

# Computational partial algebras and structural clones

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# Outline

- Partial algebras
- Partial clones
- Computing all partial clones generated by  $n$ -ary operations
- Natural duality for partial algebras
- Structural clones
- Computing all structural clones on  $\{0,1\}$  generated by binary operations
- ISP(a 2-element partial algebra)

First part mostly based on *The lattice of alter egos*



by Brian Davey



Jane Pitkethly



and Ross Willard

## Abstract

We describe an algorithm for calculating finitely generated structural clones, defined by [Davey Pitkethly Willard: The lattice of alter egos, IJAC 22, 2012] as partial clones that are closed under equalizers and restriction to domain.

The algorithm has been implemented in Python and JavaScript and is used to calculate all 1693 structural clones on a two-element set generated by a set of partial binary operations (excluding the trivial operation with empty domain).

These results are related to the dualizability of partial algebras since if two partial algebras generate the same structural clone then the partial algebras are either both dualizable or both non-dualizable.

## Partial algebras

A **partial  $n$ -ary operation** on  $A$  is a map  $f : D_f \rightarrow A$  where  $D_f \subseteq A^n$   
 $\langle A, F \rangle$  is a **partial algebra** if  $F$  is a set of **partial** operations

If  $A = \{a_1, \dots, a_n\}$ , then a partial algebra can be defined by a list of **partially filled out** operation tables

	$f$	$g$	$a_1$	$\cdots$	$a_n$	$h$	$a_1$	$\cdots$	$a_n$	$\cdots$	$a_1$	$\cdots$	$a_n$
$a_1$	$b_1$	$a_1$	$c_1$	$\cdots$	$-$	$a_1$	$e_1$	$\cdots$	$-$	$\cdots$	$-$	$\cdots$	$l_n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
$a_n$	$-$	$a_n$	$-$	$\cdots$	$d_n$	$a_n$	$e_n$	$\cdots$	$-$	$\cdots$	$k_n$	$\cdots$	$-$

**Convention:** every **(total) algebra** is a partial algebra

Also, every **relational structure** is a partial algebra

E.g. a relation  $R \subseteq A^n$  is given by a **restricted projection**  $p_R : D_{p_R} \rightarrow A$

$$p_R(x_1, \dots, x_n) = \begin{cases} x_n & \text{if } R(x_1, \dots, x_n) \\ \text{undefined} & \text{otherwise} \end{cases}$$

## The category of partial algebras of fixed type

The **type** of a partial algebra is a set  $\mathcal{F}$  of (partial) **function symbols**, each with an associated finite arity

$h: \mathbf{A} \rightarrow \mathbf{B}$  is a **homomorphism** if  $h(f^{\mathbf{A}}(x_1, \dots, x_n)) = f^{\mathbf{B}}(h(x_1), \dots, h(x_n))$  for all  $(x_1, \dots, x_n) \in D_{f^{\mathbf{A}}}$  and all  $f \in \mathcal{F}$

Note:  $h(p_R^{\mathbf{A}}(x_1, \dots, x_n)) = p_R^{\mathbf{B}}(h(x_1), \dots, h(x_n))$

iff  $R^{\mathbf{A}}(x_1, \dots, x_n) \implies R^{\mathbf{B}}(h(x_1), \dots, h(x_n))$

so homomorphisms agree for relational structures and their corresponding partial algebras

The  $i^{\text{th}}$   $n$ -ary projection is  $p_{i,n}(x_1, \dots, x_n) = x_i$  (a total operation)

$\Pi$  is the set of all projections of finite arity

## Partial clones

Let  $f$  be  $n$ -ary and  $g_1, \dots, g_n$   $m$ -ary partial operations

The **composition**  $f \circ [g_1, \dots, g_n]$  is the  $m$ -ary partial operation  $h$  defined by

$$h(x_1, \dots, x_m) = f(g_1(x_1, \dots, x_m), \dots, g_n(x_1, \dots, x_m))$$

$\mathcal{P}_A$  is the **set of all partial operations** on  $A$

A **partial clone** is a subset of  $\mathcal{P}_A$  that contains all projections and is closed under composition

$\text{PClo}(F)$  is the **partial clone generated** by a set  $F$  of partial functions

= the **smallest** set that contains  $F \cup \Pi$  and is **closed under composition**

$\text{PClo}_n(F) = \{\text{the } n\text{-ary partial operations in } \text{PClo}(F)\}$

$\text{Clo}(F) = \{\text{total operations in } \text{PClo}(F)\}$

## The lattice of partial clones

$\mathbb{L}(\mathcal{P}_A)$  is the algebraic lattice of partial clones on a finite set  $A$

$\mathbb{L}(\mathcal{T}_A)$  is the algebraic lattice of total clones on  $A$  ( $\mathcal{T}_A =$  all total ops)

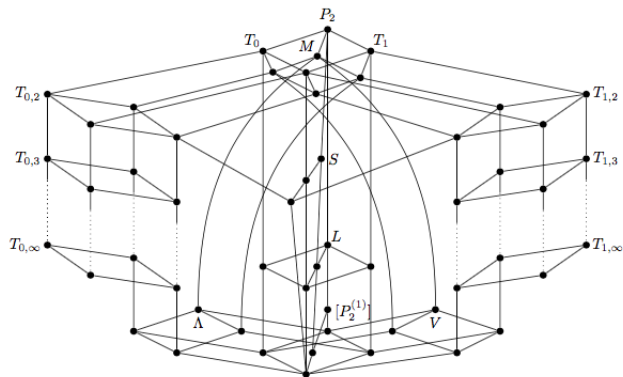
$\mathbb{L}(\mathcal{T}_A)$  is a complete sublattice of  $\mathbb{L}(\mathcal{P}_A)$  since any composition of total operations is total

Every partial algebra can be extended to a total algebra  $\tilde{\mathbf{A}}$  by adding one element  $\infty \notin A$

$$\tilde{f}(x_1, \dots, x_n) = \begin{cases} f(x_1, \dots, x_n) & \text{if } f(x_1, \dots, x_n) \text{ is defined (i.e., exists)} \\ \infty & \text{otherwise} \end{cases}$$

Therefore  $\mathbb{L}(\mathcal{P}_A)$  is embedded in  $\mathbb{L}(\mathcal{T}_{A \cup \{\infty\}})$

# Post lattice of all total clones on $\{0,1\}$



$\mathbb{L}(\mathcal{T}_{\{0,1\}})$  is countable (picture from Schölzel 2010)

$\mathbb{L}(\mathcal{P}_{\{0,1\}})$  is uncountable (true even for strong partial clones = partial clones closed under restriction to any subset of the domain)



## Computing partial clones

$\mathbb{L}_n(\mathcal{P}_A)$  = lattice of partial clones generated by  $n$ -ary partial operations

There are  $(|A| + 1)^{|A|^n}$   $n$ -ary partial operations

hence  $|\mathbb{L}_n(\mathcal{P}_A)| < 2^{(|A|+1)^{|A|^n}}$

E. g.  $|\mathbb{L}_1(\mathcal{P}_{\{0,1\}})| < 2^{3^{2^1}} = 2^9 = 512$  and  $|\mathbb{L}_2(\mathcal{P}_{\{0,1\}})| < 2^{3^{2^2}} = 2^{81} \approx 10^{24}$

$\text{Clo}_n(F) = \text{Free}_{\langle A, F \rangle}(n)$ , and the same result holds for finite partial algebras

So can use Birkhoff's subpower algorithm to calculate  $\text{PClo}_n(\mathbf{A})$  as a subalgebra of  $(A \cup \{\infty\})^{|A|^n}$

Most efficient implementation is in UACalc

## An algorithm for computing $\mathbb{L}_n(\mathcal{P}_A)$

1. Let  $\mathcal{C} = \{\text{PClo}_n(\{f\}) : f \in \mathcal{P}_A \text{ with arity } n\}$
2. Let  $\mathcal{D}$  and  $\mathcal{L}$  be copies of  $\mathcal{C}$

Repeat

3. Let  $\mathcal{E} = \{\text{PClo}_n(F \cup G) : F \in \mathcal{C} \text{ and } G \in \mathcal{D}\}$
4. Let  $\mathcal{D} = \mathcal{E} \setminus \mathcal{D}$
5. Let  $\mathcal{L} = \mathcal{L} \cup \mathcal{D}$

Until  $\mathcal{D} = \emptyset$

6.  $\mathcal{L}$  contains all  $n$ -ary partial clones, each one determined by a generating set of minimal size

## A Galois connection for partial operations

[DPW12] Davey, Pitkethly and Willard: *The lattice of alter egos*, *Intl. J. of Algebra and Comp.*, 2012

For a matrix  $[a_{ij}] \in A^{m \times n}$  denote the rows by  $a_{i*}$  and the columns by  $a_{*j}$

Let  $f, g$  be partial operations on  $A$  with arity  $m, n$  respectively and define

$f \sim g$  iff for all  $[a_{ij}] \in A^{m \times n}$  ( $a_{i*} \in D_g$  and  $a_{*j} \in D_f$  for all  $i \leq m, j \leq n$ )

$$\implies \exists b \in A \text{ s.t. } f(g(a_{1*}), \dots, g(a_{m*})) = b = g(f(a_{*1}), \dots, f(a_{*n}))$$

$$\begin{array}{ccc} \left[ \begin{array}{ccc} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{array} \right] & \begin{array}{l} \rightarrow \\ \rightarrow \end{array} & \begin{array}{l} g(a_{1*}) \\ g(a_{m*}) \end{array} \\ & & f, g \text{ are compatible} \\ \downarrow & & \downarrow \\ f(a_{*1}) \cdots f(a_{*n}) & \Rightarrow & f(g(a_{1*}), \dots, g(a_{m*})) = g(f(a_{*1}), \dots, f(a_{*n})) \end{array}$$

## Characterizing the Galois closed sets

Note:  $f \sim p_R$  is equivalent to the standard notion  $f$  **preserves**  $R$

For  $F \subseteq \mathcal{P}_A$  let  $F^\diamond = \{g \in \mathcal{P}_A : f \sim g \text{ for all } f \in F\}$

A **conjunct-atomic formula** is of the form  $\psi_1(\mathbf{v}) \& \cdots \& \psi_n(\mathbf{v})$  with  $\psi_i$  atomic

A  $k$ -ary relation  $R$  is **conjunct-atomic definable** if

$R = \{\mathbf{a} \in A^k : \psi_1(\mathbf{a}) \& \cdots \& \psi_n(\mathbf{a})\}$  for some atomic formulas  $\psi_i$

$\text{Def}_{\text{ca}}(F) = \{\text{conjunct-atomic relations definable from } F\}$

**Theorem** [DPW12]: (i)  $g \in F^{\diamond\diamond}$  if and only if

$D_g \in \text{Def}_{\text{ca}}(F)$  and  $g = f \upharpoonright E$  for some  $f \in \text{PClo}(F)$  and  $E \subseteq A$

(ii)  $F = F^{\diamond\diamond}$  if and only if  $F$  is a partial clone on  $A$  closed under restriction of domain to relations in  $\text{Def}_{\text{ca}}(F)$

## Structural clones

The Galois closed sets of the closure operator  $\diamond\diamond$  are called **structural clones**

Notation:  $\text{SClo}(F) = F^{\diamond\diamond}$  and  $\text{SClo}_n(F) = n$ -ary members

The lattice of structural clones is denoted  $\mathbb{S}(\mathcal{P}_A)$

Since  $\sim$  is a symmetric relation,  $\mathbb{S}(\mathcal{P}_A)$  is a self-dual algebraic lattice

Partial algebras  $\mathbf{A}_1, \mathbf{A}_2$  on the same set  $A$  are **structurally equivalent** if  $\text{SClo}(\mathbf{A}_1) = \text{SClo}(\mathbf{A}_2)$

For algebras, this agrees with **clone equivalence**

## Natural duality (briefly)

- Duality theory aims to find **categorical (dual) equivalences** between two categories
- **Natural dualities** provide a framework using homomorphisms into a generating object
- E.g. **Stone duality**  $D : \mathbf{BA} \rightarrow \mathbf{Stone}$ ,  $E : \mathbf{Stone} \rightarrow \mathbf{BA}$  given by

$D(\mathbf{A}) = \mathbf{Hom}(\mathbf{A}, \mathbf{2})$  with product topology from  $\mathbf{2}^{\mathbf{A}}$ ,  $D(h)(x) = x \circ h$   
 $E(\mathbf{X}) = \mathbf{Hom}(\mathbf{X}, \mathbf{2})$  with operations inherited from  $\mathbf{2}^{\mathbf{X}}$ ,  $E(k)(a) = a \circ k$

- Or **Priestley duality**  $D : \mathbf{BDL} \rightarrow \mathbf{Pri}$ ,  $E : \mathbf{Pri} \rightarrow \mathbf{BDL}$  given by

$D(\mathbf{A}) = \mathbf{Hom}(\mathbf{A}, \mathbf{C}_2)$  with product topology from  $\mathbf{C}_2^{\mathbf{A}}$ ,  $D(h)(x) = x \circ h$   
 $E(\mathbf{X}) = \mathbf{Hom}(\mathbf{X}, \mathbf{C}_2)$  with operations inherited from  $\mathbf{C}_2^{\mathbf{X}}$ ,  $E(k)(a) = a \circ k$

- Then  $E(D(\mathbf{A})) \cong \mathbf{A}$  and  $D(E(\mathbf{X})) \cong \mathbf{X}$  via the **natural evaluation embeddings**

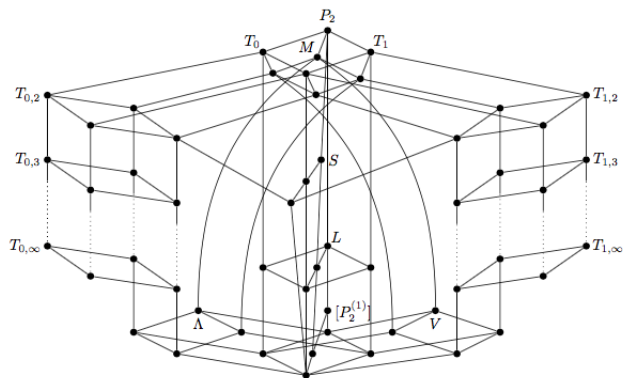
# Natural duality for partial algebras

- Davey [2006] extends natural dualities to **categories of partial algebras** and **relational structures**
- Davey, Pitkethly and Willard [2012] give the **symmetric** formulation  $f \sim g$
- Two partial algebras  $\mathbf{P}, \underline{\mathbf{P}}$  on the same finite set  $P$  are compatible if  $\text{SClo}(\mathbf{P}) = \underline{\mathbf{P}}^\diamond$
- In that case  $\underline{\mathbf{P}}$  is called an alter ego of  $\mathbf{P}$
- $\mathbf{P}$  is fully dualizable if  $E(D(\mathbf{A})) \cong \mathbf{A}$  and  $D(E(\mathbf{X})) \cong \mathbf{X}$  via the evaluation embeddings for all  $\mathbf{A} \in \mathbf{ISP}^+(\mathbf{P})$  and  $\mathbf{X} \in \mathbf{IS}_c\mathbf{P}^+(\underline{\mathbf{P}})$

# Dualizability of 2-element algebras

## Theorem

*[Clark, Davey 1998] All 2-element (total) algebras are dualizable, except for the 8 that are limits of descending chains*





## An algorithm for computing $\text{SClo}(F) = F^{\diamond\diamond}$

Let  $\mathbf{A} = \langle A, F \rangle$  be a finite partial algebra with  $F$  finite

To compute all  $n$ -ary partial functions in  $F^{\diamond\diamond}$ , compute the  $n$ -ary **partial clone**  $C = \text{PClo}_n(F) = \text{subalgebra of } (\mathbf{A} \cup_{\infty})^{A^n}$  generated by  $F \cup \{\pi_1, \dots, \pi_n\}$  the  $k$ -ary projections

Next, close under **equalizers of partial functions**, i.e.,  $f, g \in C$  implies

$$E(f, g) \in C \text{ where } E(f, g)(\mathbf{a}) = \begin{cases} f(\mathbf{a}) & \text{if } f(\mathbf{a}) = g(\mathbf{a}) \\ \text{undef.} & \text{otherwise} \end{cases}$$

Finally, close under  $R(f, g) = f \upharpoonright_g =$  the **restriction of  $f$  to the domain of  $g$** , for all  $f, g \in C$

### Theorem

$\text{SClo}_n(F) = \mathbf{R}(\mathbf{E}(\text{PClo}_n(F)))$  where  $\mathbf{E}(C) = \text{closure under } E, \text{ same for } \mathbf{R}$

Programs for computing all unary and binary (structural) clones on  $\{0, 1\}$  in a web browser are at <http://mathv.chapman.edu/~jipsen/uajs>

## Computing structural clones

Note: DPW'12 **exclude** operations with **empty domain**

There are 17 unary and 1693 binary structural clones on  $\{0,1\}$  compared to 6 unary and 26 binary (total) clones on  $\{0,1\}$

Let  $\mathbf{P}_1 = \langle \{0,1\}, +, \mathbf{0} \rangle$  where

+	0	1
0	0	1
1	1	-

$\mathbf{P}_1$  is halfway between the 2-element semilattice and the 2-element group

$x$	$y$	$0$	$x+y$	$2x$	$2y$	$2x+y$	$x+2y$	$2x+2y$	$E(x,y)$	$x \upharpoonright_{x+y}$	$y \upharpoonright_{x+y}$	$0 \upharpoonright_{x+y}$	$0 \upharpoonright_{E(x,y)}$
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	1	0	-	1	-	-	-	0	1	0	-
1	0	0	1	-	0	-	1	-	-	1	0	0	-
1	1	0	-	-	-	-	-	-	1	-	-	-	0

## $P_1$ is dualizable at the finite level

### Theorem

(Joint with M. A. Moshier)  $\mathbf{P}_1 = \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & - \end{array}$  is dualizable at the finite level

Let  $\underline{\mathbf{P}}_1 = \langle \{0, 1\}, G \rangle$  where  $G = \{+, 0\}^\diamond$ , so  $g \in G$  if  $g(0, \dots, 0) = 0$  and if  $g(x_1, \dots, x_k), g(y_1, \dots, y_k)$ , and  $x_i + y_i$  **defined** for  $i = 1, \dots, k$  then  $g(x_1 + y_1, \dots, x_k + y_k) = g(x_1, \dots, x_k) + g(y_1, \dots, y_k)$  (both defined)

Show for all finite  $\mathbf{A} \in \mathbf{ISP}(\mathbf{P}_1)$  we have  $E(D(\mathbf{A})) \cong \mathbf{A}$

**Problem:** Is  $P_1$  dualizable in general?

## Subalgebras, products, homomorphisms

- $\mathbf{B} \subseteq \mathbf{A}$  is a **(partial) subalgebra** if  $\mathbf{B}$  is closed under the partial operations of  $\mathbf{A}$
- $\prod_{i \in I} \mathbf{A}_i$  is the **product**; operations are defined pointwise; exist **iff** they exist in all coords
- Note that  $\tilde{\mathbf{A}} \times \tilde{\mathbf{B}} \not\cong \mathbf{A} \times \mathbf{B}$  (since e.g.  $9 \neq 5$ )
- $h: \mathbf{A} \rightarrow \mathbf{B}$  is a **homomorphism** if  $h(f^{\mathbf{A}}(x_1, \dots, x_n)) = f^{\mathbf{B}}(h(x_1), \dots, h(x_n))$  for all  $(x_1, \dots, x_n) \in D_f$
- **HSP** is defined using these operations
- Not much is known about **HSP**( $P_1$ )

## Identities and quasiidentities

The **signature** of a partial algebra is a set  $F$  of (partial) **function symbols**, each with an associated finite arity

The **interpretation** of  $f$  in a partial algebra  $\mathbf{A}$  is denoted  $f^{\mathbf{A}}$

**Terms** and **formulas** are defined as usual

A term  $t(a_1, \dots, a_n)$  is defined iff **all subterms are defined**

An identity  $s(x_1, \dots, x_n) = t(x_1, \dots, x_n)$  holds in a partial algebra  $\mathbf{A}$  if for all  $x_1, \dots, x_n \in A$  either **both sides are undefined**, or they are **defined and equal**

A **quasiidentity**  $s_1 = t_1 \& \dots \& s_n = t_n \implies s = t$  holds in  $\mathbf{A}$  if for all assignments that make **the premises defined and equal**,  $s, t$  are **defined and equal**

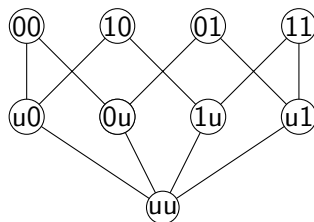
## Properties that hold in $\text{ISP}(P_1)$

- $x + y = y + x$  (commutative)
- $(x + y) + z = x + (y + z)$  (associative)
- $x + 0 = x$  (0 is the identity)
- $x + z = y + z \implies x = y$  (cancellative)
- $x + x = x + x \implies x = 0$  (orthogonal)
- $P_1$  is **coherent**, i.e., if  $x + y$ ,  $x + z$  and  $y + z$  are defined, so is  $(x + y) + z$

# Examples of algebras in $\text{ISP}(P_1)$

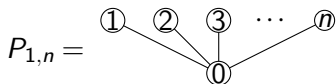
- $P_{2,2} =$

+	$uu$	$u0$	$u1$	$0u$	$1u$	$00$	$01$	$10$	$11$
$uu$	$uu$	$u0$	$u1$	$0u$	$1u$	$00$	$01$	$10$	$11$
$u0$	$u0$			$00$	$10$				
$u1$	$u1$			$01$	$11$				
$0u$	$0u$	$00$	$01$						
$1u$	$1u$	$10$	$11$						
$00$	$00$								
$01$	$01$								
$10$	$10$								
$11$	$11$								



- Define  $x \leq y$  if  $x + z = y$  for some  $z$  (the **natural order**)
- Can you find another (smaller) example? Guess what!  $P_1 = P_{1,1}$

$$P_1 = \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array} = \begin{array}{c} \textcircled{1} \\ | \\ \textcircled{0} \end{array}$$



## ISP( $P_1$ ) is not finitely axiomatizable

### Theorem

**ISP( $P_1$ ) is not finitely axiomatizable.**

**Proof.** Consider the following quasiidentities  $q_n$ :

$$\&_{i=0}^{n-1}(x_{2i} + x_{2i+1} = x_{2i+2} + x_{2i+3}) \& \&_{i=0}^{n-1}(x_{2i+1} + x_{2i+2} = y_i) \implies x_0 = x_2$$

where index addition is modulo  $2n$ .

We also define a partial algebra  $Q_n = \{0, a_0, a_1, \dots, a_{2n-1}, b_0, b_1, \dots, b_n\}$

by  $0 + x = 0 = x + 0$ ,  $a_{2i} + a_{2i+1} = b_n$ , and  $a_{2i+1} + a_{2i+2} = b_i$  (index addition mod  $2n$ )

with all other sums undefined.



**Claim 1.** For all  $n > 1$  the formula  $q_n$  holds in  $\mathbf{ISP}(P_1)$  but fails in  $Q_n$ .

**Proof.**

Suppose the premises hold in  $P_1$  but  $x_0 \neq x_2$ .

If  $x_0 = 0$  then  $x_2 = 1$ , and since  $x_1 + x_2$  is defined, it follows that  $x_1 = 0$ .

However, this contradicts  $x_0 + x_1 = x_2 + x_3$ .

If  $x_0 = 1$  then  $x_1 = 0$  since  $x_0 + x_1$  is defined, and  $x_2 = 0$  since we are assuming  $x_0 \neq x_2$ .

Now  $x_0 + x_1 = x_2 + x_3$  implies  $x_3 = 1$ , and since  $x_3 + x_4$  is defined, we have  $x_4 = 0$ .

If  $n = 2$  then  $x_4 = x_0$  since indices are calculated modulo 4, but this contradicts  $x_0 = 1$ .

Assume we have shown  $x_{2i-1} = 1$  and  $x_{2i} = 0$ .

Then  $x_{2i-2} + x_{2i-1} = x_{2i} + x_{2i+1}$  implies  $x_{2i+1} = 1$ , hence  $x_{2i+2} = 0$ .

By induction we have  $x_{2n} = 0$ , which again contradicts  $x_0 = 1$ .

To see that  $q_n$  fails in  $Q_n$ , let  $x_i = a_i$ . □

**Claim 2.** The ultraproduct  $(\prod_{n \in \omega} Q_n) / \mathcal{U}$  is in **ISP**( $P_1$ ) for any nonprincipal ultrafilter  $\mathcal{U}$  on  $\omega$ , hence **ISP**( $P_1$ ) is not finitely axiomatizable.

**Proof.**

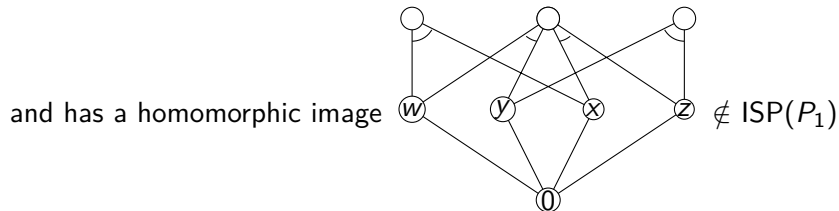
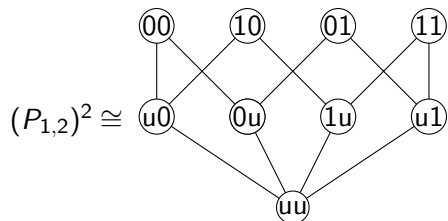
(outline) In each  $Q_n$ , the term  $a_i + a_j$  is defined iff  $j = i \pm 1 \pmod{2n}$ , and the terms  $a_{2i} + a_{2i+1}$  are all equal to  $b_n$ .

This same structure holds in the ultraproduct, except that the index addition is now done in  $\mathbb{Z}$ .

To see that the ultraproduct is in **ISP**( $P_1$ ), it suffices to embed this algebra in the powerset algebra  $\mathcal{P}(\omega)$  with disjoint union as partial operation and the empty set as identity.

Let  $a_0 = 2\omega$  (= even numbers) and  $a_1 = \omega - a_0$ . In general, let  $a_k = 2k\omega$  and  $a_{k+1} = \omega - a_k$ , and check that this map is an embedding. □

ISP( $P_1$ ) is not closed under H



## Algebras in $\text{ISP}(P_1)$ satisfy no congruence equations

$+$	$0$	$1$	$\dots$	$n$
$0$	$0$	$1$	$\dots$	$n$
$1$	$1$	$-$	$\dots$	$-$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$n$	$n$	$-$	$\dots$	$-$

- Consider the partial algebra  $P_{1,n} =$
- Can identify any two non-zero elements without collapsing any others
- Can identify any non-zero element with 0 without collapsing any others
- Therefore  $\text{Con}(P_{1,n}) = \text{Eq}(n) =$  the lattice of equivalence relations on an  $n$  element set
- Any lattice equation fails in  $\text{Eq}(n)$  for some  $n$