Example Combinatorial Proofs

October 28, 2016

Theorem 1. *For all $n \geq k \geq 0$,*

$$\binom{n}{k} = \binom{n}{n-k}$$

Illustration: Subsets of size 2 from $S = \{a, b, c, d, e\}$. ($k = 2$, $n = 5$)

Subset	k elements chosen	$n - k$ elements not chosen
{a,b}	a,b	c,d,e
{a,c}	a,c	b,d,e
{a,d}	a,d	b,c,e
{a,e}	a,e	b,c,d
{b,c}	b,c	a,d,e
{b,d}	b,d	a,c,e
{b,e}	b,e	a,c,d
{c,d}	c,d	a,b,e
{c,e}	c,e	a,b,d
{d,e}	d,e	a,b,c

Proof. We will show that both sides of the equation count the number of ways to choose a subset of size k from a set of size n.

The left hand side of the equation counts this by definition.

Now we consider the right hand side. To choose a subset of size k, we can instead choose the $n - k$ elements to exclude from the subset. There are $\binom{n}{n-k}$ ways to do this. Therefore the right hand side also counts the desired quantity. $\square$

Theorem 2. *For all $n \geq k \geq 1$,*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Illustration: Subsets of size 3 from $S = \{a, b, c, d, e\}$ ($k = 3$, $n = 5$).

Subsets	Subsets containing a	Subsets not containing a
{a,b,c}	{a,b,c}	
{a,b,d}	{a,b,d}	
{a,b,e}	{a,b,e}	
{a,c,d}	{a,c,d}	
{a,c,e}	{a,c,e}	
{a,d,e}	{a,d,e}	
{b,c,d}		{b,c,d}
{b,c,e}		{b,c,e}
{b,d,e}		{b,d,e}
{c,d,e}		{c,d,e}
$10 = \binom{5}{3}$	$6 = \binom{4}{2}$	$4 = \binom{4}{3}$

Proof. I have broken the proof under three headings to highlight its structure.

1. **QUESTION:** We will show that both sides of the equation count the number of ways to choose a subset of size k from a set S of size n.

2. **LEFT:** The left hand side of the equation counts this by definition.

3. **RIGHT:** Let $s \in S$ be a fixed element. We will show that the right hand side counts the desired quantity by conditioning on whether s is in the subset.

   First, we will count how many subsets of size k include s. Since such a subset includes s, there are $k-1$ other elements in the subset, which must be chosen from the remaining $n-1$ elements of S. Therefore there are $\binom{n-1}{k-1}$ such subsets.

   Second, we will count how many subsets of size k do not include s. Since the subset does not include s, all of its k elements are chosen from the remaining $n-1$ elements of S. Therefore there are $\binom{n-1}{k}$ such subsets.

   Since any subset of size k either includes s or does not (but not both), the total number of subsets is the sum of the counts in the two cases.

$\square$