Counting by independent choices & overcounting

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1 Counting with factorials and falling factorials

1. How many ways can you arrange seven different books on a bookshelf?
2. How many ways can you arrange two of the seven different books on the bookshelf?
3. How many ways can you arrange three of the seven different books on the bookshelf?
4. How many ways can you arrange six of the seven different books on the bookshelf?
5. Getting bored yet? Write a general formula for how to arrange k of n different books on a bookshelf.

2 Counting Committees, Revisited

1. Let's return to the counting problems we were working on last class. How many different ways can we pick a committee from our class of 27 students? This time, let's assume a committee can have any number, from 0 to 27, of people. Committee members aren't ordered, so (Ted,Joe) is the same as (Joe,Ted), as a committee.
2. Now, suppose we want to pick a committee of zero people from the class. How many ways can we do that?
3. How many ways can we pick a committee of one person from the class?
4. How many ways can you line up two people from the class in the front, left to right? (There are MORE ways to do this than just to pick a committee; this is like an ordered committee, so (Ted,Joe) is NOT the same as (Joe,Ted).)
5. Now, use your answer to the last problem to solve this one: how many ways can we pick a committee of two people from the class? (Hint: should this answer be larger or smaller than the last answer? what's the relationship?)
6. How many ways can we pick a committee of three people from the class?
7. How many ways can we pick a committee of four people from the class?
8. How many ways can we pick a committee of n people from the class, if $0 \leq n \leq 27$? Write a nice formula for your answer using factorials. Make sure to justify your formula. Don't just guess it from the special cases we did. Give a general argument.

9. Now, use your answer from the last question to answer the question 1 of this section again, given in the form of summation notation. If you don't remember summation notation (e.g. calc 2), ask me for a quick review.

10. Now, you've answered question 1 two ways. You have, in fact, proven a theorem. State the theorem you have proven. It should say that two different-looking formulas are equal.

11. If you answered the last question with a theorem about the number 27, state a more general one you can obtain by considering two answers to the question 'How many ways can you pick a committee from a room of n people?' This is as much an exercise in notation as anything else.

3 More counting!

In the previous problems, you had some factorials divided by factorials. We can make a convenient notation:

$$\binom{n}{r} \text{ is defined to be } \frac{n!}{r!(n-r)!}$$

1. Prove that
$$\binom{n}{r} = \binom{n}{n-r}.$$
by using the definition and doing algebra.

2. Prove it by arguing that both sides actually count the same thing two different ways (hint: think about committees).

3. Prove that
$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.$$
by arguing that both sides actually count the same thing two different ways (hint: think about committees). It is much more unsatisfying to prove this by doing algebra on factorials.