The Rectangularity Theorem of Libor Barto and Marcin Kozik

with applications to small CIBs

William DeMeo

williamdemeo@gmail.com

joint work with

Cliff Bergman and Josh Thompson

Iowa State University

Workshop on Algebras and Algorithms

University of Colorado, Boulder, May 19–22

slides available at

https://github.com/williamdemeo/Talks

Definition of CSP
(naive version)

Input

- *variables:* $\mathcal{V} = \{v_1, v_2, \ldots\}$
- *domain:* $\mathcal{D}$
- *constraints:* $C_1, C_2, \ldots$

Output

- "yes" if there is a *solution*

  $f : \mathcal{V} \to \mathcal{D}$ (an assignment of values to variables that satisfies all C_i)

- "no" otherwise

DEFINITION OF CSP
(JADED VERSION)

$\mathbf{A} = \langle A, \mathcal{F} \rangle$ is a finite idempotent algebra, Sub$(\mathbf{A})$ is all subuniverses of $\mathbf{A}$.

In this talk $\text{CSP}(\mathbf{A})$ denotes the following decision problem:

An *instance of degree* n of $\text{CSP}(\mathbf{A})$ is the tuple $\langle \mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{R} \rangle$

- *variables* $\mathcal{V} = \{0, 1, \dots, n-1\}$;
- *domains* $\mathcal{A} = \{\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_{n-1}\} \subset$ Sub$(\mathbf{A})$ (one for each variable)
- *scope functions* $\mathcal{S} = (\mathbf{s}_0, \mathbf{s}_1, \dots, \mathbf{s}_{p-1})$ with *constraint arities* $\text{ar}(\mathcal{S}) = (m_0, m_1, \dots, m_{p-1})$;
- *constraint relations* $\mathcal{R} = (\mathbf{R}_0, \mathbf{R}_1, \dots, \mathbf{R}_{p-1})$, where

$$\mathbf{R}_i \leqslant \mathbf{A}_{\mathbf{s}_i(0)} \times \mathbf{A}_{\mathbf{s}_i(1)} \times \cdots \times \mathbf{A}_{\mathbf{s}_i(m_i-1)}.$$

A *solution* to $\langle \mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{R} \rangle$ is an assignment $f : \mathcal{V} \to A$ of values to variables that satisfies all constraints. That is,

$$f \in \Pi_{\mathcal{V}} A_j \quad \text{and} \quad \text{Proj}_{\mathbf{s}_i} f \in \mathbf{R}_i, \quad \text{for each } 0 \leqslant i < p.$$

Definition of CSP

(jaded version)

$\mathbf{A} = \langle A, \mathcal{F} \rangle$ is a finite idempotent algebra, Sub$(\mathbf{A})$ is all subuniverses of $\mathbf{A}$.

In this talk $\mathrm{CSP}(\mathbf{A})$ denotes the following decision problem:

An *instance of degree* n of $\mathrm{CSP}(\mathbf{A})$ is the tuple $\langle \mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{R} \rangle$

- *variables* $\mathcal{V} = \{0, 1, \ldots, n-1\}$;
- *domains* $\mathcal{A} = \{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{n-1}\} \subset$ Sub$(\mathbf{A})$ (one for each variable)
- *scope functions* $\mathcal{S} = (\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_{p-1})$ with *constraint arities* $\mathrm{ar}(\mathcal{S}) = (m_0, m_1, \ldots, m_{p-1})$;
- *constraint relations* $\mathcal{R} = (\mathbf{R}_0, \mathbf{R}_1, \ldots, \mathbf{R}_{p-1})$, where

$$\mathbf{R}_i \leqslant \mathbf{A}_{\mathbf{s}_i(0)} \times \mathbf{A}_{\mathbf{s}_i(1)} \times \cdots \times \mathbf{A}_{\mathbf{s}_i(m_i-1)}.$$

A *solution* to $\langle \mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{R} \rangle$ is an assignment $f : \mathcal{V} \to A$ of values to variables that satisfies all constraints. That is,

$$f \in \Pi_{\mathcal{V}} A_j \quad \text{and} \quad \mathrm{Proj}_{\mathbf{s}_i} f \in \mathbf{R}_i, \quad \text{for each } 0 \leqslant i < p.$$

Notation: $\underline{n} = \{0, 1, \ldots, n-1\}$, so the i-th scope has type $\mathbf{s}_i : \underline{m_i} \to \underline{n}$ and

$$\mathrm{Proj}_{\mathbf{s}_i} f = f \circ \mathbf{s}_i$$

Example 1

...thanks, Ross!

Let $\mathbf{A} = \langle\{0,1\},\{f\}\rangle$, where

$$f(x,y,z) = x+y+z \pmod 2.$$

Consider the ternary relations

$$R_0 = \{(0,0,0),(1,1,0),(1,0,1),(0,1,1)\}$$

$$R_1 = \{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$$

EXAMPLE 1

...THANKS, ROSS!

Let $\mathbf{A} = \langle \{0,1\}, \{f\} \rangle$, where

$$f(x,y,z) = x + y + z \pmod 2.$$

Consider the ternary relations

$$R_0 = \{(0,0,0), (1,1,0), (1,0,1), (0,1,1)\}$$

$$R_1 = \{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}$$

Each $\mathbf{R}_i = \langle R_i, \{f\} \rangle$ is a subalgebra of $\mathbf{A}^3$

Example 1

...thanks, Ross!

Let $\mathbf{A} = \langle \{0,1\}, \{f\} \rangle$, where

$$f(x,y,z) = x + y + z \pmod 2.$$

Consider the ternary relations

$$R_0 = \{(0,0,0),(1,1,0),(1,0,1),(0,1,1)\}$$

$$R_1 = \{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$$

Each $\mathbf{R}_i = \langle R_i, \{f\} \rangle$ is a subalgebra of $\mathbf{A}^3$ ...in fact, they're subdirect.

notation: $\mathbf{R}_i \leqslant_{\mathrm{sd}} \mathbf{A}^3$

Example 1

...thanks, Ross!

Let $\mathbf{A} = \langle \{0,1\}, \{f\} \rangle$, where

$$f(x,y,z) = x + y + z \pmod 2.$$

Consider the ternary relations

$$R_0 = \{(0,0,0), (1,1,0), (1,0,1), (0,1,1)\}$$

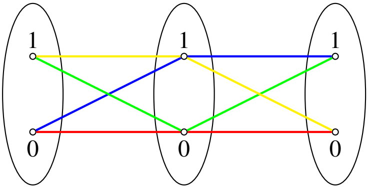
$$R_1 = \{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}$$

Each $\mathbf{R}_i = \langle R_i, \{f\} \rangle$ is a subalgebra of $\mathbf{A}^3$ ...in fact, they're subdirect.

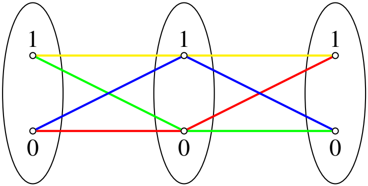
notation: $\mathbf{R}_i \leqslant_{\mathrm{sd}} \mathbf{A}^3$

So we have a degree 3 instance of $\mathrm{CSP}(\mathbf{A})$, where

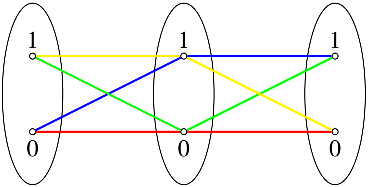
- variables: $\mathcal{V} = \{0,1,2\}$
- domains: $A_i = \{0,1\}, \quad i = 0,1,2$
- scope functions: the identity on $\{0,1,2\}$
- constraint relations: $\mathbf{R}_0$ and $\mathbf{R}_1$



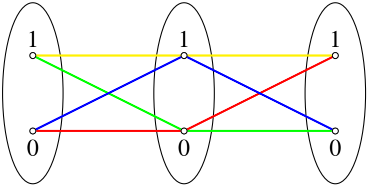
(lines are elements of R_0)



(lines are elements of R_1)



(lines are elements of R_0)



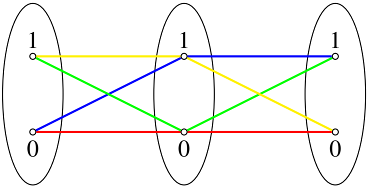
(lines are elements of R_1)

Notice for all $i, j \in \{0, 1, 2\}$,

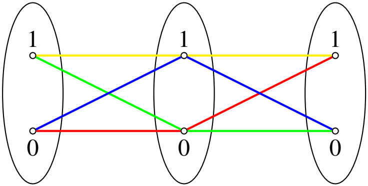
$$\mathrm{Proj}_{ij} R_0 = \mathrm{Proj}_{ij} R_1$$

EXAMPLE 1

∩ AND POTATOES



(lines are elements of R_0)



(lines are elements of R_1)

Notice for all $i, j \in \{0, 1, 2\}$,

$$\operatorname{Proj}_{ij} R_0 = \operatorname{Proj}_{ij} R_1$$

...yet $R_0 \cap R_1 = \emptyset$.

EXAMPLE 2

...THANKS, CLIFF!

Let $\mathbf{A} = \langle \{0,1\}, \{m\} \rangle$, where $m : A^3 \to A$ is a majority operation,

$$m(x,x,y) \approx m(x,y,x) \approx m(y,x,x) \approx x.$$

Let $\mathbf{R}_0, \mathbf{R}_1 \leqslant_{\mathrm{sd}} \mathbf{A}^3$ with universes

$$R_0 = \{(0,0,0), (0,0,1), (0,1,0), (1,0,0)\},$$
$$R_1 = \{(0,1,1), (1,0,1), (1,1,0), (1,1,1)\}.$$

This describes the instance of $\mathrm{CSP}(\mathbf{A})$ with

- variables: $\mathcal{V} = \{0,1,2\}$
- domains: $A_i = \{0,1\}, \quad i = 0,1,2$
- scope functions: the identity on $\{0,1,2\}$
- constraint relations: $\mathbf{R}_0$ and $\mathbf{R}_1$

Some conveniences

Restrict attention to instances where all constraint relation are subdirect,

$$\mathbf{R}_i \leqslant_{\mathrm{sd}} \mathbf{A}_{\mathbf{s}_i(0)} \times \mathbf{A}_{\mathbf{s}_i(1)} \times \cdots \times \mathbf{A}_{\mathbf{s}_i(m_i-1)}$$

SOME CONVENIENCES

Restrict attention to instances where all constraint relation are subdirect,

$$\mathbf{R}_i \leqslant_{\mathrm{sd}} \mathbf{A}_{\mathbf{s}_i(0)} \times \mathbf{A}_{\mathbf{s}_i(1)} \times \cdots \times \mathbf{A}_{\mathbf{s}_i(m_i-1)}$$

Could visualize $(\mathbf{s}_i, \mathbf{R}_i)$ as specifying a subalgebra of the full product $\Pi_{\mathcal{V}} \mathbf{A}_j$

$$[\![\mathbf{s}_i, \mathbf{R}_i]\!] = \{\mathbf{a} \in \Pi_{j \in \mathcal{V}} A_j \mid \mathrm{Proj}_{\mathbf{s}_i} \mathbf{a} \in \mathbf{R}_i\}$$

(thanks again, Ross!)

Convenient because now solutions are the elements in $\bigcap_{i \in V} [\![\mathbf{s}_i, \mathbf{R}_i]\!]$.

SOME CONVENIENCES

Restrict attention to instances where all constraint relation are subdirect,

$$\mathbf{R}_i \leqslant_{\text{sd}} \mathbf{A}_{\mathbf{s}_i(0)} \times \mathbf{A}_{\mathbf{s}_i(1)} \times \cdots \times \mathbf{A}_{\mathbf{s}_i(m_i-1)}$$

Could visualize $(\mathbf{s}_i, \mathbf{R}_i)$ as specifying a subalgebra of the full product $\Pi_{\mathcal{V}}\mathbf{A}_j$

$$[\![\mathbf{s}_i, \mathbf{R}_i]\!] = \{\mathbf{a} \in \Pi_{j\in\mathcal{V}} A_j \mid \mathrm{Proj}_{\mathbf{s}_i} \mathbf{a} \in \mathbf{R}_i\}$$

(thanks again, Ross!)

Convenient because now solutions are the elements in $\bigcap_{i\in V}[\![\mathbf{s}_i, \mathbf{R}_i]\!]$.

BUT input size is not a function of these “full” subdirect products!

(Input size could be defined as the length of a string of all tuples in scopes and constraint relations of the instance.)

End of Act I

First Intermission

pause...

...draw more potatoes...

...give audience chance to escape.

Absorption Theory (for mortals)

Let $\mathbf{A} = \langle A, F^{\mathbf{A}} \rangle$ be a finite algebra in a Taylor variety.

Let $t \in \mathrm{Clo}(\mathbf{A})$ be a k-ary term operation.

A subalgebra $\mathbf{B} \leqslant \mathbf{A}$ is *absorbing in* $\mathbf{A}$ *with respect to* t if

$$a \in A,\ b_i \in B \quad \Longrightarrow \quad t^{\mathbf{A}}(b_0, \dots, b_{j-1}, a, b_{j+1}, \dots, b_{k-1}) \in B \quad (\text{all } j \in \underline{k})$$

Absorption Theory (for mortals)

Let $\mathbf{A} = \langle A, F^{\mathbf{A}} \rangle$ be a finite algebra in a Taylor variety.

Let $t \in \operatorname{Clo}(\mathbf{A})$ be a k-ary term operation.

A subalgebra $\mathbf{B} \leqslant \mathbf{A}$ is *absorbing in* $\mathbf{A}$ *with respect to* t if

$$a \in A,\ b_i \in B \quad \Longrightarrow \quad t^{\mathbf{A}}(b_0, \dots, b_{j-1}, a, b_{j+1}, \dots, b_{k-1}) \in B \quad (\text{all } j \in \underline{k})$$

Equivalently, $t^{\mathbf{A}}[B^{j-1} \times A \times B^{k-j}] \subseteq B$, for all $0 \leqslant j < k$, that is,

$$(\mathbf{b}, \mathbf{a}, \mathbf{b}') \in B^{j-1} \times A \times B^{k-j} \quad \Longrightarrow \quad t^{\mathbf{A}}(\mathbf{b}, \mathbf{a}, \mathbf{b}') \in B.$$

Absorption Theory (for mortals)

Let $\mathbf{A} = \langle A, F^{\mathbf{A}} \rangle$ be a finite algebra in a Taylor variety.

Let $t \in \mathrm{Clo}(\mathbf{A})$ be a k-ary term operation.

A subalgebra $\mathbf{B} \leqslant \mathbf{A}$ is *absorbing in* $\mathbf{A}$ *with respect to* t if

$$a \in A,\ b_i \in B \quad \Longrightarrow \quad t^{\mathbf{A}}(b_0, \dots, b_{j-1}, a, b_{j+1}, \dots, b_{k-1}) \in B \quad (\text{all } j \in \underline{k})$$

Equivalently, $t^{\mathbf{A}}[B^{j-1} \times A \times B^{k-j}] \subseteq B$, for all $0 \leqslant j < k$, that is,

$$(\mathbf{b}, \mathbf{a}, \mathbf{b}') \in B^{j-1} \times A \times B^{k-j} \quad \Longrightarrow \quad t^{\mathbf{A}}(\mathbf{b}, \mathbf{a}, \mathbf{b}') \in B.$$

Notation:

$\mathbf{B} \triangleleft \mathbf{A}$ means $\mathbf{B}$ is absorbing in $\mathbf{A}$ with respect to some term.

To be explicit about the term, $\mathbf{B} \triangleleft_t \mathbf{A}$.

$\mathbf{B} \triangleleft\!\triangleleft \mathbf{A}$ means $\mathbf{B} \triangleleft \mathbf{A}$ and B is minimal (with respect to inclusion) among absorbing subuniverses of $\mathbf{A}$.

An algebra is *absorption-free* (AF) if it has no proper absorbing subalgebras.

Where are we going with this?

“The Absorption Theorem” of Barto and Kozik (LMCS 2012)

Concerns the special class of “linked” subdirect products.

Identifies some special cases in which a subdirect product is the full product!

Where are we going with this?

"The Absorption Theorem" of Barto and Kozik (LMCS 2012)

Concerns the special class of "linked" subdirect products.

Identifies some special cases in which a subdirect product is the full product!

Theorem (Absorption Theorem)

If $\mathcal{V}$ is an idempotent locally finite variety, then TFAE

- $\mathcal{V}$ *is a Taylor variety;*
- *if* $\mathbf{A}_0, \mathbf{A}_1 \in \mathcal{V}$ *are finite idempotent absorption-free algebras and* $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ *is linked, then* $\mathbf{R} = \mathbf{A}_0 \times \mathbf{A}_1$*.*

Where are we going with this?

"The Absorption Theorem" of Barto and Kozik (LMCS 2012)

Concerns the special class of "linked" subdirect products.

Identifies some special cases in which a subdirect product is the full product!

Theorem (Absorption Theorem)

If $\mathcal{V}$ is an idempotent locally finite variety, then TFAE

- *$\mathcal{V}$ is a Taylor variety;*
- *if $\mathbf{A}_0, \mathbf{A}_1 \in \mathcal{V}$ are finite idempotent absorption-free algebras and $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ is linked, then $\mathbf{R} = \mathbf{A}_0 \times \mathbf{A}_1$.*

At Vanderbilt Shanks Workshop (2015), Barto presented more joint work with Kozik generalizing the Absorption Theorem to more than two factors.

The "Rectangularity Theorem" says roughly*, a subdirect product of simple nonabelian algebras contains the full product of minimal absorbing subalgebras.

WHERE ARE WE GOING WITH THIS?

"The Absorption Theorem" of Barto and Kozik (LMCS 2012)

Concerns the special class of "linked" subdirect products.

Identifies some special cases in which a subdirect product is the full product!

THEOREM (ABSORPTION THEOREM)

If $\mathcal{V}$ is an idempotent locally finite variety, then TFAE

- *$\mathcal{V}$ is a Taylor variety;*
- *if $\mathbf{A}_0, \mathbf{A}_1 \in \mathcal{V}$ are finite idempotent absorption-free algebras and $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ is linked, then $\mathbf{R} = \mathbf{A}_0 \times \mathbf{A}_1$.*

At Vanderbilt Shanks Workshop (2015), Barto presented more joint work with Kozik generalizing the Absorption Theorem to more than two factors.

The "Rectangularity Theorem" says roughly*, a subdirect product of simple nonabelian algebras contains the full product of minimal absorbing subalgebras.

*assuming suitable conditions under which the theorem is true.

Linked subdirect products

A subdirect product $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ is linked if for all $a, a' \in \mathrm{Proj}_0\, R$,

$$\exists\, c_0, c_2, \ldots, c_{2n} \in A_0, \quad \exists\, c_1, c_3, \ldots, c_{2n+1} \in A_1$$

such that

$$a = c_0, \quad (c_{2i}, c_{2i+1}) \in R, \quad (c_{2i+2}, c_{2i+1}) \in R, \quad c_{2n} = a'$$

LINKED SUBDIRECT PRODUCTS

A subdirect product $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ is linked if for all $a, a' \in \operatorname{Proj}_0 R$,

$$\exists\, c_0, c_2, \ldots, c_{2n} \in A_0, \quad \exists\, c_1, c_3, \ldots, c_{2n+1} \in A_1$$

such that

$$a = c_0, \quad (c_{2i}, c_{2i+1}) \in R, \quad (c_{2i+2}, c_{2i+1}) \in R, \quad c_{2n} = a'$$

[todo: insert potato diagram]

Notation:

For $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$, let η_i denote the kernel of the i-th projection of $\mathbf{R}$. That is,

$$\eta_i = \ker(\mathbf{R} \twoheadrightarrow \mathbf{A}_i) = \{(\mathbf{r}, \mathbf{r}') \in R^2 \mid \operatorname{Proj}_i \mathbf{r} = \operatorname{Proj}_i \mathbf{r}'\}$$

Let $R^{-1} = \{(y, x) \in A_1 \times A_0 \mid (x, y) \in R\}$.

Notation:

For $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$, let η_i denote the kernel of the i-th projection of $\mathbf{R}$. That is,

$$\eta_i = \ker(\mathbf{R} \twoheadrightarrow \mathbf{A}_i) = \{(\mathbf{r}, \mathbf{r}') \in R^2 \mid \operatorname{Proj}_i \mathbf{r} = \operatorname{Proj}_i \mathbf{r}'\}$$

Let $R^{-1} = \{(y, x) \in A_1 \times A_0 \mid (x, y) \in R\}$.

The following are equivalent:

- $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ is linked;
- $\eta_0 \vee \eta_1 = 1_R$;
- if $a, a' \in \operatorname{Proj}_0 R$, then (a, a') is in the transitive closure of $R \circ R^{-1}$.

Properties of Absorption I

Absorption has nice properties...

- (transitivity) $\mathbf{C} \triangleleft \mathbf{B} \triangleleft \mathbf{A} \implies \mathbf{C} \triangleleft \mathbf{A}$
- (closure under nonempty $\cap$ and finite products)

  If $\mathbf{B} \triangleleft_f \mathbf{A}$ and $\mathbf{C} \triangleleft_g \mathbf{A}$ and $B \cap C \neq \emptyset$, then $\mathbf{B} \cap \mathbf{C} \triangleleft \mathbf{A}$.

  If $\mathbf{B}_0 \triangleleft_f \mathbf{A}_0$ and $\mathbf{B}_1 \triangleleft_g \mathbf{A}_1$, then $\mathbf{B}_0 \times \mathbf{B}_1 \triangleleft_t \mathbf{A}_0 \times \mathbf{A}_1$.

  ...with respect to $t = f \star g$ in both cases.

Properties of Absorption I

Absorption has nice properties...

- (transitivity) $\mathbf{C} \triangleleft \mathbf{B} \triangleleft \mathbf{A} \implies \mathbf{C} \triangleleft \mathbf{A}$
- (closure under nonempty $\cap$ and finite products)

  If $\mathbf{B} \triangleleft_f \mathbf{A}$ and $\mathbf{C} \triangleleft_g \mathbf{A}$ and $B \cap C \neq \emptyset$, then $\mathbf{B} \cap \mathbf{C} \triangleleft \mathbf{A}$.

  If $\mathbf{B}_0 \triangleleft_f \mathbf{A}_0$ and $\mathbf{B}_1 \triangleleft_g \mathbf{A}_1$, then $\mathbf{B}_0 \times \mathbf{B}_1 \triangleleft_t \mathbf{A}_0 \times \mathbf{A}_1$.

  ...with respect to $t = f \star g$ in both cases.

If $f : A^\ell \to A$ and $g : A^m \to A$, then $f \star g$ is the ℓm-ary operation

$$f(g(a_{11}, \ldots, a_{1m}), g(a_{21}, \ldots, a_{2m}), \ldots, g(a_{\ell 1}, \ldots, a_{\ell m}))$$

Properties of Absorption I

Absorption has nice properties...

- (transitivity) $\mathbf{C} \triangleleft \mathbf{B} \triangleleft \mathbf{A} \implies \mathbf{C} \triangleleft \mathbf{A}$
- (closure under nonempty $\cap$ and finite products)

  If $\mathbf{B} \triangleleft_f \mathbf{A}$ and $\mathbf{C} \triangleleft_g \mathbf{A}$ and $B \cap C \neq \emptyset$, then $\mathbf{B} \cap \mathbf{C} \triangleleft \mathbf{A}$.

  If $\mathbf{B}_0 \triangleleft_f \mathbf{A}_0$ and $\mathbf{B}_1 \triangleleft_g \mathbf{A}_1$, then $\mathbf{B}_0 \times \mathbf{B}_1 \triangleleft_t \mathbf{A}_0 \times \mathbf{A}_1$.

  ...with respect to $t = f \star g$ in both cases.

More generally, if $\mathbf{B}_i \triangleleft_{t_i} \mathbf{A}_i$ for $0 \leqslant i < n$, then $\Pi\mathbf{B}_i \triangleleft_s \Pi\mathbf{A}_i$.

...with respect to $s = t_0 \star t_1 \star \cdots \star t_{n-1}$.

An obvious but important consequence:

A finite product of finite idempotent algebras is AF if each factor is AF.

Properties of Absorption I

Absorption has nice properties...

- (transitivity) $\mathbf{C} \lhd \mathbf{B} \lhd \mathbf{A} \implies \mathbf{C} \lhd \mathbf{A}$
- (closure under nonempty $\cap$ and finite products)

  If $\mathbf{B} \lhd_f \mathbf{A}$ and $\mathbf{C} \lhd_g \mathbf{A}$ and $B \cap C \neq \emptyset$, then $\mathbf{B} \cap \mathbf{C} \lhd \mathbf{A}$.

  If $\mathbf{B}_0 \lhd_f \mathbf{A}_0$ and $\mathbf{B}_1 \lhd_g \mathbf{A}_1$, then $\mathbf{B}_0 \times \mathbf{B}_1 \lhd_t \mathbf{A}_0 \times \mathbf{A}_1$.

  ...with respect to $t = f \star g$ in both cases.

More generally, if $\mathbf{B}_i \lhd_{t_i} \mathbf{A}_i$ for $0 \leqslant i < n$, then $\Pi\mathbf{B}_i \lhd_s \Pi\mathbf{A}_i$.

...with respect to $s = t_0 \star t_1 \star \cdots \star t_{n-1}$.

An obvious but important consequence:

A finite product of finite idempotent algebras is AF if each factor is AF.

Restriction Lemma.
If $\mathbf{B} \lhd_t \mathbf{A}$ and $\mathbf{C} \leqslant \mathbf{A}$ and $D = B \cap C \neq \emptyset$, then $\mathbf{D} \lhd \mathbf{C}$ with respect to the restriction of t to C.

Lemma (LSD 1)

If $\mathbf{B}_i \triangleleft \mathbf{A}_i$ *and* $\mathbf{R} \leqslant \Pi_i \mathbf{A}_i$ *and* $R' := R \cap \Pi_i B_i \neq \emptyset$*, then* $\mathbf{R}' \triangleleft \mathbf{R}$*.*

Proof. $\Pi \mathbf{B}_i \triangleleft_t \Pi \mathbf{A}_i$, so follows Restriction Lemma if we put $C = R$.

Lemma (LSD 2)

Suppose $\mathbf{B}_i \triangleleft\!\triangleleft \mathbf{A}_i$ *and* $\mathbf{R} \leqslant_{\mathrm{sd}} \Pi \mathbf{A}_i$*. If* $R' := R \cap \Pi B_i \neq \emptyset$*, then* $\mathbf{R}' \leqslant_{\mathrm{sd}} \Pi \mathbf{B}_i$*.*

Lemma (LSD 2)

If $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ *is linked and* $\mathbf{S} \triangleleft \mathbf{R}$*, then* $\mathbf{S}$ *is linked.*

In some simple cases we get linking from LSD Lemmas along with the following elementary

Fact. Suppose $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ and let $\eta_i = \ker(\mathbf{R} \twoheadrightarrow \mathbf{A}_i)$.

1. If $\mathbf{A}_0$ is simple, then either $\eta_0 \vee \eta_1 = 1_R$ or $\eta_0 \geqslant \eta_1$.
2. If $\mathbf{A}_0$ and $\mathbf{A}_1$ are both simple, then either $\eta_0 \vee \eta_1 = 1_R$ or $\eta_0 = 0_R = \eta_1$.

...so, if both factors are simple, then $\eta_0 \neq \eta_1$ gives the linking...

Cor 1. Let $\mathbf{A}_0$ and $\mathbf{A}_1$ be simple. If $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ and $\eta_0 \neq \eta_1$, then $\mathbf{R}$ is linked.

...and if one factor is simple nonabelian and the other abelian, linking is free!

Cor 2. If $\mathbf{A}_0$ is simple nonabelian and $\mathbf{A}_1$ abelian, then every subdirect product of $\mathbf{A}_0 \times \mathbf{A}_1$ is linked.

Absorption Theorem: Application

Suppose we add to the respective contexts of the last three results the hypothesis that the algebras live in an idempotent variety with a Taylor term...

(We will refer to such varieties as “Taylor varieties” and we call the algebras they contain “Taylor algebras.”)

...then the Absorption Theorem (in combination with facts above) yields

Lemma: Let $\mathbf{A}_0$ and $\mathbf{A}_1$ be finite Taylor algebras with $\mathbf{B}_i \lll \mathbf{A}_i$ $(i = 0, 1)$ and suppose $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ and $\eta_0 \neq \eta_1$.

(I) If $\mathbf{A}_0$ and $\mathbf{A}_1$ are simple and $R \cap (B_0 \times B_1) \neq \emptyset$, then $\mathbf{B}_0 \times \mathbf{B}_1 \leqslant \mathbf{R}$.

(II) If $\mathbf{A}_0$ is simple nonabelian and $\mathbf{A}_1$ is abelian, then $\mathbf{B}_0 \times \mathbf{A}_1 \leqslant \mathbf{R}$.

Absorption Theorem: Application

Suppose we add to the respective contexts of the last three results the hypothesis that the algebras live in an idempotent variety with a Taylor term...

(We will refer to such varieties as “Taylor varieties” and we call the algebras they contain “Taylor algebras.”)

...then the Absorption Theorem (in combination with facts above) yields

Lemma: Let $\mathbf{A}_0$ and $\mathbf{A}_1$ be finite Taylor algebras with $\mathbf{B}_i \lhd\!\lhd \mathbf{A}_i$ $(i = 0, 1)$ and suppose $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ and $\eta_0 \neq \eta_1$.

(I) If $\mathbf{A}_0$ and $\mathbf{A}_1$ are simple and $R \cap (B_0 \times B_1) \neq \emptyset$, then $\mathbf{B}_0 \times \mathbf{B}_1 \leqslant \mathbf{R}$.

(II) If $\mathbf{A}_0$ is simple nonabelian and $\mathbf{A}_1$ is abelian, then $\mathbf{B}_0 \times \mathbf{A}_1 \leqslant \mathbf{R}$.

How is this relevant to CSP?

Absorption Theorem: Application

Suppose we add to the respective contexts of the last three results the hypothesis that the algebras live in an idempotent variety with a Taylor term...

(We will refer to such varieties as "Taylor varieties" and we call the algebras they contain "Taylor algebras.")

...then the Absorption Theorem (in combination with facts above) yields

Lemma: Let $\mathbf{A}_0$ and $\mathbf{A}_1$ be finite Taylor algebras with $\mathbf{B}_i \lhd\!\lhd \mathbf{A}_i$ $(i = 0, 1)$ and suppose $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ and $\eta_0 \neq \eta_1$.

(I) If $\mathbf{A}_0$ and $\mathbf{A}_1$ are simple and $R \cap (B_0 \times B_1) \neq \emptyset$, then $\mathbf{B}_0 \times \mathbf{B}_1 \leqslant \mathbf{R}$.

(II) If $\mathbf{A}_0$ is simple nonabelian and $\mathbf{A}_1$ is abelian, then $\mathbf{B}_0 \times \mathbf{A}_1 \leqslant \mathbf{R}$.

How is this relevant to CSP?

Abelian potatoes can all go in the same sack...

Absorption Theorem: Application

Suppose we add to the respective contexts of the last three results the hypothesis that the algebras live in an idempotent variety with a Taylor term...

(We will refer to such varieties as "Taylor varieties" and we call the algebras they contain "Taylor algebras.")

...then the Absorption Theorem (in combination with facts above) yields

Lemma: Let $\mathbf{A}_0$ and $\mathbf{A}_1$ be finite Taylor algebras with $\mathbf{B}_i \lhd\!\lhd \mathbf{A}_i$ $(i = 0, 1)$ and suppose $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{A}_1$ and $\eta_0 \neq \eta_1$.

(I) If $\mathbf{A}_0$ and $\mathbf{A}_1$ are simple and $R \cap (B_0 \times B_1) \neq \emptyset$, then $\mathbf{B}_0 \times \mathbf{B}_1 \leqslant \mathbf{R}$.

(II) If $\mathbf{A}_0$ is simple nonabelian and $\mathbf{A}_1$ is abelian, then $\mathbf{B}_0 \times \mathbf{A}_1 \leqslant \mathbf{R}$.

How is this relevant to CSP?

Abelian potatoes can all go in the same sack...

...simple nonabelian potatoes cannot.

Barto and Kozik generalized the Absorption Theorem to multiple simple nonabelian factors. This is...

Barto and Kozik generalized the Absorption Theorem to multiple simple nonabelian factors. This is...

The Rectangularity Theorem.

Let $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{n-1}$ be finite algebras in a Taylor variety, $\mathbf{B}_i \lhd\!\lhd \mathbf{A}_i$, and

- at most one $\mathbf{A}_i$ abelian, and all nonabelian factors simple,
- $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1}$,
- $\eta_i \neq \eta_j$ for all $i \neq j$.
- $\mathbf{R}' = \mathbf{R} \cap (\mathbf{B}_0 \times \mathbf{B}_1 \times \cdots \times \mathbf{B}_{n-1})$ is nonempty.

Then $\mathbf{R}' = \mathbf{B}_0 \times \mathbf{B}_1 \times \cdots \times \mathbf{B}_{n-1}$.

Let $\underline{n} = \{0, 1, 2, \dots, n-1\}$.

Let $\sigma' = \underline{n} - \sigma$, when σ is a subset of $\underline{n}$.

For $\mathbf{R} \leqslant_{\mathrm{sd}} \Pi_{\underline{n}} \mathbf{A}_i$ let

$$\eta_\sigma = \ker(R \twoheadrightarrow \Pi_\sigma A_i) = \{(\mathbf{r}, \mathbf{r}') \in R^2 \mid \mathrm{Proj}_\sigma \mathbf{r} = \mathrm{Proj}_\sigma \mathbf{r}'\},$$

If $\sigma \subseteq \underline{n}$, then by $\mathbf{R} \leqslant_{\mathrm{sd}} \Pi_\sigma \mathbf{A}_i \times \Pi_{\sigma'} \mathbf{A}_i$ we mean

$$\mathbf{R} \leqslant \Pi_{\underline{n}} \mathbf{A}_i, \quad \mathrm{Proj}_\sigma \mathbf{R} = \Pi_\sigma \mathbf{A}_i, \quad \text{and} \quad \mathrm{Proj}_{\sigma'} \mathbf{R} = \Pi_{\sigma'} \mathbf{A}_i.$$

and we say that $\mathbf{R}$ is a *subdirect product of* $\Pi_\sigma \mathbf{A}_i$ and $\Pi_{\sigma'} \mathbf{A}_i$ in this case.

The subdirect product $\mathbf{R} \leqslant_{\mathrm{sd}} \Pi_\sigma \mathbf{A}_i \times \Pi_{\sigma'} \mathbf{A}_i$ is said to be *linked* if $\eta_\sigma \vee \eta_{\sigma'} = 1_R$.

We may use $\mathbf{R}_\sigma$ for $\mathrm{Proj}_\sigma \mathbf{R}$, the projection of $\mathbf{R}$ onto coordinates in σ.

From now on, *all algebras are finite and belong to the same Taylor variety*.

From now on, *all algebras are finite and belong to the same Taylor variety*.

Lemma 1.

Let $\mathbf{B}_i \lhd\!\lhd \mathbf{A}_i$ for each $i \in \underline{n}$, and let $\underline{n} = \sigma \cup \sigma'$ be a disjoint union.

Assume $\mathbf{R}$ is a *linked* subdirect product of $\Pi_\sigma \mathbf{A}_i$ and $\Pi_{\sigma'} \mathbf{A}_i$.

Suppose $R' = R \cap \Pi_i B_i \neq \emptyset$. Then $\mathbf{R}' = \Pi_i \mathbf{B}_i$.

From now on, *all algebras are finite and belong to the same Taylor variety*.

Lemma 1.

Let $\mathbf{B}_i \lll \mathbf{A}_i$ for each $i \in \underline{n}$, and let $\underline{n} = \sigma \cup \sigma'$ be a disjoint union.
Assume $\mathbf{R}$ is a *linked* subdirect product of $\Pi_\sigma \mathbf{A}_i$ and $\Pi_{\sigma'} \mathbf{A}_i$.
Suppose $R' = R \cap \Pi_i B_i \neq \emptyset$. Then $\mathbf{R}' = \Pi_i \mathbf{B}_i$.

Lemma 2. [Kearnes-Kiss, Thm 3.27]

Suppose α and β are congruences of a Taylor algebra. Then

$$\mathsf{C}(\alpha, \alpha; \alpha \wedge \beta) \quad \Longleftrightarrow \quad \mathsf{C}(\alpha \vee \beta, \alpha \vee \beta; \beta).$$

From now on, *all algebras are finite and belong to the same Taylor variety*.

Lemma 1.

Let $\mathbf{B}_i \lll \mathbf{A}_i$ for each $i \in \underline{n}$, and let $\underline{n} = \sigma \cup \sigma'$ be a disjoint union.
Assume $\mathbf{R}$ is a *linked* subdirect product of $\Pi_\sigma \mathbf{A}_i$ and $\Pi_{\sigma'} \mathbf{A}_i$.
Suppose $R' = R \cap \Pi_i B_i \neq \emptyset$. Then $\mathbf{R}' = \Pi_i \mathbf{B}_i$.

Lemma 2. [Kearnes-Kiss, Thm 3.27]

Suppose α and β are congruences of a Taylor algebra. Then

$$\mathsf{C}(\alpha, \alpha; \alpha \wedge \beta) \quad \Longleftrightarrow \quad \mathsf{C}(\alpha \vee \beta, \alpha \vee \beta; \beta).$$

Lemma 3. [Linking Lemma]

Let $n \geqslant 2$ and $\mathbf{B}_i \lll \mathbf{A}_i$ for all $i \in \underline{n}$. Suppose

- at most one $\mathbf{A}_i$ abelian, all nonabelian factors simple
- $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1}$,
- $\eta_i \neq \eta_j$ for all $i \neq j$.

Then there exists k such that $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_k \times \mathbf{R}_{k'}$ is linked.

The Theorem. Assume $\mathbf{A}_i$ are finite Taylor algebras with $\mathbf{B}_i \lhd\!\lhd \mathbf{A}_i$, and

- at most one $\mathbf{A}_i$ abelian, all nonabelian factors simple,
- $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1}$, with $\eta_i \neq \eta_j$ for $i \neq j$,
- $\mathbf{R}' = \mathbf{R} \cap (\mathbf{B}_0 \times \mathbf{B}_1 \times \cdots \times \mathbf{B}_{n-1})$ nonempty.

Then $\mathbf{R}' = \mathbf{B}_0 \times \mathbf{B}_1 \times \cdots \times \mathbf{B}_{n-1}$.

Proof sketch.
Induct on the number of factors in the product $\mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1}$.

For $n = 2$ the result holds by an earlier Lemma (slide 16).

Fix $n > 2$ and assume for all $2 \leqslant k < n$ the result holds for k factors. We prove it for subdirect products of n factors.

Fix $\emptyset \subsetneq \sigma \subsetneq \underline{n}$.

Then $\mathbf{R}_\sigma = \mathrm{Proj}_\sigma \mathbf{R}$ and $\mathbf{R}_{\sigma'} = \mathrm{Proj}_{\sigma'} \mathbf{R}$ satisfy assumptions of RT.

Induction hypothesis implies $\Pi_\sigma \mathbf{B}_i \leqslant \mathbf{R}_\sigma$ and $\Pi_{\sigma'} \mathbf{B}_i \leqslant \mathbf{R}_{\sigma'}$.

A few more easy steps gives, for all $\emptyset \subsetneq \sigma \subsetneq \underline{n}$,

$$\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{R}_\sigma \times \mathbf{R}_{\sigma'}, \quad \Pi_\sigma \mathbf{B}_i \lhd\!\lhd \mathbf{R}_\sigma, \quad \Pi_{\sigma'} \mathbf{B}_i \lhd\!\lhd \mathbf{R}_{\sigma'}.$$

By Linking Lemma and Absorption Theorem, the proof is complete.

What if there is more than one abelian factor?

Cor. 1 Let $\mathbf{A}_i$ be finite Taylor algebras with $\mathbf{B}_i \lhd\!\lhd \mathbf{A}_i$ $(i \in \underline{n})$.
Let $\mathbf{B}_i \lhd\!\lhd \mathbf{A}_i$ $(i \in \underline{n})$ and $\alpha \subseteq \underline{n}$. Suppose

- $\mathbf{A}_i$ is abelian for each $i \in \alpha$,
- $\mathbf{A}_i$ is nonabelian and simple for each $i \in \alpha'$,
- $\mathbf{R} \leqslant_{\text{sd}} \mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1}$,
- $\eta_i \neq \eta_j$ for all $i \neq j$,
- $R' := R \cap (B_0 \times B_1 \times \cdots \times B_{n-1}) \neq \emptyset$.

Then $\mathbf{R}' = \mathbf{R}_\alpha \times \Pi_{\alpha'} \mathbf{B}_i$.

Proof.
Suppose $\alpha' = \{i_0, i_1, \ldots, i_{m-1}\}$. Clearly, $\mathbf{R} \leqslant_{\text{sd}} \mathbf{R}_\alpha \times \mathbf{A}_{i_0} \times \mathbf{A}_{i_1} \times \cdots \times \mathbf{A}_{i_{m-1}}$. If $\alpha \neq \emptyset$, then the product has a single abelian factor $\mathbf{R}_\alpha \leqslant \Pi_\alpha \mathbf{A}_i$. If $\alpha = \emptyset$, then the product has no abelian factors. In either case, the result follows from the RT Theorem.

Two more observations facilitate application to CSP problems.

Cor. 2 Let $\mathbf{A}_i$ be finite Taylor algebras with $\mathbf{B}_i \lll \mathbf{A}_i$ $(i \in \underline{n})$.
Let $\mathbf{B}_i \lll \mathbf{A}_i$ for each $i \in \underline{n}$ and suppose $\mathbf{R}$ and $\mathbf{S}$ are subdirect products of $\Pi_{\underline{n}}\mathbf{A}_i$. Let $\alpha \subseteq \underline{n}$ and assume

- $\mathbf{A}_i$ is abelian for each $i \in \alpha$,
- $\mathbf{A}_i$ is nonabelian and simple for each $i \notin \alpha$,
- $\eta_i \neq \eta_j$ for all $i \neq j$,
- R and S both intersect $\Pi_{\underline{n}} B_i$ nontrivially,
- there exists $\mathbf{x} \in R_\alpha \cap S_\alpha$.

Then $R \cap S \neq \emptyset$.
Proof.
By Cor 1, $\mathbf{R}' = \mathbf{R}_\alpha \times \Pi_{\alpha'}\mathbf{B}_i$ and $\mathbf{S}' = \mathbf{S}_\alpha \times \Pi_{\alpha'}\mathbf{B}_i$. Therefore, since $\mathbf{x} \in R_\alpha \cap S_\alpha$, we have $\{\mathbf{x}\} \times \Pi_{\alpha'}\mathbf{B}_i \subseteq R \cap S$.

Generalizing to more than two relations is easy...

Cor. 3 Let $\mathbf{A}_i$ be finite Taylor algebras with $\mathbf{B}_i \lll \mathbf{A}_i$ $(i \in \underline{n})$.
Suppose $\{\mathbf{R}_\ell : 0 \leqslant \ell < m\}$ are subdirect products of $\Pi_{\underline{n}}\mathbf{A}_i$.
Let $\alpha \subseteq \underline{n}$, and assume

- $\mathbf{A}_i$ is abelian for $i \in \alpha$ and nonabelian simple for $i \notin \alpha$,
- $\forall \ell \in \underline{m}$, $\forall i \neq j$, $\eta_i^\ell \neq \eta_j^\ell$ (where $\eta_i^\ell := \ker(\mathbf{R}_\ell \twoheadrightarrow \mathbf{A}_i)$),
- each R_ℓ intersects ΠB_i nontrivially,
- there exists $\mathbf{x} \in \bigcap \operatorname{Proj}_\alpha R_\ell$.

Then $\bigcap R_\ell \neq \emptyset$.

1. **Nonabelian factors must be simple.** This is the most obvious limitation of the theorem and we don't yet have a way to overcome it that works in general. However, we have some ideas and tools for special cases.

1. **Nonabelian factors must be simple.** This is the most obvious limitation of the theorem and we don't yet have a way to overcome it that works in general. However, we have some ideas and tools for special cases.
2. **Abelian factors must have easy partial solutions.** The last two corollaries assume that when the given constraint relations are projected onto the abelian factors, we can solve the "partial instance"—that is, an element that satisfies all constraint relations after projecting these relations onto the abelian factors of the full product. This is not a problem. Abelian algebras are tractable! (cf. Theorem 7.12 of Hobby & McKenzie)

1. **Nonabelian factors must be simple.** This is the most obvious limitation of the theorem and we don't yet have a way to overcome it that works in general. However, we have some ideas and tools for special cases.
2. **Abelian factors must have easy partial solutions.** The last two corollaries assume that when the given constraint relations are projected onto the abelian factors, we can solve the "partial instance"—that is, an element that satisfies all constraint relations after projecting these relations onto the abelian factors of the full product. This is not a problem. Abelian algebras are tractable! (cf. Theorem 7.12 of Hobby & McKenzie)
3. **Intersecting mass products.** RT and corollaries assume that the universe R of the subdirect product in question intersects nontrivially with a product ΠB_i of minimal absorbing subuniverses (or "mass product").

   In a CSP instance, there are typically many constraint relations. To apply Rectangularity, we have to be sure they all intersect a single mass product.

The Rectangularity Theorem states that under certain hypotheses (including nontrivial intersections with a single mass product) the given instance has a solution.

Consider the converse. That is, suppose $\mathcal{R} = (\mathbf{R}_0, \mathbf{R}_1, \dots, \mathbf{R}_{p-1})$ is a list of subdirect products, the full intersection of which is nonempty $\mathbf{x} \in \bigcap_{\underline{p}} R_i$.

Does it follow that a single mass product intersects nontrivially with $\bigcap_{\underline{p}} R_i$?

If the answer to this question is yes, then for each CSP instance either there's a mass product intersecting nontrivially with all constraint relations, or the instance has no solution.

What's the complexity of deciding whether all relations intersect a common mass product? Surely easier than deciding whether they intersect at all.

Concluding Remarks

Some final observations

The Rectangularity Theorem states that under certain hypotheses (including nontrivial intersections with a single mass product) the given instance has a solution.

Consider the converse. That is, suppose $\mathcal{R} = (\mathbf{R}_0, \mathbf{R}_1, \ldots, \mathbf{R}_{p-1})$ is a list of subdirect products, the full intersection of which is nonempty $\mathbf{x} \in \bigcap_{\underline{p}} R_i$.

Does it follow that a single mass product intersects nontrivially with $\bigcap_{\underline{p}} R_i$?

If the answer to this question is yes, then for each CSP instance either there's a mass product intersecting nontrivially with all constraint relations, or the instance has no solution.

What's the complexity of deciding whether all relations intersect a common mass product? Surely easier than deciding whether they intersect at all.

Thank you!