Algorithms For Integral Matrix Groups

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Matrix Group Calculations

Matrix groups over commutative ring, given by (finite number) of generating matrices.

What can we say about such groups?

Computers crave finiteness !

Over finite fields: ***matrix group recognition***

Uses: Divide-and-conquer approach. Data structure ***composition tree***. Reduction to simple groups.

Effective Homomorphisms, recursion to kernel, image.

Hasse Principle

Instead of working (globally) over $\mathbb{Z}$, work (locally) modulo different coprime numbers, combine (Paradigm: Chinese Remainder Theorem)

The purpose of this talk is to show this principle applies to a certain class of integral matrix groups.

Matrix Groups Over $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$

First consider $m = p^2$. ($m = p^a$ ditto.)

Reduction mod p gives hom. φ: $SL_n(\mathbb{Z}_m) \to SL_n(\mathbb{Z}_p)$.

Kernel $\{I + pA | A \in \mathbb{Z}_p{}^{n \times n}\}$. Note: $\det(I + pA) = 1 + p \cdot \mathrm{Tr}(A)$.

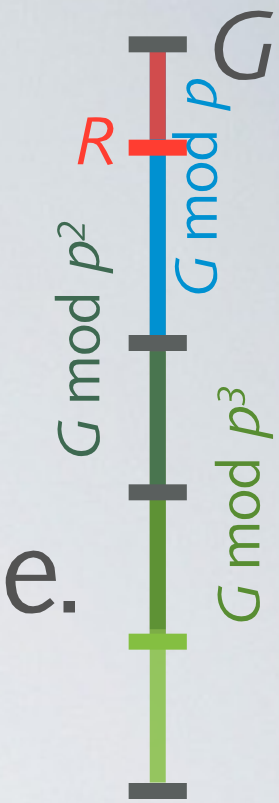
Multiplication:

$$(I+pA)(I+pB) = I + p(A+B) + p^2 \ldots \equiv I + p(A+B) \bmod m$$

is by addition of the the A-parts modulo p.

(Under map $A \mapsto I + pA$, $\ker\varphi$ is adjoint module in Lie-sense.)

Working With $G \leq GL_n(\mathbb{Z}_m)$



If $m=p^a$, first consider image H of G modulo p.

- Matrix group recognition on H. Get comp. tree. Split in radical factor and solvable radical.
- Presentation gives (module) generators of kernel. Consider p/p^2 layer as F_p-vector space. Basis with Spinning Algorithm.
- Combine to presentation of G mod p^2
- Iterate on p^2/p^3 kernel etc.

Multiple Primes

If m is product or multiple primes, G is a subdirect product of its images modulo prime powers.

To get standard solvable radical data structure:

- Consider images H_p modulo each prime.
- Combine radical factor homomorphisms ρ_p for different primes to direct product of images.
- Combine the PCGS for the radicals for different primes.
- Extend PCGS through the extra layers if there are higher prime powers in m. (Take new kernel generators each time, linear algebra on $1/p(I-x)$.)

Result: Data structure, in particular order, for $G \leq GL_n(\mathbb{Z}_m)$.

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PCGS: polycyclic generating set — data structure for solvable group that combines bases for different vector space layers into object that can get coefficients.

Multiple Primes

Careful: This can still be subdirect.

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Result: Data structure, in particular order, for $G \leq GL_n(\mathbb{Z}_m)$.

```
gap> LoadPackage("matgrp"); # available for GAP 4.8.3
   [...]
gap> g:=SL(3,Integers mod 1040);
SL(3,Z/1040Z)
gap> ff:=FittingFreeLiftSetup(g);;
gap> Size(g);
849852961151281790976000
gap> Collected(RelativeOrders(ff.pcgs));
[ [ 2, 24 ], [ 3, 1 ] ]
gap> m:=MaximalSubgroupClassReps(g);;time;
24631 #24 seconds
gap> List(m,x->Size(g)/Size(x));
[ 256, 7, 7, 8, 183, 183, 938119, 1476384, 3752476,
  123708, 123708, 123708, 31, 31, 3100, 3875, 4000 ]
```

Arithmetic Groups

Roughly: Discrete subgroup of Lie Group, defined by arithmetic properties on matrix entries(e.g. det=1, preserve form).

Definition: G linear algebraic group, over number field K. An *arithmetic group* is $\Gamma < G$, such that for integers $\mathcal{O} < K$ the intersection $\Gamma \cap G(\mathcal{O})$ has finite index in both intersectants.

Prototype: Subgroups of $SL_n(\mathbb{Z})$, $Sp_{2n}(\mathbb{Z})$ of finite index.

Applications: Number Theory (Automorphic Forms), Topology, Expander Graphs, String theory, ...

Theoretical algorithms for problems, such as conjugacy, known, but infeasible in practice.

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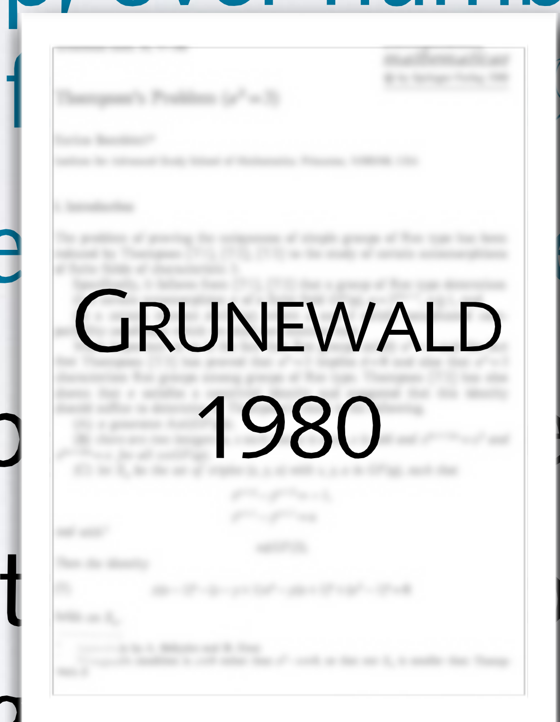

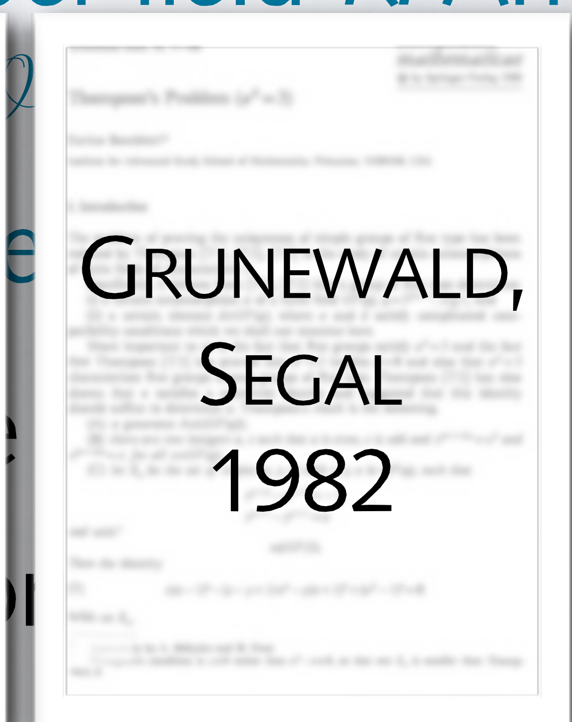

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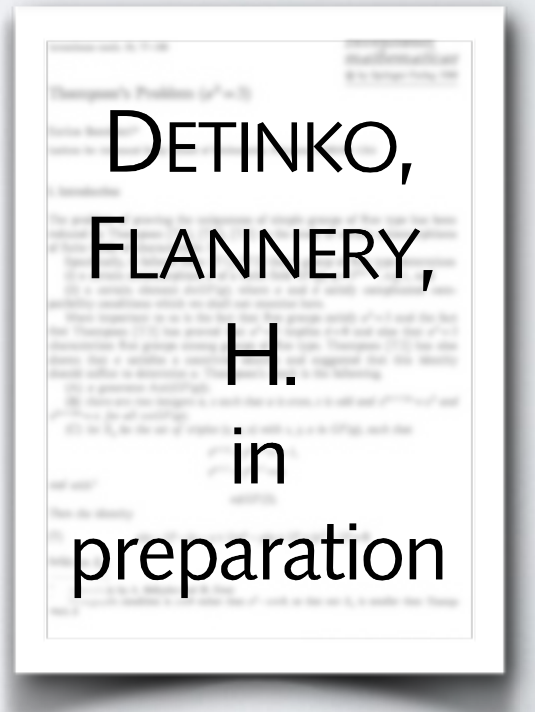
Subgroups Of $SL_n(\mathbb{Z})$, $Sp_{2n}(\mathbb{Z})$

Take subgroup $G < SL_n(\mathbb{Z})$ (or Sp_{2n}) given by finite set of generators. G is arithmetic if it has finite index.

- Can we determine whether G has finite index?
- If G has finite index, can we determine it?

Here: Only SL case.

Joint work with ALLA DETINKO, DANE FLANNERY (St. Andrews / NUI Galway).



Proving Finite Index

Consider $SL_n(\mathbb{Z})$ as finitely presented group.

Generators: Elementary matrices.

Relators (obvious ones: orders of products, commutators of generators) are known.

Write generators of G as words in these generators (Gaussian Elimination. Often better: Words in images mod m for sufficiently large m).

Enumerate cosets (Todd-Coxeter). If the index is finite this process will terminate, and give the correct index.

Proving Finite Index

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Caveat: Coset enumeration is a *method*, not an *algorithm*: run-time cannot be bounded. If index is infinite the process does not terminate.

Enumerate cosets (Todd-Coxeter). If the index is finite this process will terminate, and give the correct index.

Second Caveat

The obstacles of coset enumeration are inherent to the problem.

$SL_n(\mathbb{Z})$ contains free subgroups if $n \geq 3$, and it is thus impossible to have an decision algorithm that is guaranteed to answer *whether* elements generate a subgroup of finite index.

Thus assume an *oracle* promises finite index (or hope to be lucky).

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on Turing-machine
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Entscheidungsproblem

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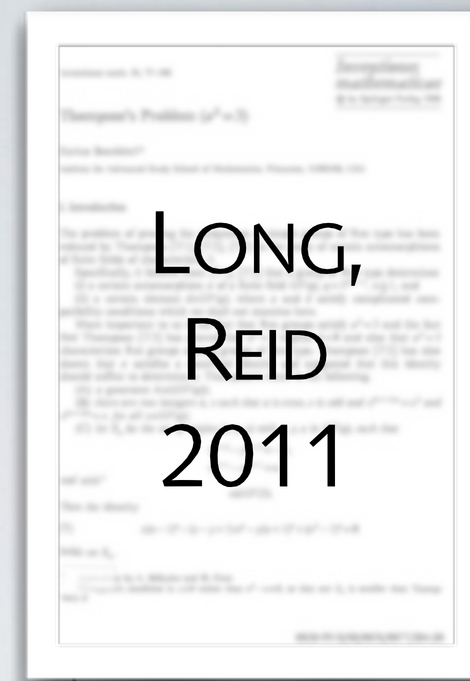
Easy Example



Let $SL_3(\mathbb{Z}) \geqq \beta_T =$

$$\left\langle \begin{pmatrix} -1+T^3 & -T & T^2 \\ 0 & -1 & 2T \\ -T & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ -T^2 & 1 & -T \\ T & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & T^2 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle,$$

then $[SL_3(\mathbb{Z}): \beta_{-2}]=3670016.$ (Barely) doable.

But $[SL_3(\mathbb{Z}): \beta_7]=24193282798937316960$

$=2^5 3^4 5 \cdot 7^{10} 19 \cdot 347821 \sim 2^{64}.$ Hopeless.

Congruence Subgroups

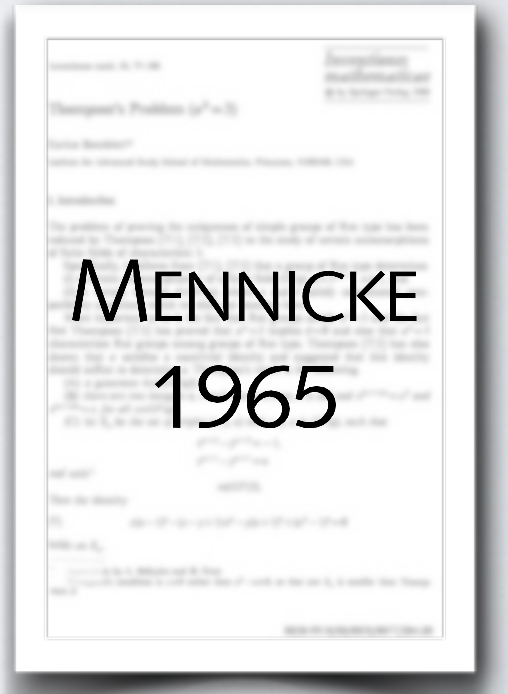
The *m*-th congruence subgroup $\Gamma_m \leq \mathrm{SL}_n(\mathbb{Z})$ is the kernel of the reduction φ_m modulo m. Image is $\mathrm{SL}_n(\mathbb{Z}_m)$.

If $G \leq \mathrm{SL}_n(\mathbb{Z})$ has finite index, there exists integer l such that $\Gamma_l \leq G$. The smallest such l is called the *level* of G.

Then $[\mathrm{SL}_n(\mathbb{Z}):G]=[\mathrm{SL}_n(\mathbb{Z}_l) : \varphi_l(G)]$.

Calculate this second index from generators of G modulo l.

Thus sufficient to find level to get index.



Strategy

Consider congruence images $\varphi_m(G) < SL_n(\mathbb{Z}_m)$ for increasing values of m to find level l of G.

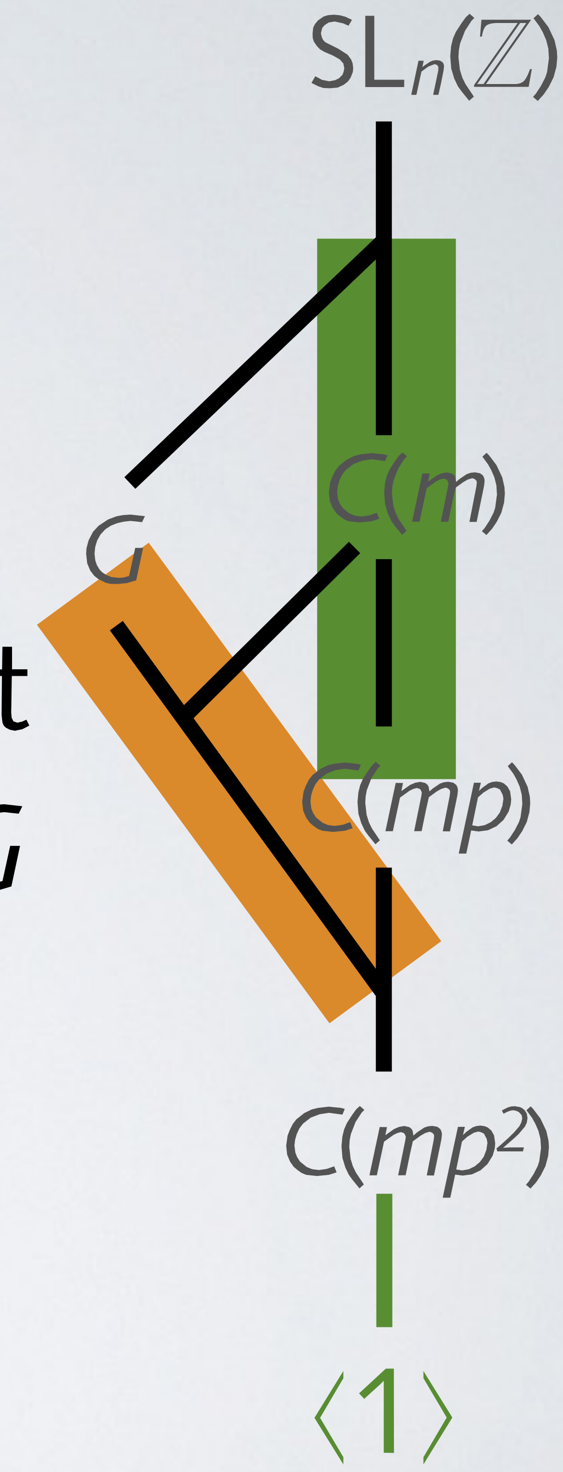
- Find the primes dividing l
- Find the prime powers dividing l
- Criterion on whether $I_m = [SL_n(\mathbb{Z}_m) : \varphi_m(G)]$ increases.

Same Index

Let $G \leq SL_n(\mathbb{Z})$ and $C(m) = \ker \varphi_m$.

If for a given m and prime p we have that $I_m = I_{mp}$ but $I_{mp} \neq I_{mp^2}$, then (modulo mp^2) G contains a supplement to $C(mp)$.

We show such supplements do not exist, thus a stable index remains stable.



Kernel Supplements

Let p be prime, $a \geq 2$, $m=p^a$ and $H=\mathrm{SL}(n,\mathbb{Z}_m)$ for $n \geq 2$ (or $H=\mathrm{Sp}(2n, \mathbb{Z}_m)$ for $n \geq 1$). Let $C(k) \triangleleft H$ kernel mod k.

Theorem: (D-F-H.) $C(p^{a+1})$ has no proper supplement in $C(p^a)$.

Theorem: (Beisiegel 1977, Weigel 1995, …,D-F-H.)

Let $a=2$. $C(p)$ has a supplement in H if and only if

(a) $H=\mathrm{SL}(2,\mathbb{Z}_4)$, $\mathrm{SL}(2,\mathbb{Z}_9)$, $\mathrm{SL}(3,\mathbb{Z}_4)$, or $\mathrm{SL}(4,\mathbb{Z}_4)$.

(b) $H=\mathrm{Sp}(2,\mathbb{Z}_4)$, $\mathrm{Sp}(2,\mathbb{Z}_9)$.

<u>Proof</u>: Small cases/counterexample by explicit calculation. Use nice elements to show supplement contains kernel.

Index Algorithm

Assume that G has (unknown) finite index and level l. Assume we know the set $\mathscr{P}$ of primes dividing l.

1. Set m=lcm(4,$\prod \mathscr{P}$).
2. While for any $p \in \mathscr{P}$ we have

   $[SL_n(\mathbb{Z}_m):\varphi_m(G)]<[SL_n(\mathbb{Z}_{pm}):\varphi_{pm}(G)]$, set $m:=pm$.
3. Repeat until index is stable, level divides m.

Show also that one can work prime-by-prime.

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   $_m(G)$], set $m:=pm$.

   A group projecting onto $\mathrm{PSL}_n(p)$ has only trivial subdirect products with subgroups of $\mathrm{PSL}_n(q)$

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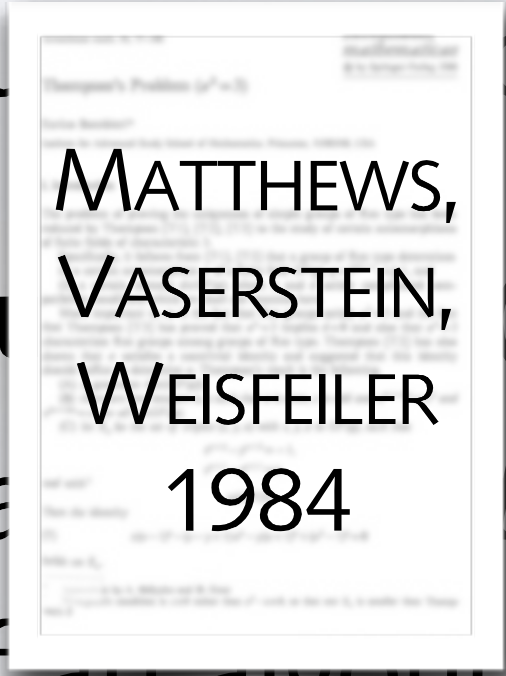
Show also that one can work prime-by-prime.

Strong Approximation

Theorem: Let $G \leq \mathrm{SL}_n(\mathbb{Z})$. If there is a prime $p > 2$ such that G mod p=$\mathrm{SL}_n(p)$, then this holds for almost all primes. Such a group is called *Zariski-dense* (which agrees with the usu … or



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Caveat: There are dense subgroups of $\mathrm{SL}_n(\mathbb{Z})$ that do not have finite index. (They are called *thin*.) This goes back to the impossibility of an algorithm that determines whether G has finite index.

Finding The Set Of Primes

Theorem: Let $n \geq 3$ and suppose G has finite index. The set $\mathscr{P}$ of primes dividing the level l consists of those primes p for which

1. $p>2$ and G mod $p \neq SL_n(p)$, or
2. $p=2$ and G mod 4 $\neq SL_n(\mathbb{Z}_4)$

Proof: If other primes divided the level, there would be a supplement modulo p^2 (or 8).

Representations

G group, F field.

A representation $\rho{:}G{\to}\mathrm{GL}_n(F)$ is *irreducible* if no proper subspace of F^n is invariant under $G\rho$.

It is *absolutely irreducible* if the same holds for subspaces of K^n for any algebraic extension K of F.

Theorem: ρ is absolutely irreducible, iff the matrices $G\rho$ span $F^{n\times n}$.

Irreducible modulo Prime

Let $\rho: G \rightarrow \mathrm{GL}_n(\mathbb{Z})$ a representation that is absolutely irreducible modulo one prime.

The $\mathbb{Z}$-lattice $L \leq \mathbb{Z}^{n\times n}$ spanned by $G\rho$ has rank n^2.

$G\rho$ is absolutely irreducible modulo each prime that does not divide discriminant of L (i.e. almost all).

To find these primes: Approximate L with (random) elements of $G\rho$ until full rank.

Transvections

Arithmeticity implies the existence of *transvections*, elements $t \in G$ with $\mathrm{rk}\,(t-1)=1$.

For such an element let $N = \langle t \rangle^G$ be the normal closure.

Let ρ be reduction modulo prime p with $G^\rho = SL_n(p)$. If $t^\rho - 1 \neq 0$, then t^ρ is transvection and N^ρ is absolutely irreducible. For odd n this implies it is SL.

Let L be the $\mathbb{Z}$-lattice spanned by (elements of) N.

Then $\mathscr{P}$ (primes with G mod $p \neq SL_n(p)$) consists of primes dividing lcm(disc.L, gcd of entries of $t-1$).

```
gap> g:=BetaT(7);
<matrix group with 3 generators>
gap> t:=b1beta(g); # transvection from Long/Reid paper
[ [-685,14,-98], [-16807,344,-2401], [2401,-49,344] ]
gap> RankMat(t-t^0);
1
gap> PrimesForDense(g,t,1);time;
[ 7, 1021 ]
60
gap> MaxPCSPrimes(g,[7,1021]);time;
Try 7 7
Try 49 7
Try 343 7
Try 343 1021
Try 350203 1021
[350203, 24193282798937316960 ] #Proven Index in SL
291395 # about 5 minutes
```

General Case

If we have no transvections but know index is finite:

Use other representations to identify primes.

Note: Representations of $SL_n(p)$ are given by polynomials on matrix entries, come from $SL_n(\mathbb{Z})$

Identify Subgroups

Take a set $\mathscr{R}$ of polynomial representations of $SL_n(\mathbb{Z})$, such that:

1. For every $\alpha \in \mathscr{R}$ the reduction α_p: $SL_n(p) \to \mathbb{Z}_p{}^{m \times m}$ modulo p is a well-defined representation.
2. For prime p sufficiently large (i.e. p>const(n)), α_p is absolutely irreducible.
3. For every maximal $M < SL_n(p)$, there exists $\alpha \in \mathscr{R}$, such that α_p is not abs. irreducible on M.

Existence: Steinberg representation.

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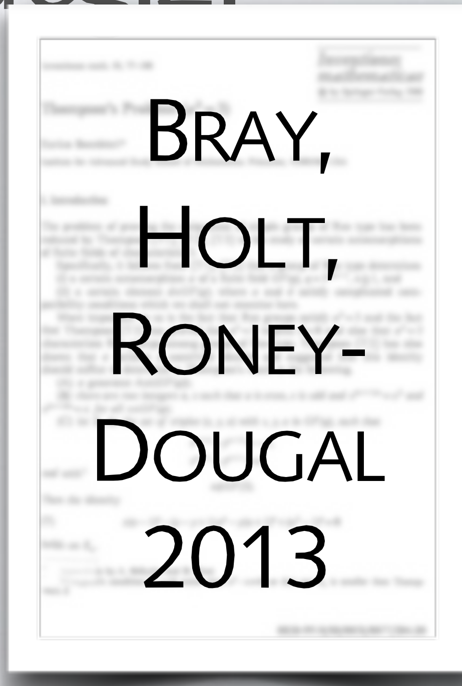
Small n

Let $\mathscr{R}$ be

1. Actions on homogeneous poly.^{s} of degree ≤ 4
2. Antisymmetric square of natural representation.
3. For n=3 (for 3.A_6) a 15-dimensional constituent of the symmetric square of polynomials deg.2.

Then the conjecture holds for $n \leqq 11$ if $p>4$.

Proof by inspection of lists of maximals.



Algorithm For Primes

For each polynomial representation $\rho \in \mathscr{R}$:

- Form (random) elements of G_ρ, span lattice L of full rank $\deg(\rho)^2$.
- Find primes dividing $\mathrm{disc}(L)$.

```
gap> g:=Group([[778,2679,665],[323,797,665],
>        [6674504920,-1557328,34062304949]],
>   [[-274290687,140904793,1960070592 ],[853,4560,294],
>        [151,930,209]]);;
gap> InterestingPrimes(g);   # about 12 hours
irrelevant prime 11
i=1 Pol1
i=2 Pol2 ->[ 53 ]
i=3 Pol3
i=4 rep15 ->[ 19 ]
[ 2, 3, 5, 19, 53 ]

gap> MaxPCSPrimes(g,[2,3,5,19,53]);
Try 1 2, Try 2 2, Try 2 3, Try 6 3, Try 6 5, Try 30 5
Try 30 19, Try 570 19, Try 570 53, Try 30210 53
Index is 5860826241898530299904=[ [ 2,13 ], [ 3,4 ],
 [ 13,3 ], [ 19,3 ], [ 31,1 ], [ 53,3 ], [ 127,1 ] ]
[ 30210, 5860826241898530299904 ]
```

Stronger Approximation, Again

As a corollary we easily get a proof of strong approximation, using our definition of Zariski dense:

Theorem: A subgroup of $SL_n(\mathbb{Z})$ is dense if and only if it surjects onto $SL_n(p)$ for at least one prime $p \geq 3$.

Proof: If G surjects onto $SL_n(p)$ for a prime, the lattice spanned by the Steinberg representation has full rank modulo p, thus full rank over $\mathbb{Z}$.

Open Questions, Directions

- Good set $\mathscr{R}$ of representations (small degree, easy construction)?
- Analog result for Sp or other classical groups? (are there polynomial representations)?
- Better arithmetic for matrices over $\mathbb{Z}_m$.
- Algorithm finds arithmetic closure for dense subgroups. Use this to *prove* finite index in certain cases more efficiently than coset enumeration?