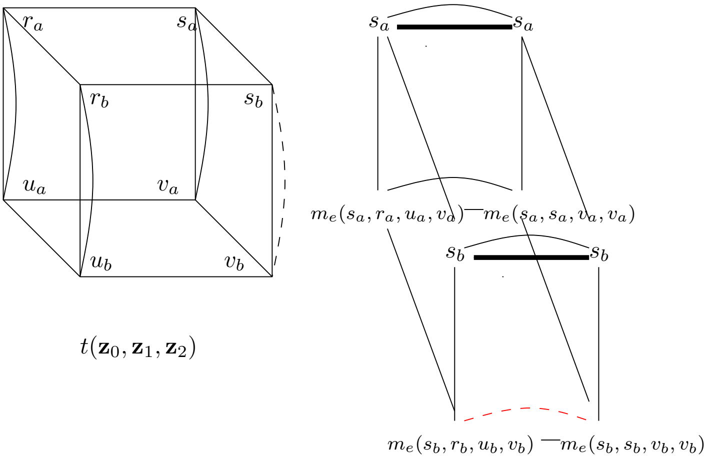
$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2)$



Symmetry of 3-ary commutator

Suppose that $C(\theta_0, \theta_2, \theta_1; \delta)$ holds. Want to show $C(\theta_1, \theta_2, \theta_0; \delta)$ holds.

Take $e \in n+1$. "Rotate" each square.

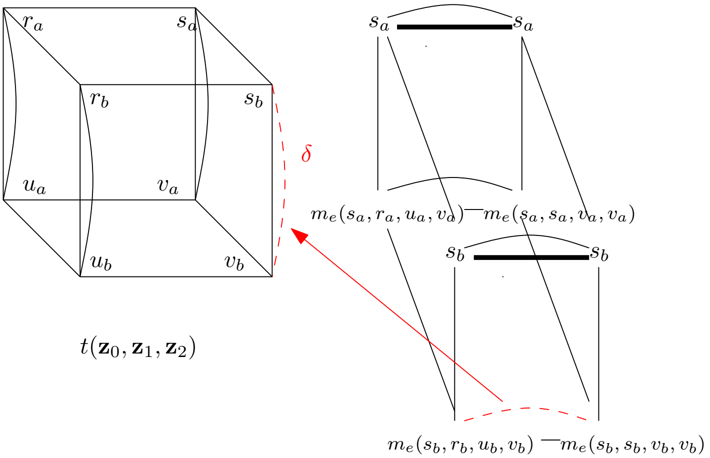


For all $e \in n+1$

Symmetry of 3-ary commutator

Suppose that $C(\theta_0, \theta_2, \theta_1; \delta)$ holds. Want to show $C(\theta_1, \theta_2, \theta_0; \delta)$ holds.

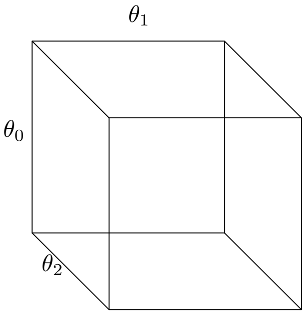
Take $e \in n+1$. "Rotate" each square.



For all $e \in n+1$

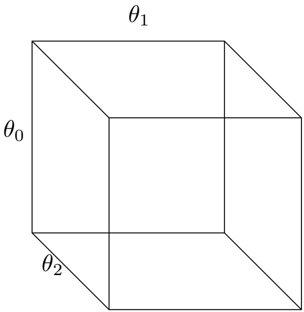
Shift Rotations for Three Dimensions

$(\theta_0, \theta_1, \theta_2)$-matrix

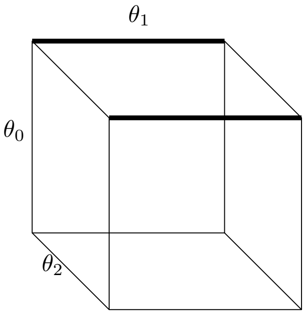


Shift Rotations for Three Dimensions

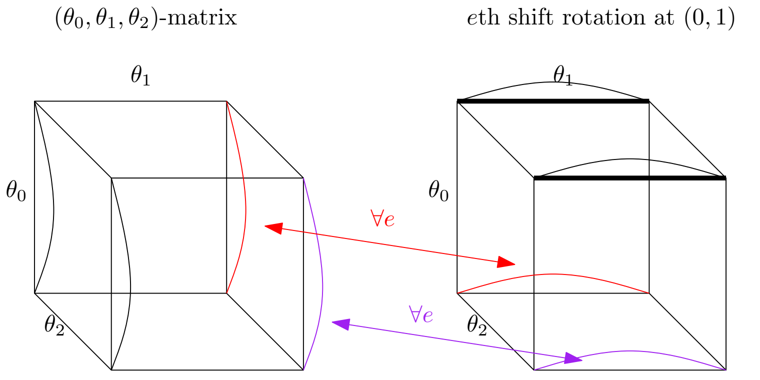
$(\theta_0, \theta_1, \theta_2)$-matrix



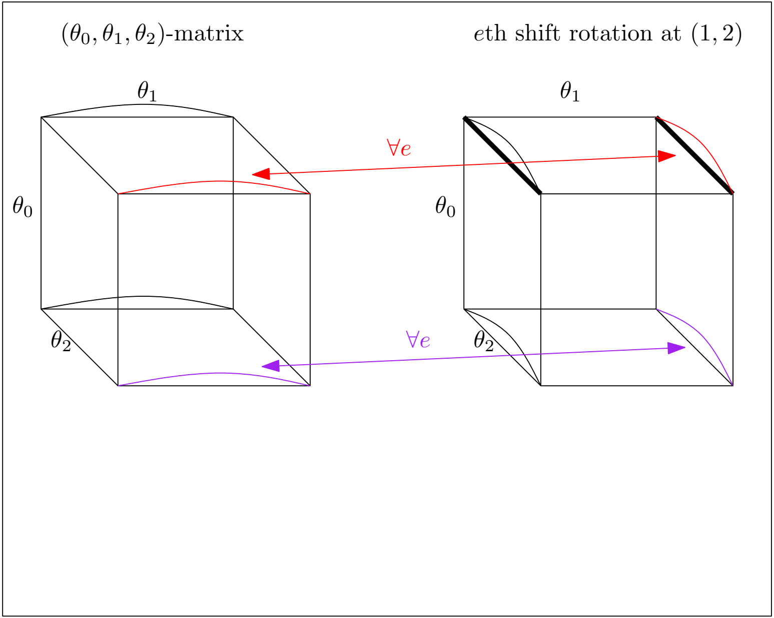
eth shift rotation at $(0, 1)$



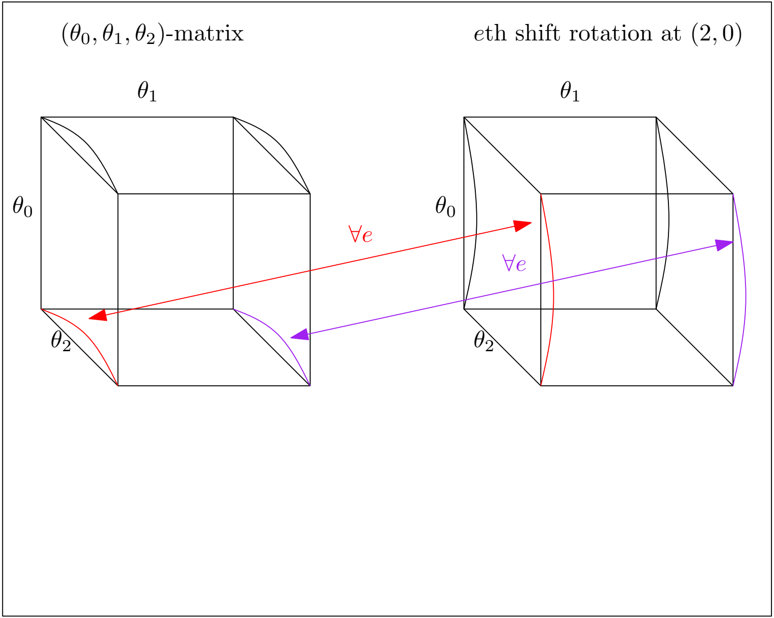
Shift Rotations for Three Dimensions



Shift Rotations for Three Dimensions

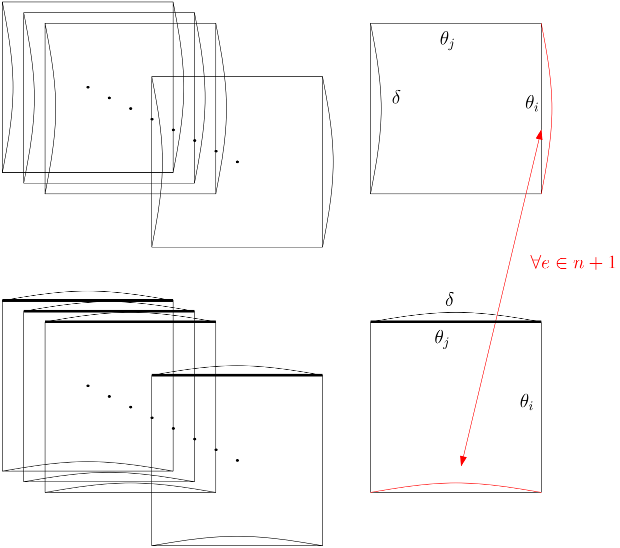


Shift Rotations for Three Dimensions



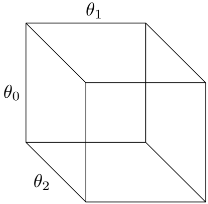
Symmetry of Higher Commutator

For any $i, j \in k$ a matrix decomposes into cross-section squares

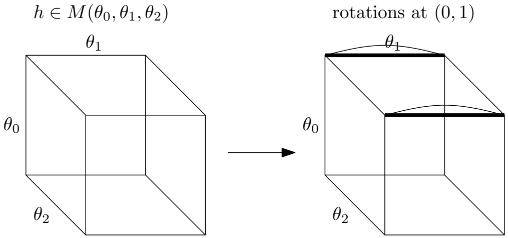


Generators for 3-ary commutator

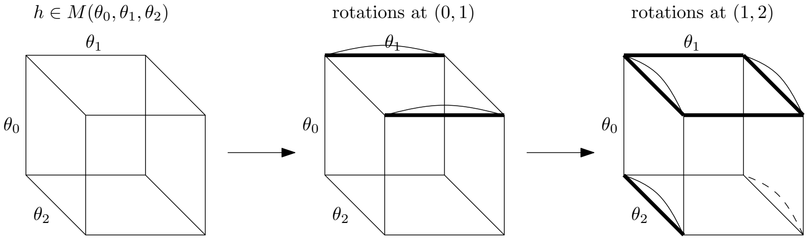
$h \in M(\theta_0, \theta_1, \theta_2)$



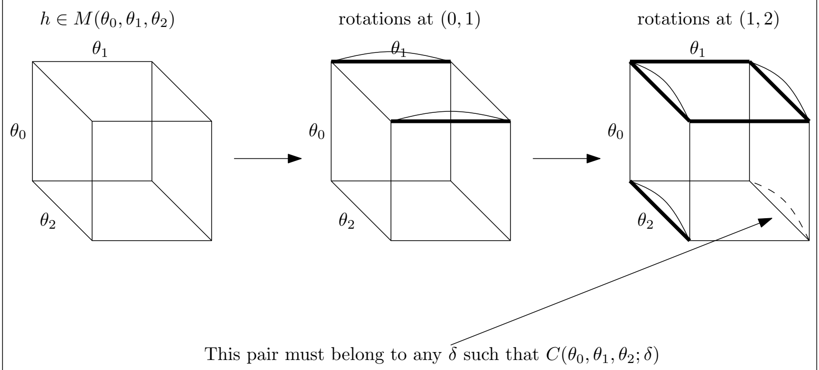
Generators for 3-ary commutator



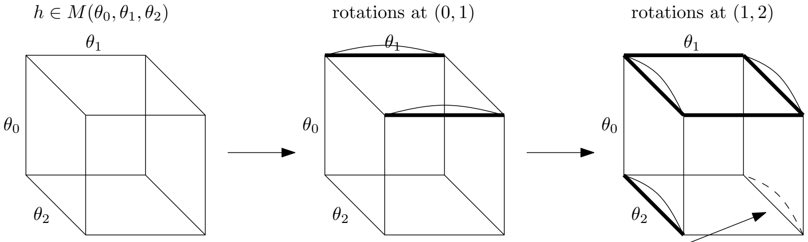
Generators for 3-ary commutator



Generators for 3-ary commutator



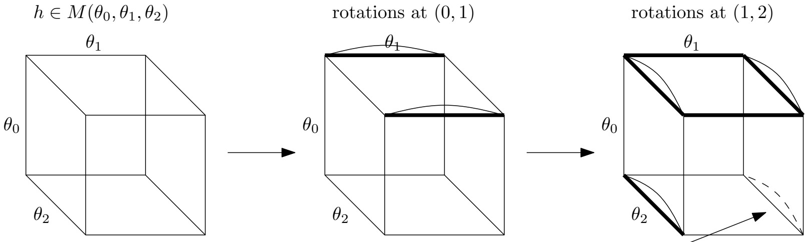
Generators for 3-ary commutator



This pair must belong to any δ such that $C(\theta_0, \theta_1, \theta_2; \delta)$

Set $X(\theta_0, \theta_1, \theta_2)$ to be the collection of all such pairs.

Generators for 3-ary commutator



This pair must belong to any δ such that $C(\theta_0, \theta_1, \theta_2; \delta)$

Set $X(\theta_0, \theta_1, \theta_2)$ to be the collection of all such pairs.

We will show that if $X(\theta_0, \theta_1, \theta_2) \subseteq \delta$ then $C(\theta_0, \theta_1, \theta_2; \delta)$ holds.

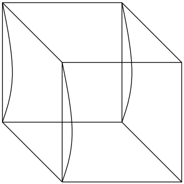
Generators for 3-ary commutator

Suppose $X(\theta_0, \theta_1, \theta_2) \subset \delta$

Generators for 3-ary commutator

Suppose $X(\theta_0, \theta_1, \theta_2) \subset \delta$

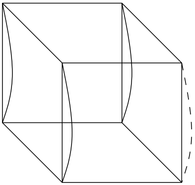
Take a $(\theta_0, \theta_1, \theta_2)$-matrix



Generators for 3-ary commutator

Suppose $X(\theta_0, \theta_1, \theta_2) \subset \delta$

Take a $(\theta_0, \theta_1, \theta_2)$-matrix

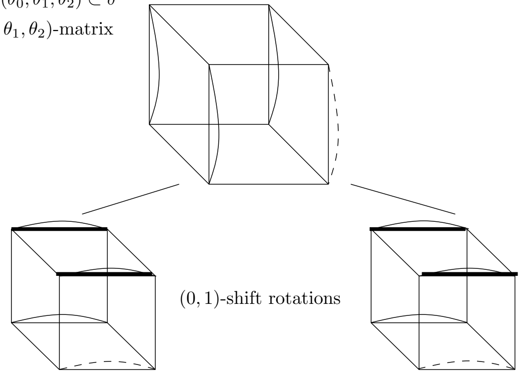


Want this pair in δ

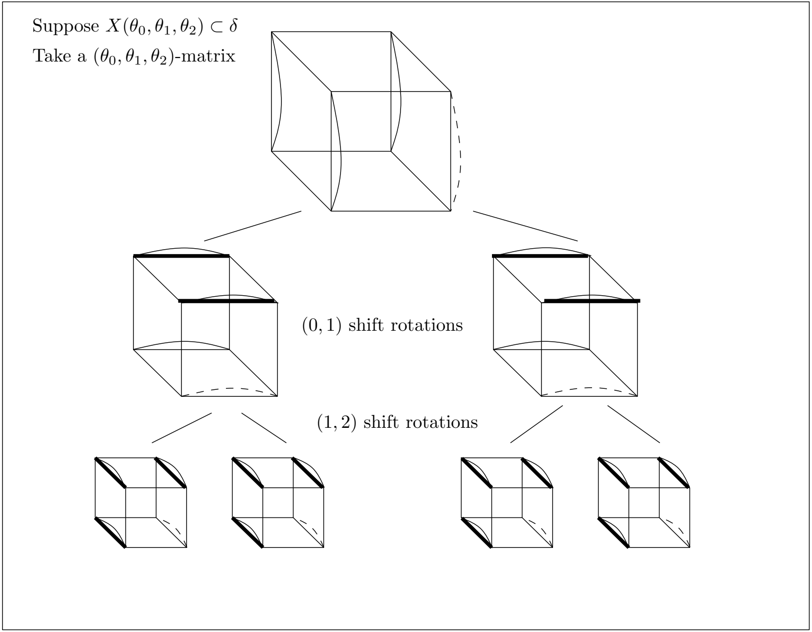
Generators for 3-ary commutator

Suppose $X(\theta_0, \theta_1, \theta_2) \subset \delta$

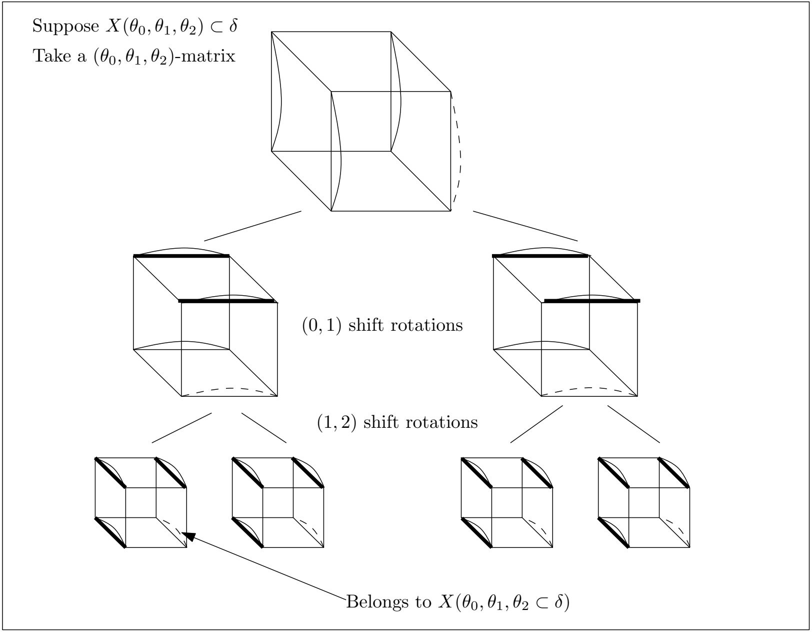
Take a $(\theta_0, \theta_1, \theta_2)$-matrix



Generators for 3-ary commutator



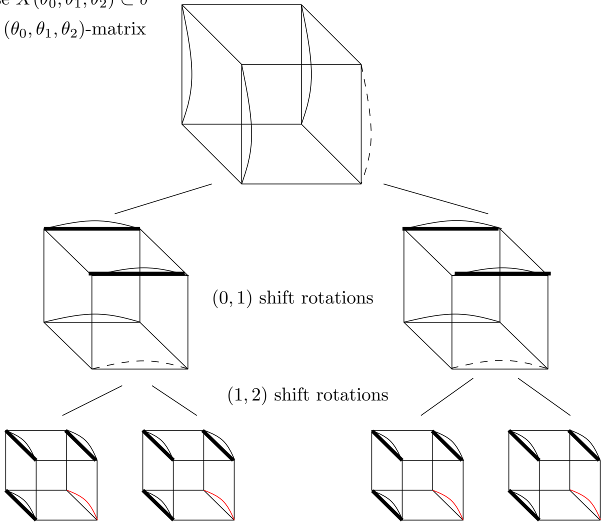
Generators for 3-ary commutator



Generators for 3-ary commutator

Suppose $X(\theta_0, \theta_1, \theta_2) \subset \delta$

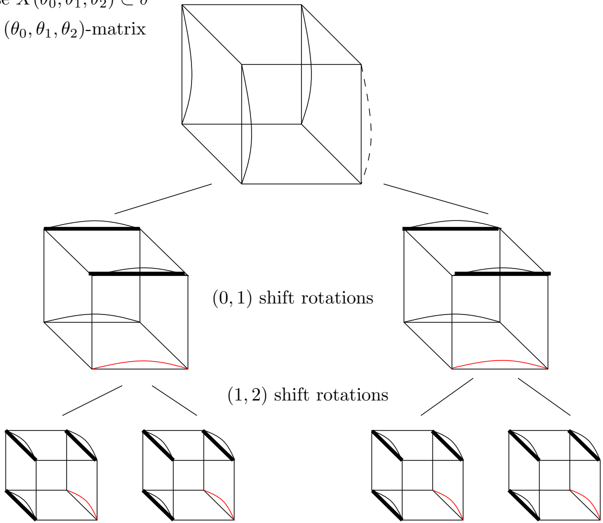
Take a $(\theta_0, \theta_1, \theta_2)$-matrix



Generators for 3-ary commutator

Suppose $X(\theta_0, \theta_1, \theta_2) \subset \delta$

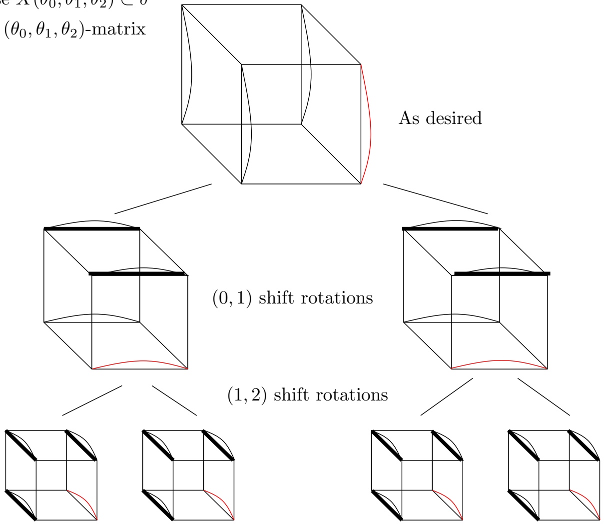
Take a $(\theta_0, \theta_1, \theta_2)$-matrix



Generators for 3-ary commutator

Suppose $X(\theta_0, \theta_1, \theta_2) \subset \delta$

Take a $(\theta_0, \theta_1, \theta_2)$-matrix

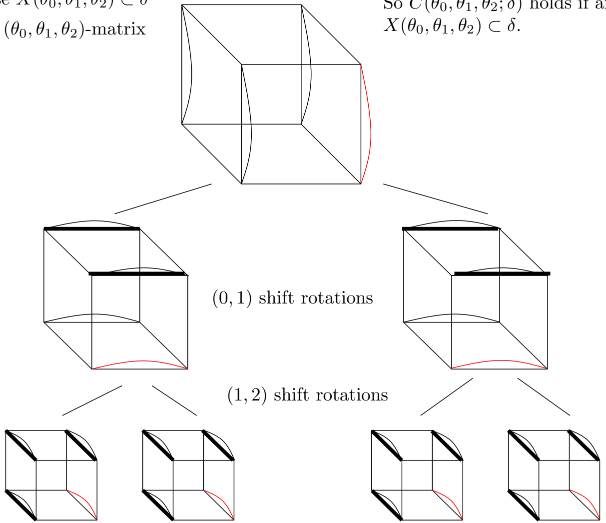


Generators for 3-ary commutator

Suppose $X(\theta_0, \theta_1, \theta_2) \subset \delta$

Take a $(\theta_0, \theta_1, \theta_2)$-matrix

So $C(\theta_0, \theta_1, \theta_2; \delta)$ holds if and only if $X(\theta_0, \theta_1, \theta_2) \subset \delta$.



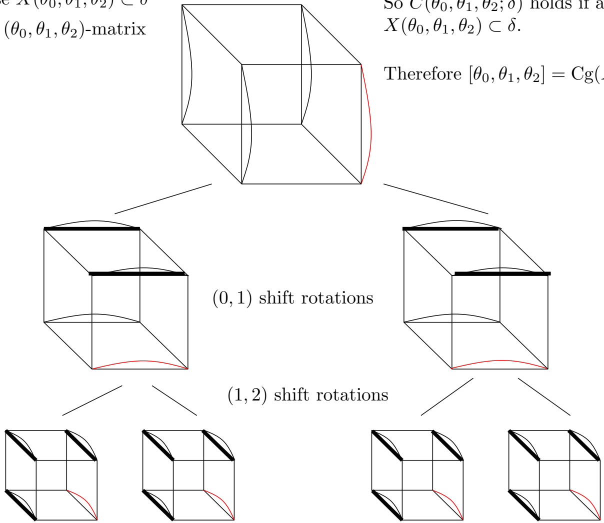
Generators for 3-ary commutator

Suppose $X(\theta_0, \theta_1, \theta_2) \subset \delta$

Take a $(\theta_0, \theta_1, \theta_2)$-matrix

So $C(\theta_0, \theta_1, \theta_2; \delta)$ holds if and only if $X(\theta_0, \theta_1, \theta_2) \subset \delta$.

Therefore $[\theta_0, \theta_1, \theta_2] = \mathrm{Cg}(X(\theta_0, \theta_1, \theta_2))$



Additivity of 3-ary commutator

Additivity of 3-ary commutator

We will show that $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] = \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

Additivity of 3-ary commutator

We will show that $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] = \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

By monotonicity, $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] \geq \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

Additivity of 3-ary commutator

We will show that $[\theta_0, \theta_1, \bigvee_{i\in I} \gamma_i] = \bigvee_{i\in I} [\theta_0, \theta_1, \gamma_i]$

By monotonicity, $[\theta_0, \theta_1, \bigvee_{i\in I} \gamma_i] \geq \bigvee_{i\in I} [\theta_0, \theta_1, \gamma_i]$

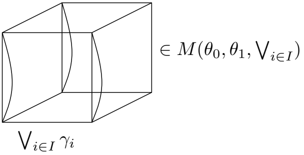
To show the other direction, it suffices to see that $C(\theta_0, \theta_1, \bigvee_{i\in I} \gamma_i; \alpha)$ holds, with $\alpha = \bigvee_{i\in I} [\theta_0, \theta_1, \gamma_i]$

Additivity of 3-ary commutator

We will show that $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] = \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

By monotonicity, $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] \geq \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

To show the other direction, it suffices to see that $C(\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i; \alpha)$ holds, with $\alpha = \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

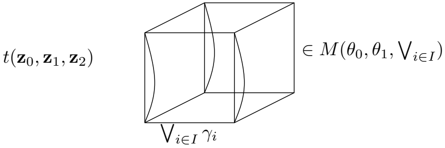


Additivity of 3-ary commutator

We will show that $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] = \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

By monotonicity, $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] \geq \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

To show the other direction, it suffices to see that $C(\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i; \alpha)$ holds, with $\alpha = \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

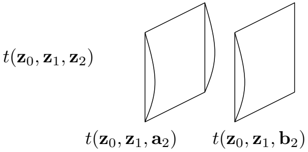


Additivity of 3-ary commutator

We will show that $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] = \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

By monotonicity, $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] \geq \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

To show the other direction, it suffices to see that $C(\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i; \alpha)$ holds, with $\alpha = \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$



where $\mathbf{a}_2 \equiv_{\bigvee \gamma_i} \mathbf{b}_2$

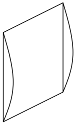
Additivity of 3-ary commutator

We will show that $[\theta_0, \theta_1, \bigvee_{i\in I} \gamma_i] = \bigvee_{i\in I}[\theta_0, \theta_1, \gamma_i]$

By monotonicity, $[\theta_0, \theta_1, \bigvee_{i\in I} \gamma_i] \geq \bigvee_{i\in I}[\theta_0, \theta_1, \gamma_i]$

To show the other direction, it suffices to see that $C(\theta_0, \theta_1, \bigvee_{i\in I} \gamma_i; \alpha)$ holds, with $\alpha = \bigvee_{i\in I}[\theta_0, \theta_1, \gamma_i]$

$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2)$



$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{a}_2)$ $\quad$ $t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{b}_2)$ $\quad$ where $\mathbf{a}_2 \equiv_{\bigvee \gamma_i} \mathbf{b}_2$

There exist tuples $\mathbf{c}_0, \ldots, \mathbf{c}_{j+1}$ such that $\mathbf{a}_2 = \mathbf{c}_0 \equiv_{\gamma_0} \mathbf{c}_1 \equiv_{\gamma_1} \ldots \equiv_{\gamma_j} \mathbf{c}_{j+1} = \mathbf{b}_2$

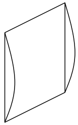
Additivity of 3-ary commutator

We will show that $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] = \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

By monotonicity, $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] \geq \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

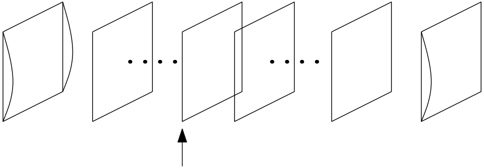
To show the other direction, it suffices to see that $C(\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i; \alpha)$ holds, with $\alpha = \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2)$



$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{a}_2)$ $t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{b}_2)$ where $\mathbf{a}_2 \equiv_{\bigvee \gamma_i} \mathbf{b}_2$

There exist tuples $\mathbf{c}_0, \ldots, \mathbf{c}_{j+1}$ such that $\mathbf{a}_2 = \mathbf{c}_0 \equiv_{\gamma_0} \mathbf{c}_1 \equiv_{\gamma_1} \ldots \equiv_{\gamma_j} \mathbf{c}_{j+1} = \mathbf{b}_2$



(θ_0, θ_1)-matrix associated with polynomial $t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{c}_l)$

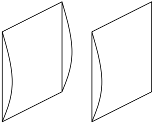
Additivity of 3-ary commutator

We will show that $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] = \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

By monotonicity, $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] \geq \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

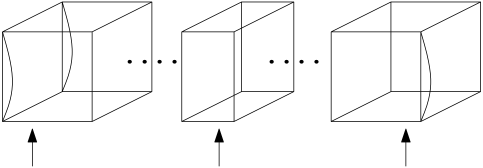
To show the other direction, it suffices to see that $C(\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i; \alpha)$ holds, with $\alpha = \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2)$



$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{a}_2)$ $\quad$ $t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{b}_2)$ $\quad$ where $\mathbf{a}_2 \equiv_{\bigvee \gamma_i} \mathbf{b}_2$

There exist tuples $\mathbf{c}_0, \ldots, \mathbf{c}_{j+1}$ such that $\mathbf{a}_2 = \mathbf{c}_0 \equiv_{\gamma_0} \mathbf{c}_1 \equiv_{\gamma_1} \ldots \equiv_{\gamma_j} \mathbf{c}_{j+1} = \mathbf{b}_2$



$(\theta_0, \theta_1, \gamma_{i_0})$-matrix $\quad$ $(\theta_0, \theta_1, \gamma_{i_s})$-matrix $\quad$ $(\theta_0, \theta_1, \gamma_{i_j})$-matrix

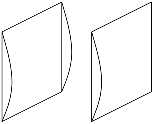
Additivity of 3-ary commutator

We will show that $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] = \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

By monotonicity, $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] \geq \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

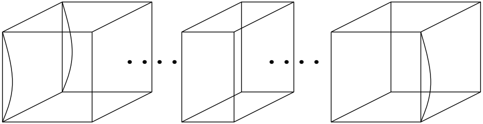
To show the other direction, it suffices to see that $C(\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i; \alpha)$ holds, with $\alpha = \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

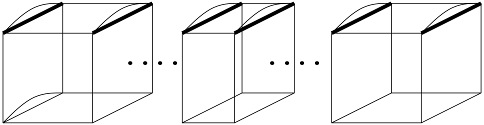
$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2)$



$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{a}_2)$ $t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{b}_2)$ where $\mathbf{a}_2 \equiv_{\bigvee \gamma_i} \mathbf{b}_2$

There exist tuples $\mathbf{c}_0, \ldots, \mathbf{c}_{j+1}$ such that $\mathbf{a}_2 = \mathbf{c}_0 \equiv_{\gamma_0} \mathbf{c}_1 \equiv_{\gamma_1} \ldots \equiv_{\gamma_j} \mathbf{c}_{j+1} = \mathbf{b}_2$





$(0,1)$ shift rotations

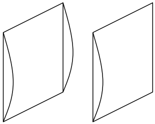
Additivity of 3-ary commutator

We will show that $[\theta_0, \theta_1, \bigvee_{i\in I} \gamma_i] = \bigvee_{i\in I} [\theta_0, \theta_1, \gamma_i]$

By monotonicity, $[\theta_0, \theta_1, \bigvee_{i\in I} \gamma_i] \geq \bigvee_{i\in I} [\theta_0, \theta_1, \gamma_i]$

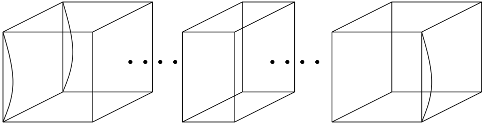
To show the other direction, it suffices to see that $C(\theta_0, \theta_1, \bigvee_{i\in I} \gamma_i; \alpha)$ holds, with $\alpha = \bigvee_{i\in I} [\theta_0, \theta_1, \gamma_i]$

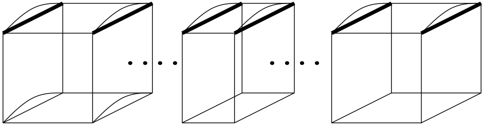
$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2)$



$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{a}_2)$ $\quad$ $t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{b}_2)$ $\quad$ where $\mathbf{a}_2 \equiv_{\bigvee \gamma_i} \mathbf{b}_2$

There exist tuples $\mathbf{c}_0, \ldots, \mathbf{c}_{j+1}$ such that $\mathbf{a}_2 = \mathbf{c}_0 \equiv_{\gamma_0} \mathbf{c}_1 \equiv_{\gamma_1} \ldots \equiv_{\gamma_j} \mathbf{c}_{j+1} = \mathbf{b}_2$





$(0, 1)$ shift rotations

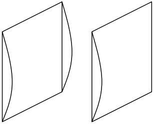
Additivity of 3-ary commutator

We will show that $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] = \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

By monotonicity, $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] \geq \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

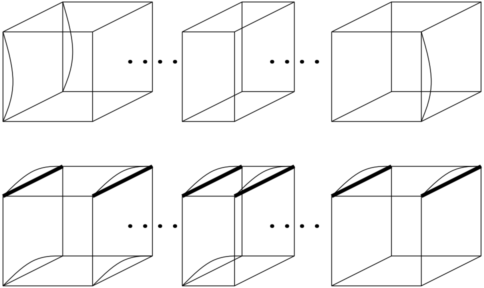
To show the other direction, it suffices to see that $C(\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i; \alpha)$ holds, with $\alpha = \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2)$



$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{a}_2)$ $t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{b}_2)$ where $\mathbf{a}_2 \equiv_{\bigvee \gamma_i} \mathbf{b}_2$

There exist tuples $\mathbf{c}_0, \ldots, \mathbf{c}_{j+1}$ such that $\mathbf{a}_2 = \mathbf{c}_0 \equiv_{\gamma_0} \mathbf{c}_1 \equiv_{\gamma_1} \ldots \equiv_{\gamma_j} \mathbf{c}_{j+1} = \mathbf{b}_2$



$(0, 1)$ shift rotations

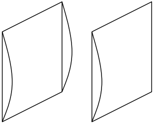
Additivity of 3-ary commutator

We will show that $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] = \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

By monotonicity, $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] \geq \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

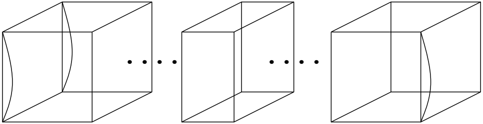
To show the other direction, it suffices to see that $C(\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i; \alpha)$ holds, with $\alpha = \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

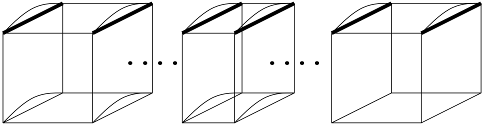
$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2)$



$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{a}_2)$ $\quad t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{b}_2)$ $\quad$ where $\mathbf{a}_2 \equiv_{\bigvee \gamma_i} \mathbf{b}_2$

There exist tuples $\mathbf{c}_0, \ldots, \mathbf{c}_{j+1}$ such that $\mathbf{a}_2 = \mathbf{c}_0 \equiv_{\gamma_0} \mathbf{c}_1 \equiv_{\gamma_1} \ldots \equiv_{\gamma_j} \mathbf{c}_{j+1} = \mathbf{b}_2$





$(0, 1)$ shift rotations

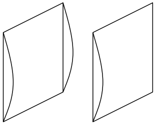
Additivity of 3-ary commutator

We will show that $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] = \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

By monotonicity, $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] \geq \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

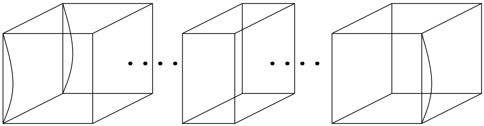
To show the other direction, it suffices to see that $C(\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i; \alpha)$ holds, with $\alpha = \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

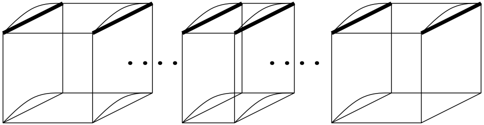
$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2)$



$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{a}_2)$ $t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{b}_2)$ where $\mathbf{a}_2 \equiv_{\bigvee \gamma_i} \mathbf{b}_2$

There exist tuples $\mathbf{c}_0, \ldots, \mathbf{c}_{j+1}$ such that $\mathbf{a}_2 = \mathbf{c}_0 \equiv_{\gamma_0} \mathbf{c}_1 \equiv_{\gamma_1} \ldots \equiv_{\gamma_j} \mathbf{c}_{j+1} = \mathbf{b}_2$





$(0, 1)$ shift rotations

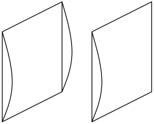
Additivity of 3-ary commutator

We will show that $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] = \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

By monotonicity, $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] \geq \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

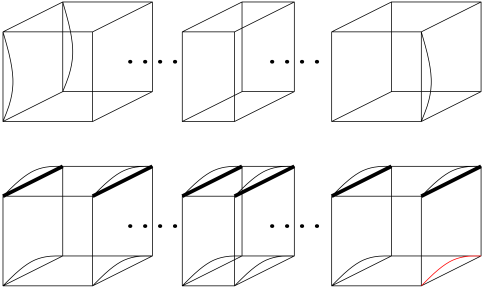
To show the other direction, it suffices to see that $C(\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i; \alpha)$ holds, with $\alpha = \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2)$



$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{a}_2)$ $t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{b}_2)$ where $\mathbf{a}_2 \equiv_{\bigvee \gamma_i} \mathbf{b}_2$

There exist tuples $\mathbf{c}_0, \dots, \mathbf{c}_{j+1}$ such that $\mathbf{a}_2 = \mathbf{c}_0 \equiv_{\gamma_0} \mathbf{c}_1 \equiv_{\gamma_1} \dots \equiv_{\gamma_j} \mathbf{c}_{j+1} = \mathbf{b}_2$



$(0, 1)$ shift rotations

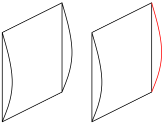
Additivity of 3-ary commutator

We will show that $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] = \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

By monotonicity, $[\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i] \geq \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

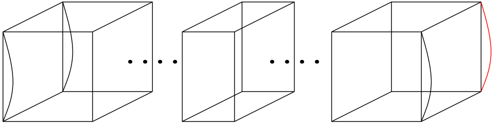
To show the other direction, it suffices to see that $C(\theta_0, \theta_1, \bigvee_{i \in I} \gamma_i; \alpha)$ holds, with $\alpha = \bigvee_{i \in I} [\theta_0, \theta_1, \gamma_i]$

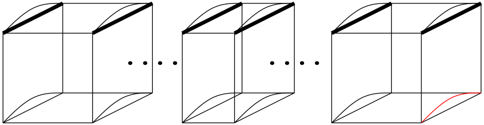
$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2)$



$t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{a}_2)$ $t(\mathbf{z}_0, \mathbf{z}_1, \mathbf{b}_2)$ where $\mathbf{a}_2 \equiv_{\bigvee \gamma_i} \mathbf{b}_2$

There exist tuples $\mathbf{c}_0, \ldots, \mathbf{c}_{j+1}$ such that $\mathbf{a}_2 = \mathbf{c}_0 \equiv_{\gamma_0} \mathbf{c}_1 \equiv_{\gamma_1} \ldots \equiv_{\gamma_j} \mathbf{c}_{j+1} = \mathbf{b}_2$





$(0, 1)$ shift rotations

Homomorphism Property

(recall) Suppose $f : \mathbb{A} \to \mathbb{B}$ is a surjective homomorphism with kernel π. Then $[\theta_0, ..., \theta_{k-1}] \vee \pi = f^{-1}([f(\theta_0 \vee \pi), ..., f(\theta_{k-1} \vee \pi))])$.

Homomorphism Property

(recall) Suppose $f : \mathbb{A} \to \mathbb{B}$ is a surjective homomorphism with kernel π. Then $[\theta_0, \ldots, \theta_{k-1}] \vee \pi = f^{-1}([f(\theta_0 \vee \pi), \ldots, f(\theta_{k-1} \vee \pi)])$.

Follows from additivity and generators.

The Δ Congruence

The Δ Congruence

The binary commutator of some α, β is actually the union of equivalence classes of a congruence $\Delta_{\alpha,\beta}$.

The Δ Congruence

The Δ Congruence

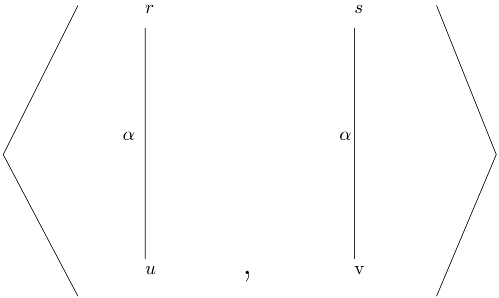
$$\begin{array}{ccc} r & \beta & s \\ \alpha & \boxed{} & \\ u & & v \end{array} \in M(\alpha, \beta)$$

The Δ Congruence

$$\begin{array}{ccc} r & \beta & s \\ \alpha & & \\ u & & \mathrm{v} \end{array} \in M(\alpha, \beta)$$

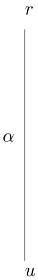
This is a pair of α-pairs

The Δ Congruence



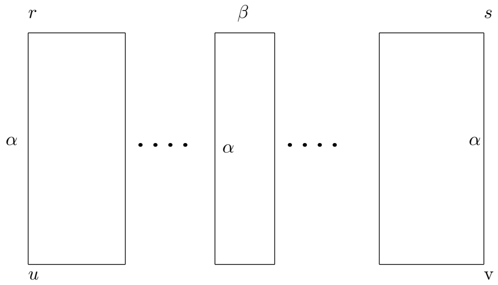
This is a pair of α-pairs

The Δ Congruence



Set $\mathbf{A}(\alpha)$ to be the collection of these columns. These are just α-pairs

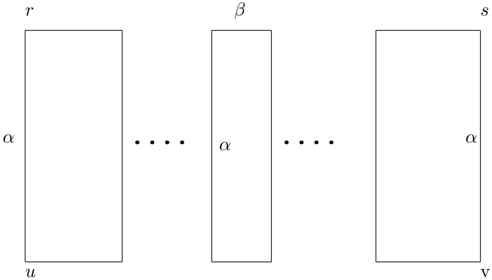
The Δ Congruence



Set $\mathbf{A}(\alpha)$ to be the collection of these columns. These are just α-pairs

Set $\Delta_{\alpha,\beta}$ to be the transitive closure of $M(\alpha, \beta)$

The Δ Congruence

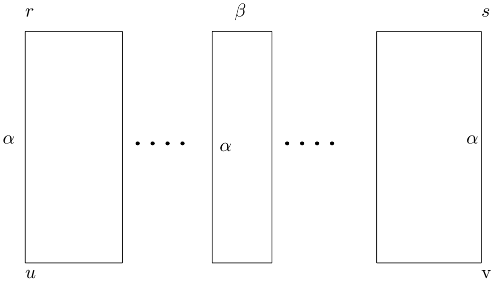


Set $\mathbf{A}(\alpha)$ to be the collection of these columns. These are just α-pairs

Set $\Delta_{\alpha,\beta}$ to be the transitive closure of $M(\alpha,\beta)$.

$\langle\langle r,u\rangle,\langle s,v\rangle\rangle \in \Delta_{\alpha,\beta}$ if and only if such a sequence of matrices exists.

The Δ Congruence



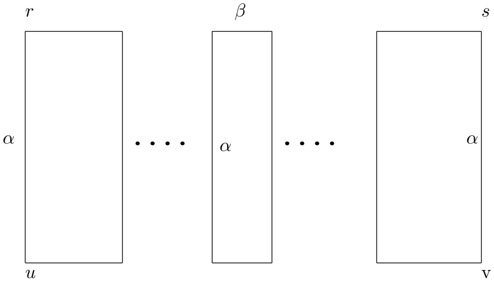
Set $\mathbf{A}(\alpha)$ to be the collection of these columns. These are just α-pairs

Set $\Delta_{\alpha,\beta}$ to be the transitive closure of $M(\alpha, \beta)$.

$\langle\langle r, u\rangle, \langle s, v\rangle\rangle \in \Delta_{\alpha,\beta}$ if and only if such a sequence of matrices exists.

$\Delta_{\alpha,\beta}$ is actually a congruence on $\mathbf{A}(\alpha)$.

The Δ Congruence



Set $\mathbf{A}(\alpha)$ to be the collection of these columns. These are just α-pairs

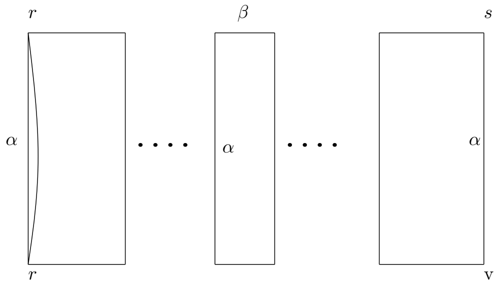
Set $\Delta_{\alpha,\beta}$ to be the transitive closure of $M(\alpha, \beta)$.

$\langle\langle r, u\rangle, \langle s, v\rangle\rangle \in \Delta_{\alpha,\beta}$ if and only if such a sequence of matrices exists.

$\Delta_{\alpha,\beta}$ is actually a congruence on $\mathbf{A}(\alpha)$.

$[\alpha, \beta]$ is the union of classes that intersect the diagonal.

The Δ Congruence



Set $\mathbf{A}(\alpha)$ to be the collection of these columns. These are just α-pairs

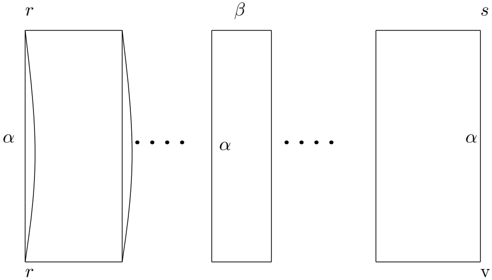
Set $\Delta_{\alpha,\beta}$ to be the transitive closure of $M(\alpha, \beta)$.

$\langle\langle r, u\rangle, \langle s, v\rangle\rangle \in \Delta_{\alpha,\beta}$ if and only if such a sequence of matrices exists.

$\Delta_{\alpha,\beta}$ is actually a congruence on $\mathbf{A}(\alpha)$.

$[\alpha, \beta]$ is the union of classes that intersect the diagonal.

The Δ Congruence



Set $\mathbf{A}(\alpha)$ to be the collection of these columns. These are just α-pairs

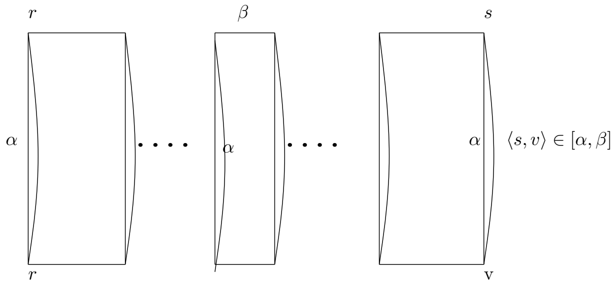
Set $\Delta_{\alpha,\beta}$ to be the transitive closure of $M(\alpha,\beta)$.

$\langle\langle r,u\rangle,\langle s,v\rangle\rangle \in \Delta_{\alpha,\beta}$ if and only if such a sequence of matrices exists.

$\Delta_{\alpha,\beta}$ is actually a congruence on $\mathbf{A}(\alpha)$.

$[\alpha,\beta]$ is the union of classes that intersect the diagonal.

The Δ Congruence



Set $\mathbf{A}(\alpha)$ to be the collection of these columns. These are just α-pairs

Set $\Delta_{\alpha,\beta}$ to be the transitive closure of $M(\alpha, \beta)$.

$\langle\langle r, u\rangle, \langle s, v\rangle\rangle \in \Delta_{\alpha,\beta}$ if and only if such a sequence of matrices exists.

$\Delta_{\alpha,\beta}$ is actually a congruence on $\mathbf{A}(\alpha)$.

$[\alpha, \beta]$ is the union of classes that intersect the diagonal.

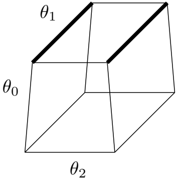
The Δ Congruence

The "greatest global operation" property is proved for the binary commutator in Freese-McKenzie using properties of $\Delta_{\alpha,\beta}$.

The Δ Congruence

The "greatest global operation" property is proved for the binary commutator in Freese-McKenzie using properties of $\Delta_{\alpha,\beta}$.
We can extend the idea of this congruence to the higher commutator.

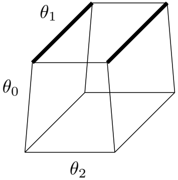
The Δ Congruence



$\in M(\theta_0, \theta_1, \theta_2)$

The Δ Congruence

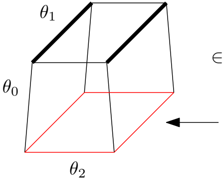
Let $C_1^2(\theta_0, \theta_1, \theta_2)$ be the set of all such matrices. Clearly $C_1^2(\theta_0, \theta_1, \theta_2) \leq M(\theta_0, \theta_1, \theta_2)$



$\in M(\theta_0, \theta_1, \theta_2)$

The Δ Congruence

Let $C_1^2(\theta_0,\theta_1,\theta_2)$ be the set of all such matrices. Clearly $C_1^2(\theta_0,\theta_1,\theta_2) \leq M(\theta_0,\theta_1,\theta_2)$

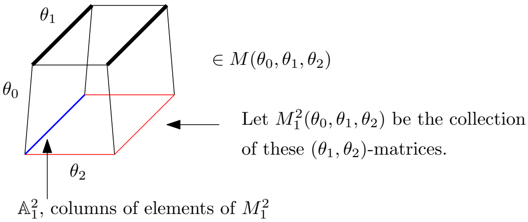


$\in M(\theta_0,\theta_1,\theta_2)$

Let $M_1^2(\theta_0,\theta_1,\theta_2)$ be the collection of these (θ_1,θ_2)-matrices.

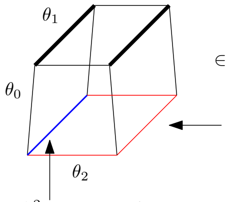
The Δ Congruence

Let $C_1^2(\theta_0, \theta_1, \theta_2)$ be the set of all such matrices. Clearly $C_1^2(\theta_0, \theta_1, \theta_2) \leq M(\theta_0, \theta_1, \theta_2)$



The Δ Congruence

Let $C_1^2(\theta_0, \theta_1, \theta_2)$ be the set of all such matrices. Clearly $C_1^2(\theta_0, \theta_1, \theta_2) \leq M(\theta_0, \theta_1, \theta_2)$



$\in M(\theta_0, \theta_1, \theta_2)$

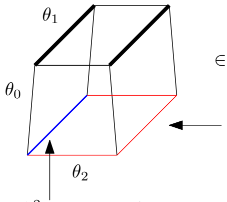
Let $M_1^2(\theta_0, \theta_1, \theta_2)$ be the collection of these (θ_1, θ_2)-matrices.

$\mathbb{A}_1^2$, columns of elements of M_1^2

Set $\Delta_1^2(\theta_0, \theta_1, \theta_2)$ to be the transitive closure of $M_1^2(\theta_0, \theta_1, \theta_2)$.

The Δ Congruence

Let $C_1^2(\theta_0, \theta_1, \theta_2)$ be the set of all such matrices. Clearly $C_1^2(\theta_0, \theta_1, \theta_2) \leq M(\theta_0, \theta_1, \theta_2)$



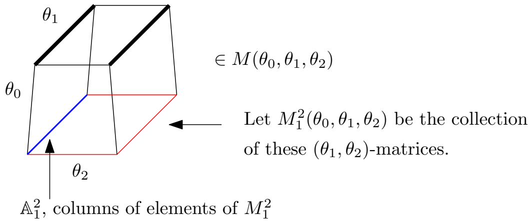
Let $M_1^2(\theta_0, \theta_1, \theta_2)$ be the collection of these (θ_1, θ_2)-matrices.

$\mathbb{A}_1^2$, columns of elements of M_1^2

Set $\Delta_1^2(\theta_0, \theta_1, \theta_2)$ to be the transitive closure of $M_1^2(\theta_0, \theta_1, \theta_2)$.
$\Delta_1^2(\theta_0, \theta_1, \theta_2)$ is a congruence of $\mathbb{A}_1^2$.

The Δ Congruence

Let $C_1^2(\theta_0, \theta_1, \theta_2)$ be the set of all such matrices. Clearly $C_1^2(\theta_0, \theta_1, \theta_2) \leq M(\theta_0, \theta_1, \theta_2)$



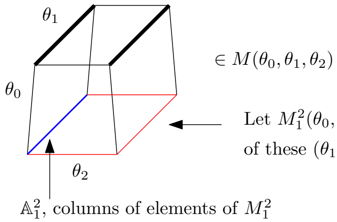
Set $\Delta_1^2(\theta_0, \theta_1, \theta_2)$ to be the transitive closure of $M_1^2(\theta_0, \theta_1, \theta_2)$.

$\Delta_1^2(\theta_0, \theta_1, \theta_2)$ is a congruence of $\mathbb{A}_1^2$.

$[\theta_0, \theta_1, \theta_2]$ is the union of congruence classes that intersect the diagonal.

The Δ Congruence

Let $C_1^2(\theta_0,\theta_1,\theta_2)$ be the set of all such matrices. Clearly $C_1^2(\theta_0,\theta_1,\theta_2) \leq M(\theta_0,\theta_1,\theta_2)$



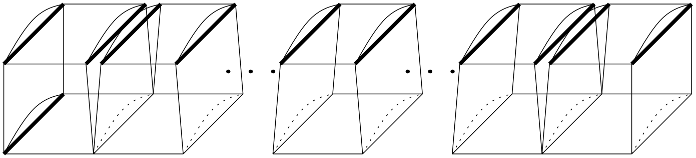
Let $M_1^2(\theta_0,\theta_1,\theta_2)$ be the collection of these (θ_1,θ_2)-matrices.

$\mathbb{A}_1^2$, columns of elements of M_1^2

Set $\Delta_1^2(\theta_0,\theta_1,\theta_2)$ to be the transitive closure of $M_1^2(\theta_0,\theta_1,\theta_2)$.

$\Delta_1^2(\theta_0,\theta_1,\theta_2)$ is a congruence of $\mathbb{A}_1^2$.

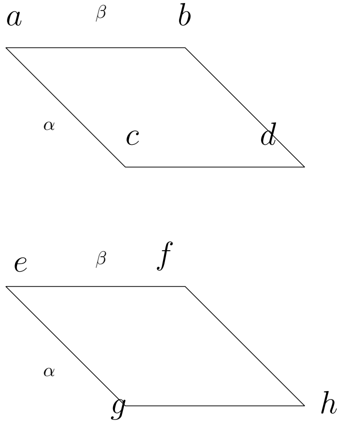
$[\theta_0,\theta_1,\theta_2]$ is the union of congruence classes that intersect the diagonal.



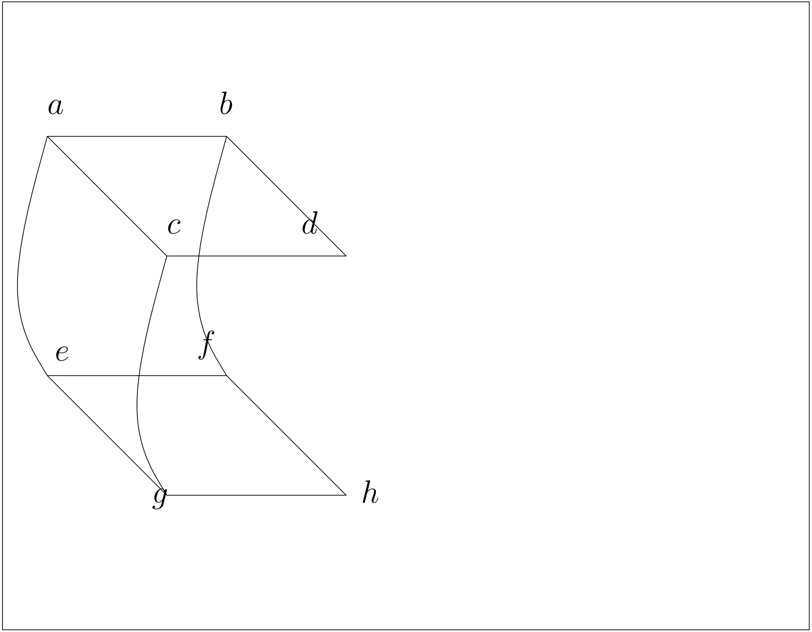
The Δ Congruence

The proof of the “greatest global operation” property given for the binary commutator is easily adopted to the higher commutator.

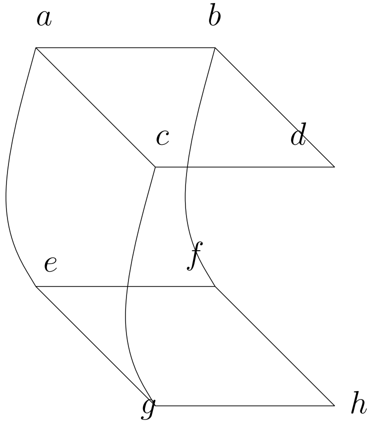
Two Term Commutator



Two Term Commutator

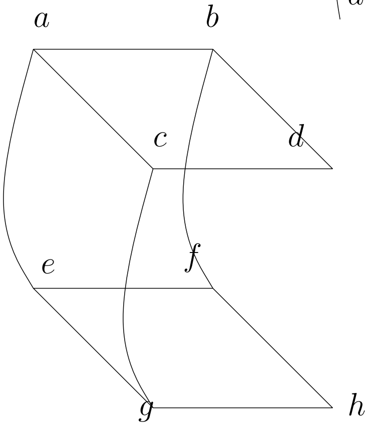


Two Term Commutator



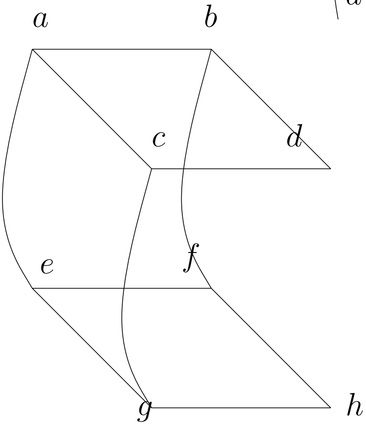
Two Term Commutator

$$m_e\begin{pmatrix} b & b & & a & b & & e & f & & f & f \\ d & d & , & c & d & , & g & h & , & h & h \end{pmatrix}$$



Two Term Commutator

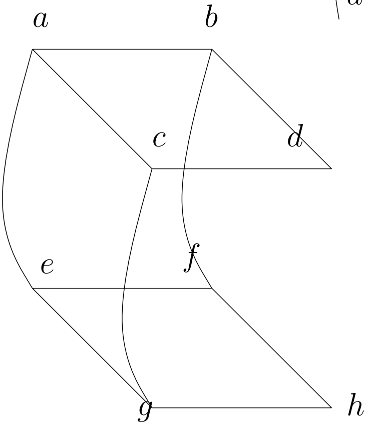
$$m_e\begin{pmatrix} b & b & & a & b & & e & f & & f & f \\ d & d & , & c & d & , & g & h & , & h & h \end{pmatrix}$$



$$\begin{tikzcd} m_e(b,a,e,f) \arrow[r, dash] \arrow[d, dash] & m_e(b,b,f,f) \arrow[d, dash] \\ m_e(d,c,g,h) \arrow[r, dash] & m_e(d,d,g,h) \end{tikzcd}$$

Two Term Commutator

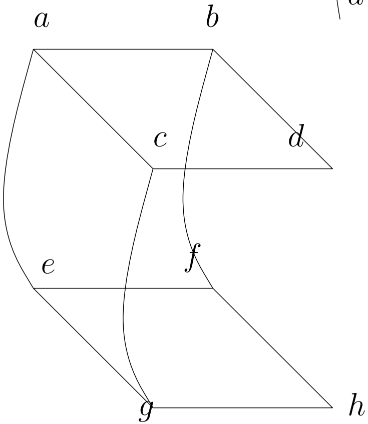
$$m_e\begin{pmatrix} b & b & & a & b & & e & f & & f & f \\ d & d & , & c & d & , & g & h & , & h & h \end{pmatrix}$$



$$\begin{tikzcd}
m_e(b,a,e,f) \arrow[r, no head] \arrow[d, no head] \arrow[r, no head, bend left] & m_e(b,b,f,f) \arrow[d, no head] \\
m_e(d,c,g,h) \arrow[r, no head] & m_e(d,d,g,h)
\end{tikzcd}$$

Two Term Commutator

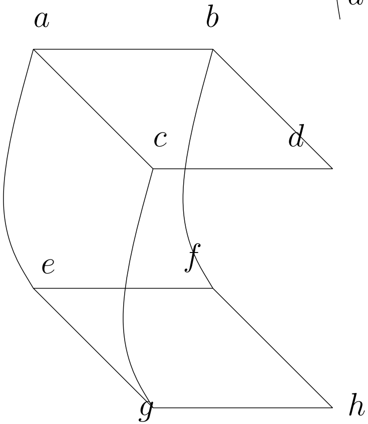
$$m_e\left(\begin{matrix} b & b \\ d & d \end{matrix}, \begin{matrix} a & b \\ c & d \end{matrix}, \begin{matrix} e & f \\ g & h \end{matrix}, \begin{matrix} f & f \\ h & h \end{matrix}\right)$$



$$\begin{tikzcd}
m_e(b,a,e,f) \arrow[r, dash] \arrow[d, dash] \arrow[r, dash, bend left] & m_e(b,b,f,f) \arrow[d, dash] \\
m_e(d,c,g,h) \arrow[r, dash] \arrow[r, dash, bend left] & m_e(d,d,g,h)
\end{tikzcd}$$

Two Term Commutator

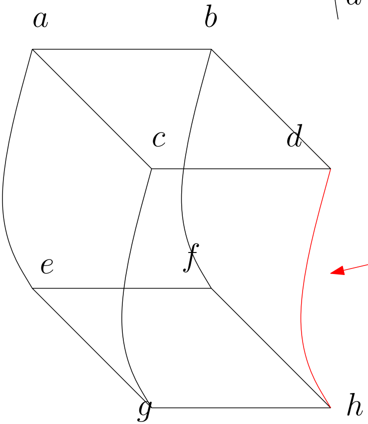
$$m_e\begin{pmatrix} b & b & & a & b & & e & f & & f & f \\ d & d & , & c & d & , & g & h & , & h & h \end{pmatrix}$$



$$\begin{tikzcd}
m_e(b,a,e,f) \arrow[r, dash] \arrow[r, dash, bend left] \arrow[d, dash] & m_e(b,b,f,f) \arrow[d, dash] \\
m_e(d,c,g,h) \arrow[r, dash] & m_e(d,d,g,h)
\end{tikzcd}$$

Two Term Commutator

$$m_e\begin{pmatrix} b & b & & a & b & & e & f & & f & f \\ d & d & , & c & d & , & g & h & , & h & h \end{pmatrix}$$



$$\begin{tikzcd} m_e(b,a,e,f) \arrow[r, dash] \arrow[d, dash] & m_e(b,b,f,f) \arrow[d, dash] \\ m_e(d,c,g,h) \arrow[r, dash] & m_e(d,d,g,h) \end{tikzcd}$$

$\forall e \in n+1$

Thank You