

Bivariant Chern Character and Longitudinal Index.

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Abstract

In this paper we consider a family of Dirac-type operators on fibration $P \rightarrow B$ equivariant with respect to an action of an étale groupoid. Such a family defines an element in the bivariant K theory. We compute the action of the bivariant Chern character of this element on the image of Connes' map Φ in the cyclic cohomology. A particular case of this result is Connes' index theorem for étale groupoids [9] in the case of fibrations.

1 Introduction

In [36,37], answering a question posed by A. Connes in [6], V. Nistor defined a bivariant Chern character for a p -summable quasimorphism and established its fundamental properties. This theory was further developed in [17,16]. Bivariant Chern character encodes extensive information related to index theory. In the present paper we compute the action of the bivariant Chern character on cyclic cohomology in geometric situations arising from equivariant families of elliptic operators.

We consider first the following geometric situation. Let $\pi : P \rightarrow B$ be a fibration. We assume that we are given a vertical Riemannian metric on this fibration, i.e. that there is a Riemannian metric on each of the fibers $P_b = \pi^{-1}(b)$, which varies smoothly with b . We assume that each P_b is a complete Riemannian manifold. Let D be a family of fiberwise Dirac type operators on

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¹ The research was supported in part by NSF grants DMS-0242780 and DMS-0400342.

Preprint submitted to Journal of Functional Analysis

this fibration acting on the section of the bundle \mathcal{E} . In this paper we consider only the even-dimensional situation, and hence the bundle \mathcal{E} is always \mathbb{Z}_2 -graded. We assume that an étale groupoid G with the unit space B acts on this fibration, and that the operator D is invariant under this action. This means that every $\gamma \in G$ defines a diffeomorphism $P_{r(\gamma)} \rightarrow P_{s(\gamma)}$, $p \mapsto p\gamma$. One also requires that these diffeomorphisms are compatible with the groupoid structure, i.e. that $(p\gamma_1)\gamma_2 = p(\gamma_1\gamma_2)$. Notice that this implies that the vertical metric is invariant under the action of G . We do not assume existence of a metric on P , invariant under the action of G .

There are two constructions in K -theory and cyclic cohomology we need to use to describe the problem. The first is Nistor's bivariant Chern character. It appears in our setting as follows. The operator D described above defines an element of the equivariant KK -theory $KK^G(C_0(P), C_0(B))$, [30,29]. Using the canonical map

$$j^G : KK^G(C_0(P), C_0(B)) \rightarrow KK(C_0(P) \rtimes G, C_0(B) \rtimes G) \quad (1)$$

we obtain a class in $KK(C_0(P) \rtimes G, C_0(B) \rtimes G)$ defined by D . This class can be represented by an explicit quasimorphism ψ_D in the sense of J. Cuntz [14,15]. Moreover, one can show that ψ_D actually defines a *p-summable* quasimorphism of smooth algebraic cross-products $C_0^\infty(P) \rtimes G$ and $C_0^\infty(B) \rtimes G$ in the sense of V. Nistor [36,37]. One can then use techniques developed in [36,37] to define the bivariant Chern character $\text{Ch}(D)$ in bivariant cyclic homology which is a morphism of complexes $CC_*(C_0^\infty(P) \rtimes G) \rightarrow CC_*(C_0^\infty(B) \rtimes G)$.

The other tool which we need is the explicit construction, due to A. Connes [8,9], of the classes in the cyclic cohomology of cross product algebras. Let M be a manifold on which an étale groupoid G acts, and let M_G be the corresponding homotopy quotient. Then Connes constructs an explicit chain map of complexes inducing an injective map in cohomology $\Phi : H_\tau^*(M_G) \rightarrow HC^*(C_0^\infty(M) \rtimes G)$. Here τ denotes a twisting by the orientation bundle of M . Introduce now the following notations. The map π induces a map $P_G \rightarrow B_G = BG$ which we also denote by π . Since the fibers of $P_G \rightarrow B_G$ are *spin*^c and hence oriented we have a pull-back map $\pi^* : H_\tau^*(B_G) \rightarrow H_\tau^*(P_G)$. Let $\widehat{A}_G(TP/B) \in H^*(P_G)$ be the equivariant \widehat{A} -genus of the vertical tangent bundle, i.e. the \widehat{A} -genus of the bundle on P_G induced by the vertical tangent bundle TP/B on P . Note that we use the conventions from [1] in the definitions of characteristic classes. Let $Ch_G(\mathcal{E}/\mathcal{S})$ be the equivariant twisting Chern character of the bundle \mathcal{E} . If the fibers of $P \rightarrow B$ have *spin* structure, and $\mathcal{E} = \mathcal{S} \otimes V$ where \mathcal{S} is the vertical spin bundle, then $\text{Ch}_G(\mathcal{E}/\mathcal{S}) = \text{Ch}_G(V)$ is just the equivariant Chern character of V . In other words it is the Chern character of the bundle on P_G induced by the bundle V on P . Set now for $c \in H_\tau^*(B_G)$ $\tilde{\pi}^*(c) = (2\pi i)^{-\frac{\dim P - \dim B}{2}} \widehat{A}_G(TP/B) \text{Ch}_G(\mathcal{E}/\mathcal{S}) \pi^*(c)$

Our main result is then the following:

Theorem 1 *The diagram*

$$\begin{array}{ccc}
H_\tau^*(P_G) & \xrightarrow{\Phi} & HC^*(C_0^\infty(P) \rtimes G) \\
\uparrow \tilde{\pi}^* & & \uparrow \text{Ch}(D)^t \\
H_\tau^*(B_G) & \xrightarrow{\Phi} & HC^*(C_0^\infty(B) \rtimes G)
\end{array} \tag{2}$$

commutes.

As an illustration consider the case when the action of G on P is free, proper and cocompact. In this case there is a Morita equivalence between the algebras $C_0^\infty(P) \rtimes G$ and $C_0^\infty(P/G)$. We have a canonical class $1 \in K_0(C_0^\infty(P/G))$, and hence the corresponding class in $K_0(C_0^\infty(P) \rtimes G)$. This class can be represented by an idempotent $e \in C_0^\infty(P) \rtimes G$. It can be described explicitly as follows. Let $\phi \in C_0^\infty(P)$ be such that for every $p \in P$ $\sum_{r(\gamma)=\pi(p)} \phi(p\gamma)^2 = 1$. We then define an element in $C_0^\infty(P) \rtimes G$ by $e(p, \gamma) = \phi(p)\phi(p\gamma)$. Since the action of G on P is proper one can define an index $\text{Ind } D$ of D in $K_0(C_0^\infty(G) \otimes \mathcal{R})$ where \mathcal{R} is the algebra of matrices with rapidly decaying entries [10]. One can represent index of D as the product in KK -theory: $\text{Ind } D = \psi_D[e]$. This, together with commutativity of (2) allows to compute the pairing of $\text{Ind } D$ with $H^*(B_G)$ in topological terms:

$$\begin{aligned}
\langle \Phi(c), \text{Ch}(\text{Ind } D) \rangle = \\
(2\pi i)^{-\frac{\dim P - \dim B}{2}} \langle \Phi(\widehat{A}_G(TP/B) \text{Ch}_G(\mathcal{E}/\mathcal{S})\pi^*(c)), \text{Ch } e \rangle \tag{3}
\end{aligned}$$

Now due to the conditions imposed on the action in this case P/G is a smooth manifold, and every cocycle $C \in C^*(G, \Omega_*(P))$ defines a class $[C] \in H_\tau^*(P/G)$. It is easy to see that

$$\langle \Phi(C), \text{Ch } e \rangle = \int_{P/G} [C]. \tag{4}$$

Hence the pairing in the right hand side equals $\int_{P/G} \widehat{A}(\mathcal{F}) \text{Ch}(\mathcal{E}/\mathcal{S})[\pi^*(c)]$, where \mathcal{F} is the foliation on P/G induced by the fibers of the fibration π . Thus in this case we recover Connes' index theorem for étale groupoids [9], for the case when $P \rightarrow B$ is a fibration. Notice that we have a constant different from the one in [9], which is due to the different conventions in defining characteristic classes. Among important particular cases of this theorem let us mention Connes'-Moscovici higher Γ -index theorem, obtained when the groupoid G is a discrete group. There are several other proofs of this theorem using the bivariant Chern character idea, compare [42,39].

Our proof of the Theorem 1 is as follows. Let $\Omega_*(B)$ denote the complex of

smooth currents, i.e. differential forms with values in the orientation bundle, on the manifold B . Then cohomology of the complex $C^*(G, \Omega_*(B))$ equals $H_\tau^*(B_G)$. We show that the diagram of complexes corresponding to (2) commutes up to the chain homotopy of complexes. To this end we use a simplicial version \mathbb{A} [18,19] of the Bismut superconnection [2] to construct a map of complexes $\Phi_{\mathbb{A}} : C^*(G, \Omega_*(B)) \rightarrow CC^*(C_0^\infty(P) \rtimes G)$ to obtain the following diagram:

$$\begin{array}{ccc}
C^*(G, \Omega_*(P)) & \xrightarrow{\Phi} & CC^*(C_0^\infty(P) \rtimes G) \\
\uparrow \tilde{\pi}^* & \nearrow \Phi_{\mathbb{A}} & \uparrow \text{Ch}(D)^t \\
C^*(G, \Omega_*(B)) & \xrightarrow{\Phi} & CC^*(C_0^\infty(B) \rtimes G)
\end{array} \tag{5}$$

We then show that each of the triangles in this diagram is commutative up to homotopy. The map $\Phi_{\mathbb{A}}$ plays the role of McKean-Singer formula in our context. To show the commutativity of the upper triangle we replace the superconnection \mathbb{A} by a rescaled superconnection \mathbb{A}_s and compute the limit when $s \rightarrow 0$. To show commutativity of the lower triangle the classical method [2] of computing the limit when $s \rightarrow \infty$ runs into serious difficulties. We avoid this difficulty by using a modification of the method from [26,27], which was inspired by [38].

We note that superconnection proofs of the Connes-Moscovici higher Γ -index theorem were obtained in [31,42]. In our work with J. Lott [26,27] we extended the methods of [31] to obtain a superconnection proof of Connes' index theorem for étale groupoids in full generality. While in the present paper also uses superconnections as the main tool, the approach here is quite different from the approach in [26,27]. We note also that an alternative approach to local index theorems is provided in the work of R. Nest and B. Tsygan [33,34].

The paper is organized as follows. In the Section 2 we construct the map $\Phi_{\mathbb{A}}$. Then, after some preparations in the Section 3, we prove commutativity of the lower triangle in the section 4. In the Section 5 we use Bismut superconnection to show commutativity of the upper triangle.

The author thanks A. Connes, X. Dai, J. Lott, R. Nest, V. Nistor, H. Moscovici, P. Piazza, R. Ponge and B. Tsygan for many helpful discussions.

2 Construction of the map $\Phi_{\mathbb{A}}$.

In this section we construct the map $\Phi_{\mathbb{A}}$, as described in the introduction. We start by considering in 2.1 the case when our groupoid is just an ordinary

manifold. Then, after reviewing in 2.2 some definitions and results about actions of groupoids on algebras, we proceed to give the general construction in 2.3.

2.1 Consider first the following situation. Let $P \rightarrow B$ be a submersion. We use the notation from [22] for the cyclic complexes. For an algebra A set $C^k(A) = (A \otimes \bar{A}^{\otimes k})'$, where $\bar{A} = A/\mathbb{C}1$. Let u be a formal variable of degree 2. We denote by $CC^*(A)$ the periodic complex of A : the complex $(C^*(A)[u, u^{-1}], b + uB)$. For the nonunital algebras we consider the reduced cyclic complex of the unitalization. Another complex we consider is the complex of smooth currents. We use notation $\Omega_k \subset (\Omega^k)'$ for smooth currents of degree k . We also adjoin u to this complex, so that $(\Omega_*)^k = \oplus_i u^i \Omega_{k-2i}$, and the differential of degree one is given by $u\partial$ where $\partial = d^t$ is the transpose of de Rham differential. In this situation we consider the complex $\text{Hom}(\Omega_*(B), CC^*(C_0^\infty(P)))$. Here we consider complex of homomorphisms of $\mathbb{C}[u, u^{-1}]$ modules. Let D be a family of Dirac-type operators on this submersion, acting on the sections of a bundle \mathcal{E} . On the base B we have an infinite-dimensional bundle $\pi_*(\mathcal{E})$ whose fiber over a point $b \in B$ is $\Gamma(\mathcal{E}, P_b)$. Consider now a superconnection \mathbb{A} adapted to the operator D , compare [1]. We assume that the superconnection has the form $\mathbb{A} = D + \mathbb{A}_{[1]} + \dots$ where $\mathbb{A}_{[1]}$ is a connection on the bundle $\pi_*(\mathcal{E})$ and $\mathbb{A}_{[i]}$ for $i > 1$ is a proper fiberwise pseudodifferential operator. We assume that the connection in a local trivialization chart has a form $d + \omega$ where ω is a 1-form on B with values in the proper fiberwise pseudodifferential operators. With such a superconnection we will associate a cochain $\Theta_{\mathbb{A}}$ of degree 0 in the complex $\text{Hom}(\Omega_*(B), CC^*(C_0^\infty(P)))$.

As a first step we have the following proposition.

Proposition 2 *The following expression defines a cocycle of degree 0 in the complex $\text{Hom}(\Omega_*(B), C^*(C_0^\infty(P))[u, u^{-1}], b + uB)$*

$$\theta_{\mathbb{A}} : c \mapsto \sum_{l \geq -\frac{\deg c}{2}} u^{-l} \theta_l(c) \quad (6)$$

where $\theta_l \in C^k(C_0^\infty(P))$, $k = \deg c + 2l$, is given by the formula:

$$\begin{aligned} & \theta_l(c)(a_0, a_1, \dots, a_k) \\ &= \left\langle c, \int_{\Delta^k} \text{Tr}_s a_0 e^{-t_0 \mathbb{A}^2} [\mathbb{A}, a_1] \dots [\mathbb{A}, a_k] e^{-t_k \mathbb{A}^2} dt_1 \dots dt_k \right\rangle \quad (7) \end{aligned}$$

Here $\Delta^k = \{(t_0, t_1, \dots, t_k) \mid \sum t_i = 1, t_i \geq 0\}$.

PROOF. It is easy to see that the expression under the trace is always a smoothing operator, and hence the trace is well defined. It is also easy to see that the expression in the formula (7) is nonzero only if $\deg c - k$ is even. The cocycle condition is verified by a direct computation compare [23].

We can now follow the method of Connes and Moscovici [11] and replace $\theta_{\mathbb{A}}$ by a finite cochain $\Theta_{\mathbb{A}} \in \text{Hom}(\Omega_*(B), CC^*(C_0^\infty(P)))$.

The construction is as follows. Let \mathbb{A}_s denote the rescaled superconnection $\mathbb{A}_s = sD + \mathbb{A}_{[0]} + s^{-1}\mathbb{A}_{[1]} + \dots$

Introduce the cochain $\tau_s \in \text{Hom}^{-1}(\Omega_*(B), C^*(C_0^\infty(P)) [u, u^{-1}], b + uB)$ defined by $\tau_s(c) = \sum_{l \geq -\frac{\deg c - 1}{2}} u^{-l} (\tau_s)_l(c)$, where $(\tau_s)_l(c) \in C^k(C_0^\infty(P))$, $k = \deg c + 2l - 1$, is given by the formula:

$$\begin{aligned} (\tau_s)_l(c)(a_0, a_1, \dots, a_k) &= \sum_{i=0}^k (-1)^i \left\langle c, \int_{\Delta_{k+1}} \text{Tr}_s a_0 e^{-t_0 \mathbb{A}_s^2} [\mathbb{A}_s, a_1] \dots \right. \\ &\quad \left. e^{-t_i \mathbb{A}_s^2} \frac{d\mathbb{A}_s}{ds} e^{-t_{i+1} \mathbb{A}_s^2} \dots [\mathbb{A}_s, a_k] e^{-t_k \mathbb{A}_s^2} dt_1 \dots dt_{k+1} \right\rangle \quad (8) \end{aligned}$$

Then we have the following:

Lemma 3

$$\frac{d}{ds} (\theta_{\mathbb{A}_s})(c) = (b + uB) \tau_s(c) + \tau_s(u\partial c) \quad (9)$$

PROOF. Consider the submersion $P \times [a, b] \rightarrow B \times [a, b]$ with superconnection $d + \mathbb{A}_s$, $s \in [a, b]$, where d is de Rham differential on $[a, b]$. Here we view the interval $[a, b]$ as a subset of \mathbb{R} for the purpose of defining smooth functions, etc. With the projection $p : B \times [a, b] \rightarrow B$ we can associate a cochain $p^* \in \text{Hom}^1(\Omega_*(B), \Omega_*(B \times [a, b]))$, which is defined on currents of degree k as $(f)^t$, where $f : \Omega_c^*(B \times [a, b]) \rightarrow \Omega^*(B)$ is the integration along the fibers of p . Denote by c_a a current on $B \times [a, b]$ whose value on a form is a composition of restriction to $B \times a$ with c , and similarly for c_b . Then if we denote by $u\partial$ differential in $\text{Hom}^1(\Omega_*(B), \Omega_*(B \times [a, b]))$ we have

$$(u\partial p^*)(c) = u(c_b - c_a). \quad (10)$$

Define also $e^* \in \text{Hom}^0(C^*(C_0^\infty(P \times [a, b])) [u, u^{-1}], C^*(C_0^\infty(P)) [u, u^{-1}])$. e^* is clearly a cocycle. The Proposition 2 implies that $\theta_{\mathbb{A}_s+d}$ is a cocycle in

$\text{Hom}(\Omega_*([0, 1] \times B), CC^*(C_0^\infty([0, 1] \times P)))$, and hence

$$\begin{aligned} (b + uB + u\partial)(e^* \circ \theta_{d+\mathbb{A}_s} \circ p^*)(c) &= e^* \circ \theta_{d+\mathbb{A}_s} \circ (u\partial p^*)(c) \\ &= ue^* \circ \theta_{d+\mathbb{A}_s}(c_b - c_a) = u(\theta_{\mathbb{A}_b} - \theta_{\mathbb{A}_a})(c), \end{aligned} \quad (11)$$

or $(\theta_{\mathbb{A}_b} - \theta_{\mathbb{A}_a})(c) = (b + uB + u\partial)u^{-1}(e^* \circ \theta_{d+\mathbb{A}_s} \circ p^*)(c)$. But it is easy to see that $u^{-1}(e^* \circ \theta_{d+\mathbb{A}_s} \circ p^*)(c)$ is exactly $\int_a^b \tau_s(c) ds$, and the statement of the Proposition follows.

We will now study behavior of $\theta_{\mathbb{A}_s}$ and τ_s near $s = 0$.

Lemma 4 *Let V_0, V_1, \dots, V_l are operators acting on sections of some vector bundle over a manifold M . We assume that V_i is a composition of a pseudodifferential operator of order v_i with a compact support with a diffeomorphism of M , lifted to act on the sections of the vector bundle, and D is a first-order selfadjoint pseudodifferential operator on M . If $v_i = \max\{\text{order } V_i, 0\}$ and $\sum v_i \leq l$ then*

$$\left| \int_{\Delta^k} \text{Tr}_s V_0 e^{-t_0 s^2 D^2} V_1 \dots V_l e^{-t_l s^2 D^2} dt_1 \dots dt_l \right| = O\left(s^{-\sum v_i - \dim M - 1}\right) \quad (12)$$

as $s \rightarrow 0$. If V_i and D depend continuously on some parameters, the estimate is uniform on the compacts.

PROOF. First, using the fact that each V_i is compactly supported integral operator and exponential decay of the heat kernel off diagonal we can replace M by a compact manifold changing our expression by at most $O(s^\infty)$. If the case when some of the operators V_i have order 1 and the other order 0 this follows from [23] and Weyl asymptotics. In general replace each V_i by $V_i'(1 + D^2)^{v_i}$ with V_i' of order 0. Then distribute powers of $(1 + D^2)$ replacing as needed $(1 + D^2)^\alpha V_i'$ by $((1 + D^2)^\alpha V_i'(1 + D^2)^{-\alpha})(1 + D^2)^\alpha$ to reduce to the case when we have at most $\sum v_i + 1$ operators V_i of order 1 and the rest are of order 0.

Remark 5 *More precise asymptotics $O\left(s^{-\sum \text{order } V_i - \dim M}\right)$ with no restrictions imposed in the previous lemma can be obtained by methods of pseudodifferential calculus, compare [40,41,24].*

Remark 6 *One can differentiate the expression in the left hand side of (12) by the parameters, obtaining again an expression of the same kind, by the Duhamel's formula. By similar methods one can show that the derivatives of order α are $O\left(s^{-\sum v_i - \dim M - 1 - 2\alpha}\right)$ as $s \rightarrow 0$.*

Proposition 7 *There exists a number N , depending on the orders of components of superconnection as pseudodifferential operators and $\dim B$ such that for all $l > N$ coefficients θ_l for u^{-l} in $\theta_{\mathbb{A}_s}$ have limit 0 as $s \rightarrow 0$, and coefficients for u^{-l} in τ_s are integrable near $s = 0$.*

PROOF. Let $\mathbb{A}^2 = \sum \mathcal{F}_{[i]}$, where $\mathcal{F}_{[i]}$ is the component of degree i . Then $\mathbb{A}_s^2 = s^2 D^2 + \sum_{i \geq 1} s^{(2-i)} \mathcal{F}_{[i]}$. Decomposing $[\mathbb{A}_s, a]$, $a \in C_0^\infty(P)$ according to the form degree looks as follows: $[\mathbb{A}_s, a] = s\alpha_{[0]}(a) + \alpha_{[1]}(a) + \dots s^{(1-i)}\alpha_{[i]}(a) + \dots$ where $\alpha_{[i]}(a)$ are pseudodifferential operators, depending on a , and order of $\alpha_{[i]}(a)$ is 0 for $i = 1, 2$, and $k_i - 1$ for $i \geq 2$. Using Duhamel's expansion we can write component of cochain θ in $\text{Hom}(\Omega_p(B), CC^k(C_0^\infty(P)))$ as a sum of finitely many terms of the form

$$\int_{\Delta^l} \text{Tr}_s V_0 e^{-t_0 D^2} V_1 \dots V_l e^{-t_l D^2} dt_1 \dots dt_l$$

where each V_i is either $\alpha_{[j]}$ for some j or $\mathcal{F}_{[j]}$ for some j . The number of the terms of the form $\alpha_{[j]}$ is k , and we denote these terms as $\alpha_{[i_1]}, \dots, \alpha_{[i_k]}$. Similarly denote the terms of the form $\mathcal{F}_{[j]}$ as $\mathcal{F}_{[j_1]}, \dots, \mathcal{F}_{[j_m]}$, for some m . If we rescale the superconnection, this term changes to

$$\int_{\Delta^l} s^Q \text{Tr}_s V_0 e^{-t_0 s^2 D^2} V_1 \dots V_l e^{-t_l s^2 D^2} dt_1 \dots dt_l$$

where $Q = (1 - i_1) + \dots + (1 - i_k) + (2 - j_1) + \dots + (2 - j_m) = k - p + 2m \geq k - p$. Now notice that among the operators V_i there are at most $\dim B$ operators of nonzero form degree, and every operator of zero form degree is bounded. Hence if v_i is the order of V_i as a pseudodifferential operator, $\sum v_i \leq v \dim B$, where $v = \max v_i$. Since $l \geq k$ we see that if $k \geq v \dim B$ we can apply the estimate of the Lemma 4 and obtain that our term is $O(s^{k-p-(\dim P - \dim B) - v \dim B - 1})$, and if $k - p > n + v \dim B / + 1$, the corresponding component has a limit 0 when $s \rightarrow 0$. Similarly one can show that with the same bound on the degree $k - p$ we obtain τ_s is integrable at $s = 0$. For the future use notice that we can take $N = \dim P + (v - 1) \dim B + 1$.

Proposition 8 *Choose any even $k \geq N$. Define the cochain $\Theta_{\mathbb{A}} \in \text{Hom}^0(\Omega_*(B), CC^*(C_0^\infty(P)))$ as $\sum u^{-l} (\Theta_{\mathbb{A}})_l$*

$$(\Theta_{\mathbb{A}})_l = \begin{cases} (\theta_{\mathbb{A}})_l & \text{for } l < k \\ (\theta_{\mathbb{A}})_k - \int_0^1 (B + \partial)(\tau_s)_{k+1} ds & \text{for } l = k \\ 0 & \text{for } l > k \end{cases} \quad (13)$$

Then this cochain is a cocycle. Its cohomology class is independent of k .

PROOF.

Integrating equation (9) and using the results of Proposition 7 we obtain

$$(\theta_{\mathbb{A}})_l = b(\delta)_{l-1} + (B + \partial)(\delta_s)_{l+1} \quad (14)$$

where $\delta_l = \int_0^1 (\tau_s)_l ds$ and $l > k$. Set also $\delta_l = 0$ for $l < k$. Then $\Theta_{\mathbb{A}} = \theta_{\mathbb{A}} - (uB + b + u\partial)\delta$ is a cocycle, since $\theta_{\mathbb{A}}$ is one. If we change k to $k', k < k'$ the cocycle $\Theta_{\mathbb{A}}$ will change by $(b + uB + u\partial)(\sum_{k < i < k'} \delta_i)$ so its cohomology class remains the same.

Assume now that we have a smooth family of superconnections $\mathbb{A}(t)$, adapted to the family D . We then have the following

Proposition 9 *Let $\mathbb{A}(t)$ be a continuous family of superconnections adapted to the family D . Then*

$$\Theta_{\mathbb{A}(1)} - \Theta_{\mathbb{A}(0)} = (b + uB + u\partial)T \quad (15)$$

where $T_{\mathbb{A}(t)} \in \text{Hom}^{-1}(\Omega_*(B), CC^*(C_0^\infty(P)))$ is defined by the formula $T_{\mathbb{A}(t)} = u^{-1} \int_{t \in [0,1]} e^* \circ \Theta_{d+\mathbb{A}(t)} \circ p^*$. Here $p^* \in \text{Hom}^1(\Omega_*(B), \Omega_*(B \times [0, 1]))$ is defined by

$$p^*(c) = \left(\int_{B \times [0,1]/B} \right)^t c, \quad (16)$$

$\Theta_{d+\mathbb{A}(t)}$ is a cocycle in $\text{Hom}^0(\Omega_*(B \times [0, 1]), CC^*(C_0^\infty(P \times [0, 1])))$ and e is the pull-back $C_0^\infty(P) \rightarrow C_0^\infty([0, 1] \times P)$. Here we use the same truncation to construct $\Theta_{d+\mathbb{A}(t)}$, $\Theta_{\mathbb{A}(1)}$ and $\Theta_{\mathbb{A}(0)}$.

PROOF. Notice that the superconnection $d + \mathbb{A}(t)$ satisfies all the conditions necessary to construct the cocycle $\Theta_{d+\mathbb{A}(t)}$ by choosing the appropriate truncation. The rest of the proof is the same as the proof of the Lemma 3.

To extend this construction to the case of nontrivial groupoid action notice that this construction can be sheafified. Consider the following sheaves on B : one is given by $U \mapsto \Omega_*(U)$, the other is the sheafification of the presheaf given by $U \mapsto CC^*(C_0^\infty(\pi^{-1}(U)))$. Both of these sheaves are fine. The map $\Theta_{\mathbb{A}}$ constructed above defines a morphism of these complexes of sheaves.

2.2 We now review some notation and definitions regarding groupoid algebras and cross-products by groupoids [28,12]. Let F be a soft sheaf on $B = G^{(0)}$ with an action of G . This means that every $\gamma \in G$ defines a map $F_{s(\gamma)} \rightarrow F_{r(\gamma)}$, which is continuous. We denote by $C^*(G, F)$ the complex of nonhomogeneous G -cochains with values in F . It can be described as follows. Set for $n \geq 1$:

$$G^{(n)} = \{(\gamma_1, \gamma_2, \dots, \gamma_n) \in G^n \mid s(\gamma_i) = r(\gamma_{i+1}), i = 1, 2, \dots, n-1\} \quad (17)$$

We have for every n a map $\varepsilon_n : G^{(n)} \rightarrow B$ defined by

$$\varepsilon_n(\gamma_1, \gamma_2, \dots, \gamma_n) = r(\gamma_1) \quad (18)$$

Set $C^n(G, F) = \Gamma(G^{(n)}; \varepsilon_n^* F)$. To define the coboundary operator we use the simplicial maps $\delta_i : G^{(n)} \rightarrow G^{(n-1)}$. They are given for $n > 1$ by the formula

$$\delta_i(\gamma_1, \gamma_2, \dots, \gamma_n) = \begin{cases} (\gamma_2, \dots, \gamma_n) & \text{if } i = 0 \\ (\gamma_1, \dots, \gamma_i \gamma_{i+1} \dots, \gamma_n) & \text{if } 1 \leq i < n-1 \\ (\gamma_1, \gamma_2, \dots, \gamma_{n-1}) & \text{if } i = n \end{cases} \quad (19)$$

For $n = 1$ $\delta_0(\gamma_1) = r(\gamma_1)$ and $\delta_1(\gamma_1) = s(\gamma_1)$. The coboundary $\delta : C^{n-1}(G, F) \rightarrow C^n(G, F)$ is given by $\sum_{i=0}^n (-1)^i \delta_i$ where the action of G on F is used to identify $\delta_0^* \varepsilon_{n-1}^* F$ with $\varepsilon_n^* F$. We can apply this construction if $F = (F^*, d)$ is a complex of sheaves with a differential $d : F^* \rightarrow F^{*+1}$. In this case $C^*(G, F^*)$ is a bicomplex; we consider only finite cochains in this bicomplex. To form a total complex of this bicomplex we equip it with the differential $\pm \delta + d$, where $\pm = (-1)^n$ on $C^m(G, F^n)$.

If A is a sheaf of algebras over G , one can form a G -sheaf $CC^*(A)$, defined as sheafification of the presheaf $U \mapsto CC^*(\Gamma_c(U; A))$. In our examples this sheaf is always fine. Hence we can define a complex $C^*(G, CC^*(A))$. On the other hand one can form a cross-product algebra $A \rtimes G$, defined as functions on G with $f(\gamma) \in A_{r(\gamma)}$. The product is given by convolution taking the action of G on A into account. When A is the sheaf of smooth functions on B one obtains the convolution algebra which we denote by $C_0^\infty(G)$. One can then define a map of complexes

$$\Phi : C^*(G, CC^*(A)) \rightarrow CC^*(A \rtimes G) \quad (20)$$

This map has been constructed in [12], based on the constructions in [5], compare also [35], [20], [22]. An important particular case is when A is the sheaf $C^\infty(B)$ of smooth functions on B . In this case there is a canonical morphism of complexes of sheaves $\iota : \Omega_*(B) \rightarrow CC^*(C_0^\infty(B))$ defined on the current c of degree m by

$$\iota(c)(a_0, a_1, \dots, a_m) = \frac{1}{m!} \langle c, a_0 da_1 \dots da_m \rangle. \quad (21)$$

Composing ι with the map in (20) one obtains Connes' map [8,9], also denoted by Φ :

$$\Phi : C^*(G, \Omega_*(B)) \rightarrow CC^*(C_0^\infty(G)). \quad (22)$$

We will use the following properties of the map Φ , compare [12]:

- If A, B two G -algebras and $f : A \rightarrow B$ is a G -homomorphism, we get a natural induced homomorphism, which we also denote by f , from $A \rtimes G$ to $B \rtimes G$. Also the induced map $f^* : CC^*(B) \rightarrow CC^*(A)$ is G -equivariant, and hence defines a map, also denoted by f^* , from $C^*(G, CC^*(B))$ to $C^*(G, CC^*(A))$. Then the following diagram commutes:

$$\begin{array}{ccc} C^*(G, CC^*(A)) & \xrightarrow{\Phi} & CC^*(A \rtimes G) \\ \uparrow f^* & & \uparrow f^* \\ C^*(G, CC^*(B)) & \xrightarrow{\Phi} & CC^*(B \rtimes G) \end{array} \quad (23)$$

- If A is a filtered algebra and the action of G preserves filtration then the complexes $C^*(G, CC^*(A))$ and $CC^*(A \rtimes G)$ also are naturally filtered. The map Φ preserves this filtration.
- Let U_i be an open cover of B and set $B' = \coprod U_i$. Let G' be the pull-back of the groupoid G by the natural projection $p : B' \rightarrow B$. Set also $A' = p^*A$. Then A' is naturally a G' -algebra. The cross-products $A \rtimes G$ and $A' \rtimes G'$ are naturally Morita equivalent. We also have a pull-back map $C^*(G, CC^*(A)) \rightarrow C^*(G', CC^*(A'))$. Then the following diagram is commutative up to homotopy:

$$\begin{array}{ccc} C^*(G', CC^*(A')) & \xrightarrow{\Phi} & CC^*(A' \rtimes G') \\ \uparrow & & \uparrow \\ C^*(G, CC^*(A)) & \xrightarrow{\Phi} & CC^*(A \rtimes G) \end{array} \quad (24)$$

Here the vertical arrows are the isomorphisms induced by the pull-back and Morita equivalence respectively.

2.3 Let G be an étale groupoid with the unit space $G^{(0)}$. We say that a manifold P is a G -space if we are given submersion $\pi : P \rightarrow G^{(0)}$, and for every $\gamma \in G$ we have a diffeomorphism $\gamma : P_{r(\gamma)} \rightarrow P_{s(\gamma)}$. Here P_b , $b \in B$ is a fiber $p^{-1}(b)$ over the point b . The diffeomorphisms γ should be compatible with the groupoid structure. We assume also that the fibers are equipped with the complete G -invariant Riemannian metric, and we are given a family of G -invariant fiberwise Dirac operators acting on the sections of G -equivariant

bundle \mathcal{E} . We define the map $\Theta_{\mathbb{A}}$ associated to the simplicial superconnection.

$$P^{(n)} = \{(p, \gamma_1, \gamma_2, \dots, \gamma_n) \in P \times G^n \mid \pi(p) = r(\gamma_1) \text{ and } s(\gamma_i) = r(\gamma_{i+1})\} \quad (25)$$

For $n = 0$ we set $P^0 = P$. Note that π induces a natural submersion $\pi_n : P^{(n)} \rightarrow G^{(n)}$:

$$\pi_n(p, \gamma_1, \gamma_2, \dots, \gamma_n) = (\gamma_1, \gamma_2, \dots, \gamma_n). \quad (26)$$

For every $0 \leq i \leq n$ we get a submersion map $\delta_i : P^{(n)} \rightarrow P^{(n-1)}$ defined by

$$\delta_i(p, \gamma_1, \gamma_2, \dots, \gamma_n) = \begin{cases} (p\gamma_1, \gamma_2, \dots, \gamma_n) & \text{if } i = 0 \\ (p, \gamma_1, \dots, \gamma_i\gamma_{i+1}, \dots, \gamma_n) & \text{if } 1 \leq i < n - 1 \\ (p, \gamma_1, \gamma_2, \dots, \gamma_{n-1}) & \text{if } i = n \end{cases} \quad (27)$$

The underlying map of the base spaces is given by the formulas (19). We introduce also submersion maps $\alpha_i : P^{(n)} \rightarrow P$, $i = 0, 1, \dots, n$ defined by

$$\alpha_i(p, \gamma_1, \dots, \gamma_n) = \begin{cases} p\gamma_1 \dots \gamma_i & \text{if } i > 0 \\ p & \text{if } i = 0 \end{cases}. \quad (28)$$

The underlying maps of the base spaces are $\alpha_i(\gamma_1, \dots, \gamma_n) = s(\gamma_1 \dots \gamma_i)$, with $\alpha_0(\gamma_1, \dots, \gamma_n) = r(\gamma_1)$. On each of the spaces $P^{(n)}$ we consider the bundle $\mathcal{E}^{(n)} = \alpha_0^* \mathcal{E}$. It is naturally isomorphic to each of the bundles $\alpha_i^* \mathcal{E}$, with the isomorphism given by the action of G on \mathcal{E} .

We now give a definition of simplicial connection. Natural imbedding of the simplex as a subset of \mathbb{R}^n allows one to talk about smooth functions, etc. on the simplex. By simplicial connection we mean a collection of connections $\nabla^{(n)}$ on the $\pi_n \times \text{id} : P^{(n)} \times \Delta^n \rightarrow B^{(n)} \times \Delta^n$ where $\Delta^n = \{\sigma_0, \dots, \sigma_n \in \mathbb{R} \mid \sum \sigma_i = 1, \sigma_i \geq 0\}$ satisfying the following compatibility conditions, compare [19]:

$$(\text{id} \times \partial)^* \nabla^{(n)} = (\delta_i \times \text{id})^* \nabla^{(n-1)} \quad (29)$$

Here $\partial_i : \Delta^{n-1} \rightarrow \Delta^n$ is the i -th face map given by $(\sigma_0, \dots, \sigma_{n-1}) \mapsto (\sigma_0, \dots, \sigma_{i-1}, 0, \sigma_i, \dots, \sigma_{n-1})$. We also require that our connection has the hollowing property: in a local chart $\nabla^{(n)}$ can be written as $d + \omega$; we require that $\omega\left(\frac{\partial}{\partial \sigma_i}\right) = 0$.

Existence of simplicial connections follows from the following construction. One starts with an arbitrary connection ∇ on the submersion $P \rightarrow G^{(0)}$. Then one can set

$$\nabla^{(n)} = \sum_{i=0}^n \sigma_i \alpha_i^* \nabla + d_{dR} \quad (30)$$

where we set $\nabla^{(0)} = \nabla$. Here d_{dR} is de Rham differential on Δ^n .

A G -invariant family D naturally defines a family of fiberwise operators on each of the submersion π_n , which we also denote by D . We then define simplicial superconnection \mathbb{A} on G -submersion $\pi : P \rightarrow B$ as a collection of superconnections on submersions $\pi_n \times \text{id} : P^{(n)} \times \Delta^n \rightarrow G^{(n)} \times \Delta^n$, adapted to D and satisfying the compatibility conditions:

$$(\text{id} \times \partial_i)^* \mathbb{A}^{(n)} = (\delta_i \times \text{id})^* \mathbb{A}^{(n-1)}. \quad (31)$$

We also require that they have the following properties. Locally the superconnection $\mathbb{A}^{(n)}$ can be written as $d + \omega$ where ω is sum of differential forms on $G^{(n)} \times \Delta^n$. We require that every component of ω which has a positive degree in $d\sigma$ variables also has a positive degree in $G^{(n)}$ variables. It follows that $(\mathbb{A}^{(n)})^2$ also has this property. In particular we see that there exists number $r > 0$ such that every component of ω or of $(\mathbb{A}^{(n)})^2$ of degree m in $d\sigma$ has degree at least m/r in $G^{(n)}$ direction. We also require that the degrees of components of $\mathbb{A}^{(n)}$ as pseudodifferential operators are bounded uniformly in n .

An example of such superconnection is given by the $\mathbb{A}_0 = D + \nabla$, with ∇ an arbitrary simplicial connection. More precisely we set $\mathbb{A}^{(n)} = D + \nabla^{(n)}$. Another example is given by the Bismut simplicial superconnection described in the section 5.

We now use simplicial superconnection \mathbb{A} to construct the map $\Phi_{\mathbb{A}}$. We start by constructing a cocycle $\{\Theta_{\mathbb{A}}^i\} \in C^*(G, \text{Hom}(\Omega_*(B), CC^*(C_0^\infty(P))))$.

Introduce the cochains $p^n \in \text{Hom}^n(\Omega_*(B), \Omega_*(B \times \Delta^n))$ by the formula

$$p^n(c) = \left(\int_{B \times \Delta^n / B} \right)^t c. \quad (32)$$

Here we view $\Omega_*(B \times \Delta^n)$ as a sheaf on B .

$$\theta_{\mathbb{A}}^n = u^{-n} (e_n)^* \circ \theta_{\mathbb{A}^{(n)}} \circ p^n \in \text{Hom}^{-n}(\Omega_*(G^{(n)}), C^*(C_0^\infty(P^{(n)})) [u, u^{-1}], b + uB). \quad (33)$$

Similarly one constructs τ_s^n by using the superconnection $\mathbb{A}^{(n)}$.

Lemma 10 *There exists number L such that $\theta_{\mathbb{A}}^n = 0$ and $\tau_s^n = 0$ for $n > L$.*

PROOF.

Let $U \in G^{(n)}$ be an open set, $c \in \Omega_*(U)$, $a_i \in C_0^\infty(\pi_n^{-1}(U))$. Then

$$\begin{aligned}
& (e^* \circ \theta_{\mathbb{A}^{(n)}})_l (p^n(c)) (a_0, a_1, \dots, a_k) \\
&= \left\langle c, \int_{\Delta^n \times G^{(n)} | G^{(n)}} \int_{\Delta^k} \text{Tr}_s a_0 e^{-t_0 (\mathbb{A}^{(n)})^2} [\mathbb{A}^{(n)}, a_1] \dots [\mathbb{A}^{(n)}, a_k] e^{-t_k (\mathbb{A}^{(n)})^2} dt_1 \dots dt_k \right\rangle.
\end{aligned} \tag{34}$$

Now due to the conditions imposed on the superconnection every component of $[\mathbb{A}^{(n)}, a_i]$ of degree m in $d\sigma$ variables has degree at least m/k in the $G^{(n)}$ direction, and the same is true about $(\mathbb{A}^{(n)})^2$. Duhamel's expansion shows that the same is true about $e^{-t(\mathbb{A}^{(n)})^2}$. Hence the component of the expression under the integral which has degree n in $d\sigma$ variables has degree at least n/r in the $G^{(n)}$ direction. Since $\dim G^{(n)} = \dim B$ this expression is 0 for $n > r \dim B$. The same argument works for τ_s^n .

From this and the Proposition 7 we immediately obtain the following:

Proposition 11 *There exists a number N , depending on the orders of components of superconnection as pseudodifferential operators and $\dim B$ and independent of n such that for all $l > N$ $\lim_{s \rightarrow 0} (\theta_{\mathbb{A}_s^n})_l = 0$ and $(\tau_s^n)_l(c)$ is integrable near $s = 0$.*

We now can construct cochains $\Theta_{\mathbb{A}}^n \in \text{Hom}^{-n}(\Omega_*(G^{(n)}), CC^*(C_0^\infty(P^{(n)})))$ as follows. Chose any $k > N$, with N as in the previous Proposition, and define the cochain $\Theta_{\mathbb{A}^{(n)}}$ as in the Proposition 8. We define then $\Theta_{\mathbb{A}}^n = u^{-n} e^* \circ \Theta_{\mathbb{A}^{(n)}} \circ p^n$. Notice that $\Theta_{\mathbb{A}}^n = 0$ for $n > r \dim B$. Hence we can view $\Theta_{\mathbb{A}}^n$ as a cochain in $C^*(G, \text{Hom}(\Omega_*(B), CC^*(C_0^\infty(P))))$. We then have the following

Lemma 12

$$(b + uB + u\partial) \Theta_{\mathbb{A}}^n = (-1)^{n-1} \delta \Theta_{\mathbb{A}}^{n-1}. \tag{35}$$

PROOF. First, notice that as in the proof of Proposition 8 we have $e^* \circ (b + uB + u\partial) \Theta_{\mathbb{A}^{(n)}} \circ p^n = 0$. Hence $(b + uB + u\partial) \Theta_{\mathbb{A}}^n = e^* \circ \Theta_{\mathbb{A}^{(n)}} \circ u\partial p^n$. We have

$$\partial p^n = (-1)^{n-1} \sum_{i=0}^n (-1)^i (\text{id} \times \partial_i)_* \circ p^{n-1}. \tag{36}$$

The result of the Proposition then follows from this formula together with the compatibility conditions (31).

Theorem 13 *The cochain $\{\Theta_{\mathbb{A}}^i\} = \sum_{n \geq 0} \Theta_{\mathbb{A}}^n \in C^*(G, CC^*(C_0^\infty(P)))$ is a cocycle. The cohomology class of this cocycle is independent of the choice of simplicial superconnection adapted to the family D .*

PROOF. The first assertion – that $\{\Theta_{\mathbb{A}}^i\}$ is a cocycle – follows immediately from the Lemma 12. To see independence of the connection let \mathbb{A} be any simplicial superconnection and let $\mathbb{A}_0 = D + \nabla$, where $\nabla = \{\nabla^{(n)}\}$ is an arbitrary simplicial connection. Define then $\mathbb{A}(t) = t\mathbb{A} + (1-t)\mathbb{A}_0$. More precisely we set $\mathbb{A}(t)^{(n)} = t\mathbb{A}^{(n)} + (1-t)\mathbb{A}_0^{(n)}$. $\mathbb{A}(t)$ is then a simplicial superconnection for every $0 \leq t \leq 1$, i.e. the equation (31) is satisfied, as well as the conditions listed after that equations. Moreover, we can use the same number r as defined there. It follows that we can use the construction of the Proposition 9 and construct for every n a cochain $T_{\mathbb{A}(t)}^n = u^{-n}e^* \circ T_{\mathbb{A}(t)^{(n)}} \circ p^n$, which is 0 for $n > r \dim B + 1$. Hence we get a cochain $\{T_{\mathbb{A}(t)}^i\} \in C^*(G, \text{Hom}(\Omega_*(B), CC^*(C_0^\infty(P))))$. As in the Lemma 12 we obtain

$$(b + uB + u\partial) T_{\mathbb{A}(t)}^n = (-1)^n \delta T_{\mathbb{A}(t)}^{n-1} + \Theta_{\mathbb{A}}^n - \Theta_{\mathbb{A}_0}^n \quad (37)$$

and hence $(b + uB + u\partial \pm \delta) \{T_{\mathbb{A}(t)}^i\} = \{\Theta_{\mathbb{A}}^i\} - \{\Theta_{\mathbb{A}_0}^i\}$. We conclude that the cocycle $\{\Theta_{\mathbb{A}}^i\}$ is cohomologous to the cocycle $\{\Theta_{\mathbb{A}_0}^i\}$ for every simplicial superconnection \mathbb{A} .

Now using the cup product

$$\cup : C^*(G, \text{Hom}(\Omega_*, CC^*(C_0^\infty(P)))) \otimes C^*(G, \Omega_*) \rightarrow C^*(G, CC^*(C_0^\infty(P))) \quad (38)$$

we construct the map $C^*(G, \Omega_*) \rightarrow C^*(G, CC^*(C_0^\infty(P)))$ defined by $\alpha \mapsto \{\Theta_{\mathbb{A}}^i\} \cup \alpha$. We obtain the map $\Phi_{\mathbb{A}}$ by composing this map with the map $\Phi : C^*(G, CC^*(C_0^\infty(P))) \rightarrow CC^*(C_0^\infty(P) \rtimes G)$. Explicitly we define

$$\Phi_{\mathbb{A}}(\alpha) = \Phi(\{\Theta_{\mathbb{A}}^i\} \cup \alpha) \quad (39)$$

The following theorem follows immediately from the Theorem 13.

Theorem 14 *Let \mathbb{A} be a simplicial superconnection adapted to the family D . Then $\Phi_{\mathbb{A}}$ defined in the equation (39) is a cochain map of complexes. If \mathbb{A}' is another simplicial superconnection adapted to the family D , the maps $\Phi_{\mathbb{A}}$ and $\Phi_{\mathbb{A}'}$ are homotopic.*

3 Fiberwise Pseudodifferential Operators and Bivariant Chern Character

In this section we consider the properties of the algebra of the proper fiberwise pseudodifferential operators which will be needed in our discussion of the bivariant Chern character in the Section 4. We start by giving the general definitions in 3.1. Then in 3.2 we define the trace map on the cyclic complex

of this algebra. In 3.3 we discuss a different construction of the trace map, involving connections. Finally in 3.4 we show that the two maps are the same up to homotopy.

3.1 Let \mathcal{F} be a G -equivariant bundle over P . We assume that \mathcal{F} is \mathbb{Z}_2 -graded with the grading given by the operator $\gamma \in \text{End}(\mathcal{F})$. In this section we consider the algebra $\Psi(\mathcal{F})$ of the fiberwise pseudodifferential operators on the submersion $P \rightarrow B$ of order 0 acting on the sections of the bundle \mathcal{F} which are even with respect to the grading and whose Schwartz kernel is compactly supported. The \mathbb{Z}_2 grading we use here is induced by the grading of \mathcal{F} .

This algebra also has a natural filtration by the order of pseudodifferential operators. This filtration induces corresponding filtration on the cyclic homology complex of the algebra $\Psi(\mathcal{F})$. We denote by $F^{-k}CC_*(\Psi(\mathcal{F}))$ the subcomplex of the cyclic complex $CC_*(\Psi^0)$ generated by the $\bigoplus_{j_0+\dots+j_l \leq -k} \Psi^{j_0}(\mathcal{F}) \otimes \dots \otimes \Psi^{j_l}(\mathcal{F})$. Thus we obtain filtration

$$CC_*(\Psi(\mathcal{F})) = F^0CC_*(\Psi(\mathcal{F})) \supset F^{-1}CC_*(\Psi(\mathcal{F})) \supset F^{-2}CC_*(\Psi(\mathcal{F})) \supset \dots$$

It follows from Goodwillie's theorem [25], compare also [17], that the inclusion $F^{-i}CC_*(\Psi(\mathcal{F})) \rightarrow F^{-1}CC_*(\Psi(\mathcal{F}))$ is a quasiisomorphism for every i .

We will also consider the dual complexes $F_kCC^*(\Psi(\mathcal{F}))$, so that we have

$$CC^*(\Psi(\mathcal{F})) = F_0CC^*(\Psi(\mathcal{F})) \subset F_1CC^*(\Psi(\mathcal{F})) \subset F_2CC^*(\Psi(\mathcal{F})) \subset \dots$$

Groupoid G acts on $\Psi(\mathcal{F})$, preserving the order filtration. We can then form the cross-product algebra $\Psi(\mathcal{F}) \rtimes G$, which inherits filtration from the algebra $\Psi(\mathcal{F})$. We use this filtration to construct the complexes $F^{-k}CC_*(\Psi(\mathcal{F}) \rtimes G)$ and $F_kCC^*(\Psi(\mathcal{F}) \rtimes G)$. Note also that the map Φ defines a map of complexes:

$$\Phi : C^*(G, F_iCC^*(\Psi(\mathcal{F}))) \rightarrow F_iCC^*(\Psi(\mathcal{F}) \rtimes G). \quad (40)$$

3.2 Now *assuming that $P \rightarrow B$ is a fibration* we construct for every $n > \dim(P) - \dim(B)$ a map

$$\tau : F^{-n}CC_*(\Psi(\mathcal{F}) \rtimes G) \rightarrow CC_*(C_0^\infty(G)). \quad (41)$$

The construction is as follows, compare [13]. Consider a covering U_i of B trivializing the fibration. Set $B' = \coprod U_i$. Also set $P' = \coprod \pi^{-1}U_i$. The pull-back groupoid G' then acts on the trivial fibration $\pi' : P' \rightarrow B'$. Morita equivalence induces an isomorphism $CC_*(\Psi(\mathcal{F}) \rtimes G) \rightarrow CC_*(\Psi(\mathcal{F}') \rtimes G')$, where \mathcal{F}' is the pull-back of \mathcal{F} to P' . This isomorphism preserves filtrations. This allows

us to assume that the fibration is trivial, $P = F \times B$ with π given by the projection on the second factor. Notice that since the fibration is trivial the action of G on B induces an action on the fibration by $(f, b) \gamma = (f, b\gamma)$. We will call this action the product action. This action is different in general from the action arising from the original action on P .

We will need to consider a bigger algebra $\Psi\Delta = \Psi\Delta(\mathcal{F}) = \Psi(\mathcal{F}) \rtimes \text{Diff}$ instead of the algebra $\Psi(\mathcal{F})$, where Diff is the group of smooth families of the fiberwise diffeomorphisms, viewed as a discrete group. An element of this algebra can be viewed as a fiberwise integral operator on the fibration with a distributional kernel. We will denote by $\Psi\Delta \rtimes_t G$ the cross product of the algebra $\Psi\Delta$ by the product action. The algebra $\Psi\Delta$ contains a dense subalgebra defined as an algebraic tensor product of the algebra K of the smoothing integral operators on the fiber with the smooth compactly supported functions on B . The corresponding cross-product algebra is the algebraic tensor product $C_0^\infty(G) \otimes K$, and we define τ_0 by

$$\tau_0((a_0 \otimes k_0) \otimes \dots \otimes (a_i \otimes k_i)) = \text{Tr}_s(k_0 \dots k_i) a_0 \otimes a_1 \dots a_i \quad (42)$$

where $a_i \in C_0^\infty(G)$, $k_i \in K$. The map τ_0 then has a unique extension to the cyclic complex $F^{-n}CC_*(\Psi\Delta \rtimes_t G)$.

We now consider the general case, when the action on the fibration is not necessarily the product. In this case the action of $\gamma \in G$ is given by

$$(f, b) \gamma = (\sigma(\gamma) f, b\gamma) \quad (43)$$

where $\sigma(\gamma)$ is a diffeomorphism of the fiber F . Notice that σ has to satisfy the cocycle condition:

$$\sigma(\gamma_1\gamma_2) = \sigma(\gamma_2)\sigma(\gamma_1) \quad (44)$$

We will denote by $\Psi\Delta \rtimes G$ the cross-product of $\Psi\Delta$ by G with the action given by (43). These two cross-products are isomorphic as filtered algebras. Explicitly the isomorphism $I: \Psi\Delta \rtimes G \rightarrow \Psi\Delta \rtimes_t G$ is given by

$$(I(f))(\gamma) = f(\gamma)\sigma(\gamma) \quad (45)$$

We can now define the trace map τ by

$$\tau = \tau_0 \circ I_* \circ i_* \quad (46)$$

where $i: \Psi(\mathcal{F}) \rtimes G \rightarrow \Psi\Delta \rtimes G$ is the inclusion map.

3.3 On the other hand a choice of simplicial connection ∇ as in the section 2 provides us with the map

$$\mathcal{T}_{\nabla} : C^*(G, \Omega_*(B)) \rightarrow F_k CC^*(\Psi) \quad (47)$$

where k depends on the choice of the superconnection ∇ , or more precisely on order of it as a vertical pseudodifferential operator. The definition is parallel to the definition of the map $\Theta_{\mathbb{A}}$ but only the connection part is used instead of the full superconnection. Since the definition is very close to the construction in the section 2, we give just a brief outline here.

Notice that $\Psi(\mathcal{F})$ naturally can be viewed as compactly supported sections of a sheaf over B $U \mapsto \Psi^U(\mathcal{F})$, where $\Psi^U(\mathcal{F})$ is the algebra of proper fiberwise pseudodifferential operators on the submersion $\pi^{-1}(U) \rightarrow U$. We can then form a sheaf $CC^*(\Psi(\mathcal{F}))$ as the sheafification of the presheaf $U \mapsto CC^*(\Psi^U(\mathcal{F}))$.

First we consider the case of the trivial action of G . In that case for any choice of connection ∇ on $P \rightarrow B$ we can define a cocycle $\mathcal{L}_{\nabla} \in \text{Hom}(\Omega_*(B), F_m CC^*(\Psi(\mathcal{F})))$ by the formula similar to the equation (7) and m is specified below in (50). Explicitly we define

$$\mathcal{L}_{\nabla} : c \mapsto \sum_{l \geq -\frac{\deg c}{2}} u^{-l} \mathcal{L}_l(c) \quad (48)$$

where $\mathcal{L}_l(c) \in C^k(\Psi(\mathcal{F}))$, $k = \deg c + 2l$, is given by the formula:

$$\begin{aligned} & \mathcal{L}_l(c)(a_0, a_1, \dots, a_k) \\ &= \left\langle c, \int_{\Delta^k} \text{Tr}_s a_0 e^{-t_0 \nabla^2} [\nabla, a_1] \dots [\nabla, a_k] e^{-t_k \nabla^2} dt_1 \dots dt_k \right\rangle \end{aligned} \quad (49)$$

Here we view ∇^2 as a 2-form on B with values in the fiberwise differential operators and define $e^{-t\nabla^2}$ by the usual series.

If ∇ locally looks like $d + \omega$ where ω is a one-form on B with values in the proper fiberwise pseudodifferential operators of order $v \geq 1$. Then ∇^2 is a vertical operator of order at most $2v$. If $a_0 \otimes \dots \otimes a_k$ above is in $F^{-m} CC_*(\Psi(\mathcal{F}))$ then the expression under the integral above is a compactly supported fiberwise pseudodifferential operator of order at most $v \dim B - m$. Hence the expression under the integral will be well-defined if

$$m > (v - 1) \dim B + \dim P. \quad (50)$$

Next given a simplicial connection $\nabla^{(n)}$ we construct a cocycle

$$\{\mathcal{L}_{\nabla}^i\} \in C^*(G, \text{Hom}(\Omega_*(B), F_m CC^*(\Psi(\mathcal{F}))))). \quad (51)$$

The n -th component $\mathcal{L}_{\nabla}^n \in \varepsilon_n^* \text{Hom}^{-n}(\Omega_*(G), F_m CC^*(\Psi(\mathcal{F})))$ is given by $\mathcal{L}_{\nabla}^n = u^{-n} e^* \circ \mathcal{L}_{\nabla^{(n)}} \circ p^n$. We assume that order of $\nabla^{(n)}$ as a pseudodifferential operator is bounded by v , independent from n . Then for \mathcal{L}_{∇}^n to be well-defined it is sufficient to have $m > (v-1)(\dim B + n) + \dim P$. However, as before we have $\mathcal{L}_{\nabla}^n = 0$ for $n > \dim B$, by exactly the same argument as in the section 2. This implies that we indeed get a cochain in $C^*(G, \text{Hom}(\Omega_*(B), F_m CC^*(\Psi(\mathcal{F}))))$, and also that any $m > 2(v-1)\dim B + \dim P$ can be used for all n . The same argument shows that this cochain is indeed a cocycle. We can now construct a map of complexes $C^*(G, \Omega_*(B)) \rightarrow CC^*(G, F_m CC^*(\Psi(\mathcal{F})))$ by taking the cup product with $\{\mathcal{L}_{\nabla}^i\}$. Following it by the map Φ we obtain our map \mathcal{T}_{∇} . It is a map of complexes and different choice of simplicial connection leads to a chain-homotopic map. Explicitly we have

$$\mathcal{T}_{\nabla}(\alpha) = \Phi(\{\mathcal{L}_{\nabla}^i\} \cup \alpha). \quad (52)$$

3.4 In this section we prove the following result:

Theorem 15 *The following diagram is commutative up to homotopy.*

$$\begin{array}{ccc} & & F_m CC^*(\Psi(\mathcal{F}) \rtimes G) \\ & \nearrow \mathcal{T}_{\nabla} & \uparrow (\tau)^t \\ C^*(G, \Omega_*(B)) & \xrightarrow{\Phi} & CC^*(C_0^\infty(G)) \end{array} \quad (53)$$

PROOF.

First we reduce this statement to the case of trivial fibration $P \rightarrow B$. Let P' , B' and G' be as in section 3.2, and Ψ' -corresponding sheaf of algebras of pseudodifferential operators on $P' \rightarrow B'$. We then have for every n a natural projection $\rho : (G')^{(n)} \rightarrow G^{(n)}$, and $(P')^{(n)}$ is a pull-back, as a bundle, of $P^{(n)}$ under this projection. We can construct a simplicial connection ∇' by setting $(\nabla')^{(n)} = \rho^* \nabla^{(n)}$. It is easy to see then that $\{\mathcal{L}_{\nabla'}^i\} = \rho^* \{\mathcal{L}_{\nabla}^i\}$. This implies that the diagram

$$\begin{array}{ccc} C^*(G', \Omega_*(B')) & \xrightarrow{\cdot \cup \{\mathcal{L}_{\nabla'}^i\}} & C^*(G', F_m CC^*(\Psi(\mathcal{F}')) \\ \uparrow \rho^* & & \uparrow \rho^* \\ C^*(G, \Omega_*(B)) & \xrightarrow{\cdot \cup \{\mathcal{L}_{\nabla}^i\}} & C^*(G, F_m CC^*(\Psi(\mathcal{F}))) \end{array} \quad (54)$$

commutes. This, together with (24) implies commutativity of the diagram

$$\begin{array}{ccc}
C^*(G', \Omega_*(B')) & \xrightarrow{\mathcal{T}_{\nabla'}} & F_m CC^*(\Psi(\mathcal{F}') \rtimes G') \\
\uparrow \rho^* & & \uparrow \\
C^*(G, \Omega_*(B)) & \xrightarrow{\mathcal{T}_{\nabla}} & F_m CC^*(\Psi(\mathcal{F}) \rtimes G)
\end{array} \quad (55)$$

up to homotopy. Here the right vertical arrow is induced by the Morita equivalence. This, together with the definition of the map τ and (24) implies that the statement in the general case follows from the statement in the case of the trivial fibration.

Next consider the case of the trivial fibration with the product action. We can choose a trivial connection on fibrations $P^{(n)} \rightarrow G^{(n)}$, i.e. the one given by the de Rham differential d with respect to the decomposition $P^{(n)} = F \times G^{(n)}$. This clearly defines a simplicial connection which we denote ∇_0 . Commutativity of the diagram (53) in this case is clear.

Consider now the general case: $P = F \times B$, the action is given by the formula (45), and let ∇_1 be an arbitrary simplicial connection. The statement will follow from the previous observations and the following result: the diagram

$$\begin{array}{ccc}
& & F_m CC^*(\Psi\Delta \rtimes G) \\
& \nearrow \mathcal{T}_{\nabla_1} & \uparrow I^* \\
C^*(G, \Omega_*(B)) & & \\
& \searrow \mathcal{T}_{\nabla_0} & F_m CC^*(\Psi\Delta \rtimes_t G)
\end{array} \quad (56)$$

is commutative up to homotopy. To prove this consider the algebra $M_2(\Psi\Delta)$. The product action of G on $\Psi\Delta$ induces an action on the algebra $M_2(\Psi\Delta)$ which we denote as $m \mapsto m^\gamma$. Consider now another action of G on this algebra:

$$\gamma(m) = \begin{bmatrix} 1 & 0 \\ 0 & \sigma(\gamma) \end{bmatrix} m^\gamma \begin{bmatrix} 1 & 0 \\ 0 & \sigma(\gamma) \end{bmatrix}^{-1} \quad (57)$$

We use this action to form the cross-product algebra $M_2(\Psi\Delta) \rtimes G$. Notice that we have homomorphisms $i_0 : \Psi\Delta \rtimes_t G \rightarrow M_2(\Psi\Delta) \rtimes G$ and $i_1 : \Psi\Delta \rtimes G \rightarrow$

$M_2(\Psi\Delta) \rtimes G$ defined by

$$i_0(f)(\gamma) = \begin{bmatrix} f(\gamma) & 0 \\ 0 & 0 \end{bmatrix} \quad (58)$$

$$i_1(f)(\gamma) = \begin{bmatrix} 0 & 0 \\ 0 & f(\gamma) \end{bmatrix} \quad (59)$$

Notice that these homomorphisms are induced by the G -homomorphisms of the corresponding algebras. Let Λ be an automorphism of $M_2(\Psi\Delta) \rtimes G$ given by

$$\Lambda(f) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} f \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1}. \quad (60)$$

Notice that

$$i_0 \circ I = \Lambda \circ i_1 \quad (61)$$

We can construct a new simplicial connection $\nabla_0 \oplus \nabla_1$ on $\pi_*(\mathcal{F}) \oplus \pi_*(\mathcal{F})$. Its commutation with elements of $M_2(\Psi\Delta)$ is given by

$$\left[\nabla_0 \oplus \nabla_1, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right] = \begin{bmatrix} [\nabla_0, a] & [\nabla_0, b] - b(\nabla_1 - \nabla_0) \\ [\nabla_0, c] + (\nabla_1 - \nabla_0)c & [\nabla_1, d] \end{bmatrix} \quad (62)$$

and the curvature is

$$(\nabla_0 \oplus \nabla_1)^2 = \begin{bmatrix} \nabla_0^2 & 0 \\ 0 & \nabla_1^2 \end{bmatrix} \quad (63)$$

One way to view this construction is to replace P with $P \sqcup P$, fibered over B , with the G action on the first copy being the product action and on the second copy being the one given by equation (45). The algebra $M_2(\Psi\Delta)$ is then the cross-product algebra for this G -fibration.

We can use connection $\nabla_0 \oplus \nabla_1$ to construct the map

$$\mathcal{T}_{\nabla_0 \oplus \nabla_1} : C_*(G, \Omega_*(B)) \rightarrow F_m CC^*(M_2(\Psi\Delta) \rtimes G) \quad (64)$$

for m large enough. We then have

$$(i_0)^* \mathcal{T}_{\nabla_0 \oplus \nabla_1} = \mathcal{T}_{\nabla_0} \quad (65)$$

$$(i_1)^* \mathcal{T}_{\nabla_0 \oplus \nabla_1} = \mathcal{T}_{\nabla_1} \quad (66)$$

Here we have used the fact that the map Φ is compatible with homomorphisms. It follows these equalities and equation (61) that

$$I^* \circ \mathcal{T}_{\nabla_0} = I^* \circ (i_0)^* \circ \mathcal{T}_{\nabla_0 \oplus \nabla_1} = (i_1)^* \circ \Lambda^* \circ \mathcal{T}_{\nabla_0 \oplus \nabla_1} \quad (67)$$

We now notice that $\Lambda^* \circ \mathcal{T}_{\nabla_0 \oplus \nabla_1}$ is homotopic to $\mathcal{T}_{\nabla_0 \oplus \nabla_1}$. Indeed, let Λ_t be an automorphism of $M_2(\Psi\Delta) \rtimes G$ given by

$$\Lambda_t(f) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} f \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \quad (68)$$

Then $\Lambda_0 = \text{id}$, $\Lambda_{\frac{\pi}{2}} = \Lambda$. Since we have $\frac{d}{dt}\Lambda_t(f) = [\lambda, \Lambda_t(a)]$, where $\lambda = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ these maps satisfy $\frac{d}{dt}\Lambda_t^* = \Lambda_t^* L_\lambda$, where L_λ is the action of the derivation ad_λ on the cyclic complex. Notice that this derivation preserves the order filtration. We then have a homotopy formula

$$L_\lambda = [(B + b), H_\lambda] \quad (69)$$

for the action of L_λ , [7,25,21]. Explicit formulas for H_λ are not important here, we just need to know that it preserves the order filtration, since L_λ does. Since $\mathcal{T}_{\nabla_0 \oplus \nabla_1}$ is a map of complexes we conclude that

$$\begin{aligned} \frac{d}{dt}\Lambda_t^* \circ \mathcal{T}_{\nabla_0 \oplus \nabla_1}(c) = \\ (b + uB)(H_\lambda \Lambda_t^* \mathcal{T}_{\nabla_0 \oplus \nabla_1})(c) - (H_\lambda \Lambda_t^* \mathcal{T}_{\nabla_0 \oplus \nabla_1})(\pm\delta + u\partial). \end{aligned} \quad (70)$$

Integrating from 0 to $\pi/2$ provides us with desired homotopy and finishes the proof.

4 Comparison with the bivariant Chern character.

We start by reviewing in 4.1 V. Nistor's construction of the bivariant Chern character. Then in 4.2 we explain the construction of the quasihomomorphism ψ_D . Finally in 4.3 we show the commutativity of the lower triangle in (5) up to homotopy.

4.1 In [36,37] V. Nistor constructed a bivariant Chern character of a quasihomomorphism. We will need the following from this construction. Given algebras A and B , where B is equipped with filtration $B = B_0 \supset B_1 \dots$, with $B_i B_j \subset B_{i+j}$. We can use this filtration introduce cyclic complexes $F^i CC_*(B)$ and $F_i CC^*(B)$ as in the section 3.1.

A quasihomomorphism Υ is a pair of homomorphisms $v_0, v_1 : A \rightarrow B$ such that

$$v_1(a) - v_0(a) \in B_1 \quad (71)$$

V. Nistor constructed a sequence of maps $c_i(\Upsilon) : CC_*(A) \rightarrow CC_*^{-i}(B)$, $i = 1, 2, \dots$, which satisfy the following properties:

- The first map $c_1(\Upsilon)$ is given by the formula

$$c_1(\Upsilon) = \frac{1}{2}((v_1)_* - (v_0)_*) \quad (72)$$

We introduce the coefficient $\frac{1}{2}$ since we consider below the symmetrized version of quasihomomorphism associated to an operator, compare [7].

- If $r_i : F^{-i}CC_*(B) \rightarrow F^{-1}CC_*(B)$ is the natural inclusion, then $r_i \circ c_i(\Upsilon)$ is homotopic to $c_1(\Upsilon)$ for $i \geq 1$.

The construction proceeds as follows. Starting with the algebra A one constructs certain canonical algebra QA with filtration, together with canonical quasihomomorphism $j = (j_0, j_1)$ from A to QA . One constructs canonical maps $s_i : F^{-1}CC_*(QA) \rightarrow F^{-i}CC_*(QA)$ as follows: $s_1 = \text{id}$, $s_{i+1} = s_i + [b + uB, H_i]$, where H_i is a certain canonical endomorphism of $CC_*(QA)$, which preserves filtration. We note that s_i defines a homotopy equivalence of the complexes $F^{-1}CC_*(QA)$ and $F^{-i}CC_*(QA)$ for every i , with the homotopy inverse given by the inclusion $F^{-i}CC_*(QA) \rightarrow F^{-1}CC_*(QA)$. One then can define the bivariant Chern character of the canonical quasihomomorphism j . This is a sequence of canonical maps $c_k(j) : CC_*(A) \rightarrow F^kCC_*(QA)$ where $c_1(j) = \frac{1}{2}((i_1)_* - (i_0)_*)$ and $c_k(j) = s_k \circ c_1$ for $k \geq 1$. Now the quasihomomorphism Υ from A to B defines canonically a homomorphism $h : QA \rightarrow B$, which preserves filtration. The bivariant Chern character of Υ then is defined as $c_i(\Upsilon) = h_* \circ c_i(j)$.

We introduce also $c^i(\Upsilon) = c_i(\Upsilon)^t$. Assume now that A and B are G -algebras, filtration on B is G -invariant and v_0, v_1 are G -equivariant. In this case these maps have the following naturality property. A G -equivariant quasihomomorphism induces a quasihomomorphism from $A \rtimes G$ to $B \rtimes G$, which we also denote by Υ . Since all the steps in the construction of the bivariant Chern character are canonical $c_i(\Upsilon)$ can be chosen to be G -equivariant. Hence its transposed induces a map $C^*(G, F_iCC^*(B)) \rightarrow C^*(G, CC^*(A))$.

Proposition 16 *The following diagram is commutative up to homotopy for every $i \geq 1$:*

$$\begin{array}{ccc} C^*(G, CC^*(A)) & \xrightarrow{\Phi} & CC^*(A \rtimes G) \\ \uparrow c^i(\Upsilon) & & \uparrow c^i(\Upsilon) \\ C^*(G, F_iCC^*(B)) & \xrightarrow{\Phi} & F_iCC^*(B \rtimes G) \end{array} \quad (73)$$

PROOF. We start by showing the commutativity of the diagram

$$\begin{array}{ccc}
C^*(G, CC^*(A)) & \xrightarrow{\Phi} & CC^*(A \rtimes G) \\
\uparrow c^i(j) & & \uparrow c^i(j) \\
C^*(G, F_i CC^*(QA)) & \xrightarrow{\Phi} & F_i CC^*(QA \rtimes G)
\end{array} \tag{74}$$

First notice that since QA is constructed canonically it is indeed a G -algebra, so the statement makes sense. For $i = 1$ the commutativity is clear. Hence it is enough to show the commutativity up to homotopy of the diagram

$$\begin{array}{ccc}
C^*(G, F_1 CC^*(QA)) & \xrightarrow{\Phi} & F_1 CC^*(QA \rtimes G) \\
\uparrow s^i & & \uparrow s^i \\
C^*(G, F_i CC^*(QA)) & \xrightarrow{\Phi} & F_i CC^*(QA \rtimes G)
\end{array} \tag{75}$$

Here both vertical arrows are transposed of the maps s_i mentioned above. The right vertical arrow is constructed for the algebra $QA \rtimes G$. The left is constructed from the transposed of s_i for the algebra QA . Since the construction is canonical it is G -equivariant, and hence defines a map $C^*(G, F_i CC^*(QA)) \rightarrow C^*(G, F_1 CC^*(QA))$. Notice that both of these maps are homotopy equivalences of complexes, with the inverses induced by inclusion. This statement follows from the Goodwillie's theorem [25] for the right vertical arrow. For the left arrow we notice that the homotopy H_i above is constructed canonically and hence is G -equivariant. As a result H_i defines a filtration-preserving endomorphism of $C^*(G, CC^*(QA))$. This implies that s^i is indeed a homotopy equivalence. Note that since we consider only the finite cochains in the complex $C^*(G, CC^*(QA))$ the fact that $s^i : F_i CC^*(QA) \rightarrow F_1 CC^*(QA)$ is a homotopy equivalence does not imply that $C^*(G, F_i CC^*(QA)) \rightarrow C^*(G, F_1 CC^*(QA))$ is a homotopy equivalence; we need the more precise version of the argument given above.

Hence the commutativity of the diagram (75) follows from the commutativity of the diagram

$$\begin{array}{ccc}
C^*(G, F_1 CC^*(QA)) & \xrightarrow{\Phi} & F_1 CC^*(QA \rtimes G) \\
\downarrow & & \downarrow \\
C^*(G, F_i CC^*(QA)) & \xrightarrow{\Phi} & F_i CC^*(QA \rtimes G)
\end{array} \tag{76}$$

where the vertical arrows are induced by the transposes of the inclusions. This is true since Φ preserves filtrations.

Now to deduce the general case we note that the homomorphism $QA \rightarrow B$ being canonical is G -equivariant. Hence the commutativity of the diagram (73) follows from the compatibility of Φ with homomorphisms.

4.2 For our bundle \mathcal{E} consider the algebra $\Psi(\mathcal{E} \oplus \mathcal{E})$ of even operators on $\mathcal{E} \oplus \mathcal{E}$. Here \mathbb{Z}_2 -grading is defined as follows. The bundle \mathcal{E} has a grading given by the operator $\gamma \in \text{End}(\mathcal{E})$. This induces the grading on $\mathcal{E} \oplus \mathcal{E}$, and hence on $\Psi(\mathcal{E} \oplus \mathcal{E})$, given by the operator

$$\Gamma = \begin{bmatrix} \gamma & 0 \\ 0 & -\gamma \end{bmatrix} \quad (77)$$

We will write elements of $\Psi(\mathcal{E} \oplus \mathcal{E})$ as 2×2 matrices of pseudodifferential operators on \mathcal{E} . Consider now the algebra Ψ_r defined as a subalgebra of $\Psi(\mathcal{E} \oplus \mathcal{E})$

whose principal zero-order symbol has a form $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, where a is a scalar endomorphism of \mathcal{E} . We associate with the family D a G -equivariant quasi-homomorphism from the algebra $C_0^\infty(P)$ to the algebra Ψ_r . Here we use the filtration on Ψ_r induced by the filtration on $\Psi(\mathcal{E} \oplus \mathcal{E})$.

We now define quasi-homomorphism ψ_D by the following formulas:

$$\psi_0(a) = U_D \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} U_D^{-1} \quad (78)$$

$$\psi_1(a) = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \quad (79)$$

Here U_D is constructed as follows. Let Q be a family of proper pseudodifferential operators forming a parametrix of D . One can always choose Q to be G -equivariant, as follows for example from explicit formulas for parametrix [32]. Then $S_0 = 1 - QD$ and $S_1 = 1 - DQ$ are proper smoothing G -equivariant fiberwise operators. Define then U_D by the formula

$$U_D = \begin{bmatrix} D & S_1 \\ S_0 & -(1 + S_0)Q \end{bmatrix} \quad (80)$$

Its inverse U_D^{-1} is given by the following explicit formula:

$$U_D^{-1} = \begin{bmatrix} (1 + S_0) Q & S_0 \\ S_1 & -D \end{bmatrix} \quad (81)$$

It is easy to see that $\psi_1(a) - \psi_0(a) \in \Psi_r^{-1}$. Moreover ψ_0 and ψ_1 are G -equivariant, since D is G -equivariant. Notice also that U_D is odd with respect to Γ :

$$U_D \Gamma = -\Gamma U_D \quad (82)$$

and in particular $\psi_0(a)$ is an even operator.

Since the quasihomomorphism ψ_D is G -equivariant, it also defines a quasihomomorphism, also denoted by ψ_D , from $C_0^\infty(P) \rtimes G$ to $\Psi_r \rtimes G$. One now gets for every i the map of complexes $c_i(\psi_D) : CC_*(C_0^\infty(P)) \rightarrow F^{-i} CC_*(\Psi_r \rtimes G)$, satisfying the above listed properties. We now define the bivariant Chern character $\text{Ch}(D) : CC_*(C_0^\infty(P) \rtimes G) \rightarrow CC_*(C_0^\infty(G))$ by the formula

$$\text{Ch}(D) = \tau \circ c_i(\psi_D) \quad (83)$$

where τ is defined in (46) and $i > \dim P - \dim B$. It is clear that different choices of i give homotopic maps.

Remark 17 Write $D = \begin{bmatrix} 0 & D^+ \\ D^- & 0 \end{bmatrix}$, where the decomposition is with respect to the grading γ . We can use above formulas applied to D^+ , omitting however the $\frac{1}{2}$ factor from (72) and using ordinary trace instead of supertrace, to construct the bivariant Chern character $\text{Ch}(D^+)$. Similarly we can construct $\text{Ch}(D^-)$. It is easy to see that

$$\text{Ch}(D) = \frac{1}{2} (\text{Ch}(D^+) - \text{Ch}(D^-)) \quad (84)$$

It is however easy to see that $\text{Ch}(D^+)$ is homotopic to $-\text{Ch}(D^-)$, and hence $\text{Ch}(D)$ and $\text{Ch}(D^+)$ are homotopic.

Different choices of Q are homotopic, and hence lead to the homotopic maps of complexes. For our purposes it will be convenient to chose Q so that it commutes with D . That this is possible again follows from the explicit construction in [32]. In this case all the entries in U_D commute with D and hence

U_D commutes with $\begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}$.

4.3 We start by proving the following:

Proposition 18 *Let \mathbb{A} be any simplicial superconnection adapted to the family D , and let ∇ be any simplicial connection on \mathcal{E} , even with respect to γ . Then the following diagram is commutative up to homotopy.*

$$\begin{array}{ccc}
& & CC^*(C_0^\infty(P) \times G) & (85) \\
& \nearrow \Phi_{\mathbb{A}} & \uparrow c^m(\psi_D) & \\
C^*(G, \Omega_*(B)) & \xrightarrow{\mathcal{T}_{\nabla}} & F^m CC^*(\Psi_r \times G) &
\end{array}$$

PROOF. First notice that by the Theorem 13 we can assume that $\mathbb{A} = D + \nabla$. Consider now the superconnection $\tilde{\mathbb{A}} = \mathbb{A} \oplus \mathbb{A}$ on the bundle $\tilde{\mathcal{E}} = \mathcal{E} \oplus \mathcal{E}$. By this we mean that we lift naturally $\tilde{\mathcal{E}}$ to $P^{(n)}$ and define a simplicial superconnection $\tilde{\mathbb{A}}$ by $\tilde{\mathbb{A}}^{(n)} = \mathbb{A}^{(n)} \oplus \mathbb{A}^{(n)}$. It is adapted to the G -invariant family $\tilde{D} = D \oplus D$. We can now construct a cocycle $\{X_{\tilde{\mathbb{A}}}^i\} \in C^*(G, \text{Hom}(\Omega_*(B), F_1 CC^*(\Psi_r)))$ by exactly the same formulas as $\{\Theta_{\mathbb{A}}^i\}$. Since for $A \in \Psi_r$ the commutator $[\tilde{D}, A]$ has order 0, all the estimates used for the construction of $\{\Theta_{\mathbb{A}}^i\}$ still hold, and the formulas make sense. We use for $\{X_{\tilde{\mathbb{A}}}^i\}$ the same truncation as for $\{\Theta_{\mathbb{A}}^i\}$. Proof of the Theorem 13 applies here as well and shows that if one replaces $\tilde{\mathbb{A}}$ by another simplicial superconnection adapted to \tilde{D} one obtains a cohomologous cocycle.

Recall that $c^1(\psi_D)$ defines a G -equivariant map $F_1 CC^*(\Psi_r) \rightarrow CC^*(C_0^\infty(P))$, and hence a map from $C^*(G, F_1 CC^*(\Psi_r))$ to $C^*(G, CC^*(C_0^\infty(P)))$. We then have the following:

Lemma 19 *The cocycles $c^1(\psi_D) \circ \{X_{\tilde{\mathbb{A}}}^i\}$ and $\{\Theta_{\mathbb{A}}^i\}$ are cohomologous.*

PROOF. First notice that both $\psi_0^* \circ \{X_{\tilde{\mathbb{A}}}^i\}$ and $\psi_1^* \circ \{X_{\tilde{\mathbb{A}}}^i\}$ are well defined, and not just their difference. It is clear that $\psi_1^* \circ \{X_{\tilde{\mathbb{A}}}^i\} = \Phi_{\mathbb{A}}$. On the other hand the identity

$$\begin{aligned}
& \text{Tr}_s U_D a_0 U_D^{-1} e^{-t_0 \mathbb{A}^2} [\mathbb{A}, U_D a_1 U_D^{-1}] \dots [\mathbb{A}, U_D a_k U_D^{-1}] e^{-t_k \mathbb{A}^2} = \\
& - \text{Tr}_s a_0 e^{-t_0 (U_D^{-1} \mathbb{A} U_D)^2} [(U_D^{-1} \mathbb{A} U_D), a_1] \dots [(U_D^{-1} \mathbb{A} U_D), a_k] e^{-t_k (U_D^{-1} \mathbb{A} U_D)^2} \quad (86)
\end{aligned}$$

together with a similar identity for components of τ_s imply that $\psi_0^* \circ \{X_{\tilde{\mathbb{A}}}^i\} = -\psi_1^* \circ \{X_{U_D^{-1} \tilde{\mathbb{A}} U_D}^i\}$. Here $U_D^{-1} \tilde{\mathbb{A}} U_D$ is the simplicial superconnection defined by $(U_D^{-1} \tilde{\mathbb{A}} U_D)^{(n)} = U_D^{-1} (\tilde{\mathbb{A}})^{(n)} U_D$. It is also adapted to the operator \tilde{D} . Explicitly we have

$$U_D^{-1} \tilde{\mathbb{A}} U_D = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} + U_D^{-1} \begin{bmatrix} \nabla & 0 \\ 0 & \nabla \end{bmatrix} U_D \quad (87)$$

Since the cohomology class of $\{X_{\tilde{\mathbb{A}}}^i\}$ is independent of the superconnection we obtain that $\psi_0^*\{X_{\tilde{\mathbb{A}}}^i\}$ is cohomologous to $-\psi_1^*\{X_{\tilde{\mathbb{A}}}^i\}$, and the statement of the Lemma follows.

Consider now a map $X : \alpha \mapsto \Phi(\{X_{\tilde{\mathbb{A}}}^i\} \cup \alpha)$. It follows that the composition of X with the map $c^1(\psi_D) : F_1CC^*(\Psi_r \rtimes G) \rightarrow CC^*(C_0^\infty(P) \rtimes G)$ is homotopic to $\Phi_{\tilde{\mathbb{A}}}$. Let $r^m : F_1CC^*(\Psi_r \rtimes G) \rightarrow F_mCC^*(\Psi_r \rtimes G)$ be the be the transposed of the inclusion r_m . Since $c^m(\psi_D) \circ r^m$ is homotopic to $c^1(\psi_D)$, to complete the proof it is sufficient to show that $r^m \circ X$ is homotopic to $\mathcal{T}_{\tilde{\mathbb{V}}}$. To establish this we consider a homotopy $\tilde{\mathbb{A}}_t = t\tilde{\mathbb{A}} + (1-t)\tilde{\mathbb{V}} = t\tilde{D} + \tilde{\mathbb{V}}$ between $\tilde{\mathbb{A}}$ and $\tilde{\mathbb{V}}$. The existence of this homotopy clearly implies that $r^m \circ \{X_{\tilde{\mathbb{A}}}^i\}$ and $\{\mathcal{L}_{\tilde{\mathbb{V}}}^i\}$ are cohomologous after one establishes an analogue of the identity (37) in this context. Explicitly, one needs to show that the analogue of the cochain T is well define. This is done exactly as before, with the only difference being that the estimate of the Lemma 4 has to be replaced by the following:

Lemma 20 *Let V_0, V_1, \dots, V_l be pseudodifferential operators with a compact support acting on sections of some vector bundle over a manifold M . Let D be a first-order selfadjoint pseudodifferential operator on M . Assume that $\sum \text{order } V_i < -\dim M$. Then the expression*

$$\left| \int_{\Delta^k} \text{Tr}_s V_0 e^{-t_0 s^2 D^2} V_1 \dots V_l e^{-t_l s^2 D^2} dt_1 \dots dt_l \right| \quad (88)$$

is bounded uniformly in s .

PROOF. Let $\phi \in C_0^\infty(M)$ be such that $\phi V_0 = V_0$. Denote $V_0 e^{-t_0 s^2 D^2} V_1 \dots V_l e^{-t_l s^2 D^2}$ by A . Then $\text{Tr}_s A = \text{Tr}_s A\phi$. If $v \sum \text{order } V_i$, then the operator $A\phi D^{-v}$ is bounded uniformly in s . Let now $\psi \in C_0^\infty(M)$ be such that $\phi\psi = \phi$. Then $|\text{Tr}_s A| = |\text{Tr}_s A\phi D^{-v} D^v \phi| \leq \|A\phi\| |\text{Tr } D^v \phi|$, which is bounded uniformly.

This estimate implies that for sufficiently small m T is well defined. This completes the proof of the Proposition.

We now can prove the main result of this section:

Theorem 21 *The diagram*

$$\begin{array}{ccc}
 & & CC^*(C_0^\infty(P) \times G) \\
 & \nearrow \Phi_{\mathbb{A}} & \uparrow \text{Ch}(D)^t \\
 C^*(G, \Omega_*(B)) & \xrightarrow{\Phi} & CC^*(C_0^\infty(G))
 \end{array} \tag{89}$$

is commutative up to homotopy.

PROOF. The Proposition 16 together with the previous Proposition immediately imply commutativity of the diagram

$$\begin{array}{ccc}
 & & CC^*(C_0^\infty(P) \times G) \\
 & \nearrow \Phi_{\mathbb{A}} & \uparrow c^m(\psi_D) \\
 C^*(G, \Omega_*(B)) & \xrightarrow{\mathcal{T}_{\nabla}} & F_m CC^*(\Psi_r \times G)
 \end{array} \tag{90}$$

Combined with the Theorem 15 this gives the proof of our Theorem.

5 Bismut Superconnection and Short-time limit

In this section we obtain a topological expression for $\Phi_{\mathbb{A}}$, thus finishing the proof of the Theorem 1. In order to do this we construct simplicial Bismut superconnection \mathbb{A} for which $\Phi_{\mathbb{A}_s}$, with \mathbb{A}_s the rescaled superconnection, has a limit when $s \rightarrow 0$.

First we will need the following result:

Proposition 22 *Let \mathbb{A} be a simplicial superconnection. Consider the family of maps $\Phi_{\mathbb{A}_s}$, where \mathbb{A}_s is the rescaled superconnection. Then the maps obtained for different values of $s > 0$ are homotopic.*

PROOF. Consider first the case of the trivial groupoid action. In this case we have in the notations of the Proposition 8

$$\frac{d}{ds} \Theta_{\mathbb{A}_s}(c) = (b + uB + u\partial) \sum_{l \geq -\frac{\deg c - 1}{2}}^k u^{-l} (\tau_s)_l(c) \tag{91}$$

where τ_s is defined in the equation (8). Denote the cochain appearing in the right hand side by $T_{\mathbb{A}_s}$. In the general case construct the cochain $\{T_{\mathbb{A}_s}^i\} \in C^*(G, \text{Hom } \Omega_*(B), CC^*(C_0^\infty(P)))$ by $T_{\mathbb{A}_s}^n = u^{-n}e^* \circ T_{\mathbb{A}_s^{(n)}} \circ p^n$. Calculations as in the Lemma 12 then show that

$$\frac{d}{ds}\{\Theta_{\mathbb{A}_s}^i\} = (b + uB + u\partial \pm \delta)\{T_{\mathbb{A}_s}^i\}. \quad (92)$$

Integration of this identity shows that cocycles $\{\Theta_{\mathbb{A}_s}^i\}$ obtained for different values of s are cohomologous. This implies that corresponding maps $\Phi_{\mathbb{A}_s}$ are homotopic.

We now proceed to construct the simplicial Bismut superconnection. Recall that we are given a G -submersion $\pi : P \rightarrow B$. The fibers are equipped with the complete G -invariant Riemannian metric, and we are given a family of G -invariant fiberwise Dirac operators acting on the sections of G -equivariant Clifford module bundle \mathcal{E} . We start by reviewing the case of the trivial groupoid action, see [1] for the details. In this case one needs to make the following choices. First one needs to choose the horizontal distribution on the submersion $P \rightarrow B$. By this we mean choice of a smooth subbundle H of the tangent bundle TP such that $TP \cong H \oplus TP/B$. Here TP/B is the vertical tangent bundle. The choice of H is, of course, not unique. At each point $p \in P$ the set of all possible H_p has a natural structure of an affine space based on the vector space $\text{Hom}(T_{\pi(p)}B, T_pP/B)$. Hence the set of all horizontal distributions has a natural structure of an affine space. A choice of a horizontal distribution H is equivalent to the choice of projection $P_H : TP \rightarrow TP/B$. We have the following relation:

$$P_{tH_1+(1-t)H_2} = tP_{H_1} + (1-t)P_{H_2} \quad (93)$$

Choice of H allows one also to construct a canonical connection $\nabla^{P/B} = \nabla^{P/B}(H)$ on TP/B , whose restriction on the fibers of π coincides with Levi-Civita connection. This connection can be defined by the relation

$$\begin{aligned} (\nabla_X^{P/B}Y, Z) &= \frac{1}{2} ((P_H[X, Y], Z) - ([Y, Z], P_HX) + \\ &\quad (P_H[Z, X], Y) + X(Y, Z) - Z(P_HX, Y) + Y(Z, P_HX)) \end{aligned} \quad (94)$$

From this equation it is clear that the correspondence $H \mapsto \nabla^{P/B}(H)$ is an affine map. One then needs to chose a connection $\nabla^\mathcal{E}$ on the bundle \mathcal{E} which is compatible with the connection $\nabla^{P/B}(H)$. Notice that here again the vertical part of this connection is determined uniquely. Now using the horizontal distribution H and a connection $\nabla^\mathcal{E}$ one can construct a connection

$\nabla = \nabla^H$ on $\pi_*\mathcal{E}$, compare [1]. The following facts are easy consequences of the definitions:

- If H_1, H_2 are two horizontal distributions and $\nabla_1^\mathcal{E}$ and $\nabla_2^\mathcal{E}$ are two connections on \mathcal{E} compatible with the connections $\nabla^{P/B}(H_1)$ and $\nabla^{P/B}(H_2)$ respectively. Then the connection $t\nabla_1^\mathcal{E} + (1-t)\nabla_2^\mathcal{E}$ is compatible with $\nabla^{P/B}(tH_1 + (1-t)H_2)$.
- Let ∇^{H_i} be constructed using connection $\nabla_i^\mathcal{E}$, $i = 1, 2$. Then the connection $\nabla^{tH_1 + (1-t)H_2}$ constructed using $t\nabla_1^\mathcal{E} + (1-t)\nabla_2^\mathcal{E}$ is equal to $t\nabla^{H_1} + (1-t)\nabla^{H_2}$.

The Bismut superconnection is then defined as

$$\mathbb{A} = D + \nabla^H - \frac{1}{4}c(T^H) \quad (95)$$

where c denotes the Clifford multiplication by a vertical vector field and T is a 2-form on B with the values in the vertical vector fields given by the curvature of the distribution H . Explicitly it is described as follows. Let X, Y be two vector fields on B , and X^H, Y^H their lifts to the horizontal vector fields on P . Then

$$T^H(X, Y) = -([X^H, Y^H] - [X, Y]^H) \quad (96)$$

We now explain how one can extend this construction to the simplicial context. We fix the horizontal distribution H . Fix also a connection $\nabla^\mathcal{E}$ compatible with $\nabla^{P/B}$. We will now use this data to construct a simplicial superconnection such that each component $\mathbb{A}^{(n)}$ is a Bismut superconnection on the corresponding submersion. We start by constructing a horizontal distribution on each of the submersions $P^{(n)} \times \Delta^n \rightarrow G^{(n)} \times \Delta^n$. Recall the maps $\alpha_i : P^{(n)} \rightarrow P$ defined in (28). Define $\tilde{\alpha}_i$ as the composition of the projection $P^{(n)} \times \Delta^n \rightarrow P^{(n)}$ with the map α_i . Let σ_i be the barycentric coordinates on Δ^n ; below we view them as functions on $P^{(n)} \times \Delta^n$. Define the distribution $H^{(n)}$ as

$$H^{(n)} = \sum_{i=0}^n \sigma_i (d\tilde{\alpha}_i)^{-1}(H) \quad (97)$$

The corresponding projection is given by the formula

$$P_{H^{(n)}} = \sum_{i=0}^n \sigma_i ((\tilde{\alpha}_i)^*(P_H) \oplus 0). \quad (98)$$

Notice that this horizontal distribution satisfies the compatibility conditions

$$(\text{id} \times \partial_i)^{-1} H^{(n)} = (\delta_i \times \text{id})^{-1} H^{(n-1)}. \quad (99)$$

Using the formula (94) and the horizontal distribution $H^{(n)}$ we can construct for every n the canonical connection $(\nabla^{P/B})^{(n)}$ on the vertical bundle for the

submersion $P^{(n)} \times \Delta^n \rightarrow G^{(n)} \times \Delta^n$. It follows from the formula (94) that these connections are given explicitly by the formula

$$\left(\nabla^{P/B}\right)^{(n)} = \sum_{i=0}^n \sigma_i \alpha_i^* \nabla^{P/B} (H) + d = \sum_{i=0}^n \sigma_i \left(\alpha_i^* \nabla^{P/B} (H) + d\right) \quad (100)$$

where d is de Rham differential on Δ^n . It is clear that these connections satisfy the simplicial compatibility conditions. It follows then that the connections on $\left(\nabla^{\mathcal{E}}\right)^{(n)}$ on $\tilde{\alpha}_0^* \mathcal{E}$ defined by the formula

$$\left(\nabla^{\mathcal{E}}\right)^{(n)} = \sum_{i=0}^n \sigma_i \alpha_i^* \nabla^{\mathcal{E}} + d \quad (101)$$

are compatible with the connections $\left(\nabla^{P/B}\right)^{(n)}$. We can now construct for every n Bismut superconnection $\mathbb{A}^{(n)}$ on the submersion $P^{(n)} \times \Delta^n \rightarrow G^{(n)} \times \Delta^n$ using the distribution $H^{(n)}$ and connection $\left(\nabla^{\mathcal{E}}\right)^{(n)}$. The simplicial compatibility conditions (31) are clearly satisfied.

Also, the only terms involving $d\sigma$ in the superconnection form are the ones arising from $T^{H^{(n)}}$. But it is easy to see that $T^{H^{(n)}} \left(\frac{\partial}{\partial \sigma_i}, \frac{\partial}{\partial \sigma_i}\right) = 0$, and hence there are no terms of positive degree in $d\sigma$ and degree 0 in $G^{(n)}$ direction. It follows that the conditions stated after the equation (31) are satisfied as well; r can be taken to be 1.

To state the next proposition we follow conventions and notations from [1]. Let $R^{(n)}$ be the curvature of the connection $\left(\nabla^{P/B}\right)^{(n)}$. One can then define the $\hat{A}(R^{(n)})$, a differential form on $P^{(n)}$, where $\hat{A}(x) = \det^{\frac{1}{2}} \left(\frac{x/2}{\sinh(x/2)}\right)$. The structure of the Clifford module on \mathcal{E} allows one to view $R^{(n)}$ as a 2-form on P with values in the endomorphisms of \mathcal{E} . The twisting curvature $F_{\mathcal{E}/S}^{(n)}$ of $\left(\nabla^{\mathcal{E}}\right)^{(n)}$ is defined as $F_{\mathcal{E}/S}^{(n)} = \left(\left(\nabla^{\mathcal{E}}\right)^{(n)}\right)^2 - R^{(n)}$

We now have the following

Proposition 23 *a) The cocycle $\Theta_{\mathbb{A}_s}$ has a limit when $s \rightarrow 0$ given by*

$$\lim_{s \rightarrow 0} \Theta_{\mathbb{A}_s}^n = \sum_l u^{-n-l} \phi_l^n \quad (102)$$

where $\phi_l^n \in \text{Hom}^{-n} \left(\Omega_* \left(G^{(n)} \right), CC^* \left(C_0^\infty \left(P^{(n)} \right) \right) \right)$ is defined by the formula

$$\frac{\phi_l^n(c)(a_0, a_1, \dots, a_m) = (2\pi i)^{-\frac{\dim P - \dim B}{2}}}{m!} \left\langle \int_{(P^{(n)} \times \Delta^n)/P^{(n)}} \widehat{A}(R^{(n)}) \text{Ch}(F_{\mathcal{E}/S}^{(n)}) \pi_n^* c, a_0 da_1 \dots da_m \right\rangle \quad (103)$$

$$m = \deg c + 2l + n.$$

b) The cochain $T_{\mathbb{A}_s}$ defined in the proof of the Proposition 22 has a limit when $s \rightarrow 0$.

PROOF. From the construction of $\Theta_{\mathbb{A}_s}^n$ it is clear that it is enough to compute $\lim_{s \rightarrow 0} \Theta_{\mathbb{A}_s}$ with $\Theta_{\mathbb{A}_s}$ defined in the Proposition 8. From the equation (13) we obtain

$$(\Theta_{\mathbb{A}_s})_l = \begin{cases} (\theta_{\mathbb{A}_s})_l & \text{for } l < k \\ (\theta_{\mathbb{A}_s})_k - \int_0^s (B + \partial)(\tau_t)_{k+1} dt & \text{for } l = k \\ 0 & \text{for } l > k \end{cases} \quad (104)$$

We see that it is enough to study the behavior of $(\theta_{\mathbb{A}_s})_l$ when $s \rightarrow 0$. Now the standard application of the Getzler's calculus [1] shows that

$$\lim_{s \rightarrow 0} (\theta_{\mathbb{A}_s})_l(c)(a_0, a_1, \dots, a_m) = \frac{(2\pi i)^{-\frac{\dim P - \dim B}{2}}}{m!} \left\langle \widehat{A}(R) \text{Ch}(F_{\mathcal{E}/S}) \pi^* c, a_0 da_1 \dots da_m \right\rangle \quad (105)$$

where R is the curvature of the connection $\nabla^{P/B}$, and $F_{\mathcal{E}/S}$ is the twisting curvature of $\nabla^{\mathcal{E}}$. The statement of the part a) follows. The proof of the part b) is analogous to the proof of the local regularity of η -forms [3]. Namely one shows that $T_{\mathbb{A}_s}$ has asymptotic expansion in powers of s , starting with s^{-1} . Then one shows that the coefficient for the leading term is 0.

Note that the above Proposition implies that we can represent $\lim_{s \rightarrow 0} \Theta_{\mathbb{A}_s}$ as a composition of two maps.

The first one is $\tilde{\pi}^* : \Omega_*(G^{(n)}) \rightarrow \Omega_*(P^{(n)})$ defined by

$$\tilde{\pi}^* : c \mapsto (2\pi i)^{-\frac{\dim P - \dim B}{2}} \int_{(P^{(n)} \times \Delta^n)/P^{(n)}} \widehat{A}(u^{-1}R^{(n)}) \text{Ch}(u^{-1}F_{\mathcal{E}/S}^{(n)}) \wedge \pi_n^*(c) \quad (106)$$

It is easy to see that this map commutes with the differential $\pm\delta$ and hence defines a map of complexes $C^*(G, \Omega_*(B)) \rightarrow C^*(G, \Omega_*(P))$ To identify this map

on the level of cohomology notice that the collection of forms $\widehat{A}(R^{(n)}) \text{Ch}(F_{\mathcal{E}/\mathcal{S}}^{(n)})$ on $P^{(n)} \times \Delta^n$ defines a closed simplicial form in the sense of [19]. The cohomology class of this form is the product of $\widehat{A}_G(TP/B)$, the equivariant \widehat{A} -genus of TP/B , and $\text{Ch}_G(\mathcal{E}/\mathcal{S})$ the equivariant twisting Chern character of \mathcal{E} , compare [4]. Hence on the level of cohomology the map $\widetilde{\pi}^*$ defines a map $H_\tau^*(B_G) \rightarrow H_\tau^*(P_G)$ which is pull-back composed with the multiplication by $(2\pi i)^{-\frac{\dim P - \dim B}{2}} \widehat{A}_G(TP/B) \text{Ch}_G(\mathcal{E}/\mathcal{S})$.

The second one is the map $\iota : \Omega_*(P^{(n)}) \rightarrow CC^*(C_0^\infty(P^{(n)}))$ from the equation (21).

The Proposition 23 immediately implies the following:

Corollary 24 *The following diagram commutes up to homotopy*

$$\begin{array}{ccc}
C^*(G, \Omega_*(P)) & \xrightarrow{\Phi} & CC^*(C_0^\infty(P) \rtimes G) \\
\uparrow \widetilde{\pi}^* & \nearrow \Phi_\Delta & \\
C^*(G, \Omega_*(B)) & &
\end{array} \quad (107)$$

Combining this result with the Theorem 21 we obtain the main result of this paper:

Theorem 25 *The following diagram commutes up to homotopy*

$$\begin{array}{ccc}
C^*(G, \Omega_*(P)) & \xrightarrow{\Phi} & CC^*(C_0^\infty(P) \rtimes G) \\
\uparrow \widetilde{\pi}^* & & \uparrow \text{Ch}(D)^t \\
C^*(G, \Omega_*(B)) & \xrightarrow{\Phi} & CC^*(C_0^\infty(G))
\end{array} \quad (108)$$

Remark 26 *In this paper we were concerned only with the smooth currents. Our result can be generalized to the nonsmooth currents as follows. Fix an arbitrary integer number N and let Ω_k^N be the space of currents of degree k which belong to the Sobolev space H_{loc}^{N+k} . We can replace the complex $\Omega_*(B)$ by the similarly constructed complex using these currents. The results of these paper remain true. In the proofs one needs to use the estimates indicated in the Remark 6 in addition to the Lemma 4.*

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