Worksheet on Functions

November 28, 2016

1 Review

1. Please have the worksheet of last class on hand, as it contains all the basic functions terminology.
2. Please take up Problem 2 of that worksheet all together as a class.
3. As a class, please prove that the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $2x + 3$ is injective, but not surjective.

2 Practice

1. Draw a graph of a function f which is injective but not surjective, which has domain and codomain $\mathbb{R}$, and satisfies $f([0, \infty)) = [1, \infty)$ and $f^{-1}((0, \infty)) = \mathbb{R}$.

2. Given two sets of equal cardinality $|A| = |B| = n$.
   (a) How many functions are there $f : A \to B$?
   (b) How many of these are bijective?
   (c) Can you construct one which is injective but not bijective (try $n = 2$)?

3. Let A and B be finite sets, and suppose $f : A \to B$.

   Fill in the table with P (possible) and I (impossible).

	$\|A\| = \|B\|$	$\|A\| > \|B\|$	$\|A\| < \|B\|$
bijective			
surjective, not injective			
injective, not surjective			
neither injective nor surjective			

3 Composition of Functions

Given two functions $f : A \to B$ and $g : B \to C$, it is possible to compose them, to obtain a new function $g \circ f : A \to C$ given by the rule

$$g \circ f(x) = g(f(x)).$$

1. Suppose $f : \mathbb{Z} \to \mathbb{N}$ (here $\mathbb{N}$ denotes non-negative integers) is given by $f(x) = x^2$ and $g : \mathbb{N} \to 2\mathbb{N}$ (here $2\mathbb{N}$ denotes $\{2x : x \in \mathbb{N}\}$) is given by $g(x) = 2x$. Determine a formula for $g \circ f$.

2. Reminder: functions do not have to have formulas. Consider the function $f : \{a, b, c\} \to \{1, 2, 3\}$ given by $f(a) = 1$, $f(b) = 2$ and $f(c) = 3$. Consider also the function $g : \{1, 2, 3\} \to \{7, 8\}$ given by $g(1) = g(2) = 7, g(3) = 8$. Determine $g \circ f$.

3. Consider two functions $f, g : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ and $g(x) = x + 1$. Determine $g \circ f$ and $f \circ g$. Are they the same?

4. Explain why in the definition of composition above, the domain of g must match the codomain of f. You may wish to give an example to illustrate.

5. Let $f : \mathbb{Z} \to \mathbb{Z}$ be given by $f(x) = x + 1$. Find a function $g : \mathbb{Z} \to \mathbb{Z}$ such that $f \circ g$ and $g \circ f$ are both the identity function on $\mathbb{Z}$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$. Find a function $g : \mathbb{R} \to \mathbb{R}$ such that $f \circ g$ and $g \circ f$ are both the identity function on $\mathbb{R}$.

7. Let $f : \mathbb{R} \to \mathbb{R}^+$ (here $\mathbb{R}^+$ is the notation for positive real numbers) be given by $f(x) = e^x$. Find a function $g : \mathbb{R}^+ \to \mathbb{R}$ such that $f \circ g : \mathbb{R}^+ \to \mathbb{R}^+$ and $g \circ f : \mathbb{R} \to \mathbb{R}$ are both identity functions.

4 Inverse Functions

Given a function $f : A \to B$, if there is a function $g : B \to A$ such that $f \circ g$ is the identity function on B and $g \circ f$ is the identity function on A, then we call g the *inverse of* f and write $g = f^{-1}$.

1. Can you find an inverse to the function $f : \mathbb{Z} \to 2\mathbb{Z}$ given by $f(x) = 2x$?

2. Can you find an inverse to the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = 2x$?

3. Exactly one of the two previous answers was "yes". Explain why a non-surjective function will not have an inverse.

4. The notation $\mathbb{R}^{\geq 0}$ denotes non-negative real numbers. Can you find an inverse to the function $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ given by $f(x) = x^2$?

5. Can you find an inverse to the function $f : \mathbb{R} \to \mathbb{R}^{\geq 0}$ given by $f(x) = x^2$?

6. Exactly one of the two previous answers was "yes". Explain why a non-injective function will not have an inverse.

5 Proofs

1. Prove that the function $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ given by $f(x) = x^2$ is bijective. You can use known facts about squares and square roots from calculus, say.

2. Prove that a function is bijective if and only if it has an inverse.