

# Deformations of gerbes on smooth manifolds

Paul Bressler\*, Alexander Gorokhovsky†, Ryszard Nest , Boris Tsygan‡

**Abstract.** We identify the 2-groupoid of deformations of a gerbe on a  $C^\infty$  manifold with the Deligne 2-groupoid of a corresponding twist of the DGLA of local Hochschild cochains on  $C^\infty$  functions.

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## 1. Introduction

In [5] we obtained a classification of formal deformations of a gerbe on a manifold ( $C^\infty$  or complex-analytic) in terms of Maurer-Cartan elements of the DGLA of Hochschild cochains twisted by the cohomology class of the gerbe. In the present paper we develop a different approach to the derivation of this classification in the setting of  $C^\infty$  manifolds, based on the differential-geometric approach of [4].

The main result of the present paper is the following theorem which we prove in Section 8.

**Theorem 1.** *Suppose that  $X$  is a  $C^\infty$  manifold and  $\mathcal{S}$  is an algebroid stack on  $X$  which is a twisted form of  $\mathcal{O}_X$ . Then, there is an equivalence of 2-groupoid valued functors of commutative Artin  $\mathbb{C}$ -algebras*

$$\mathrm{Def}_X(\mathcal{S}) \cong \mathrm{MC}^2(\mathfrak{g}_{\mathrm{DR}}(\mathcal{J}_X)_{[\mathcal{S}]}) .$$

Notations in the statement of Theorem 1 and the rest of the paper are as follows. We consider a paracompact  $C^\infty$ -manifold  $X$  with the structure sheaf  $\mathcal{O}_X$  of complex valued smooth functions. Let  $\mathcal{S}$  be a twisted form of  $\mathcal{O}_X$ , as defined in Section 4.5. Twisted forms of  $\mathcal{O}_X$  are in bijective correspondence with  $\mathcal{O}_X^\times$ -gerbes and are classified up to equivalence by  $H^2(X; \mathcal{O}^\times) \cong H^3(X; \mathbb{Z})$ .

One can formulate the formal deformation theory of algebroid stacks ([17, 16]) which leads to the 2-groupoid valued functor  $\mathrm{Def}_X(\mathcal{S})$  of commutative Artin  $\mathbb{C}$ -algebras. We discuss deformations of algebroid stacks in the Section 6. It is natural

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to expect that the deformation theory of algebroid pre-stacks is “controlled” by a suitably constructed differential graded Lie algebra (DGLA) well-defined up to isomorphism in the derived category of DGLA. The content of Theorem 1 can be stated as the existence of such a DGLA, namely  $\mathfrak{g}(\mathcal{J}_X)_{[\mathcal{S}]}$ , which “controls” the formal deformation theory of the algebroid stack  $\mathcal{S}$  in the following sense.

To a nilpotent DGLA  $\mathfrak{g}$  which satisfies  $\mathfrak{g}^i = 0$  for  $i < -1$  one can associate its Deligne 2-groupoid which we denote  $\mathrm{MC}^2(\mathfrak{g})$ , see [11, 10] and references therein. We review this construction in the Section 3. Then the Theorem 1 asserts equivalence of 2-groupoids  $\mathrm{Def}_X(\mathcal{S})$  and  $\mathrm{MC}^2(\mathfrak{g}_{\mathrm{DR}}(\mathcal{J}_X)_{[\mathcal{S}]})$ .

The DGLA  $\mathfrak{g}(\mathcal{J}_X)_{[\mathcal{S}]}$  is defined as the  $[\mathcal{S}]$ -twist of the DGLA

$$\mathfrak{g}_{\mathrm{DR}}(\mathcal{J}_X) := \Gamma(X; \mathrm{DR}(\overline{\mathcal{C}}^\bullet(\mathcal{J}_X))[1]) .$$

Here,  $\mathcal{J}_X$  is the sheaf of infinite jets of functions on  $X$ , considered as a sheaf of topological  $\mathcal{O}_X$ -algebras with the canonical flat connection  $\nabla^{\mathrm{can}}$ . The shifted normalized Hochschild complex  $\overline{\mathcal{C}}^\bullet(\mathcal{J}_X)[1]$  is understood to comprise locally defined  $\mathcal{O}_X$ -linear continuous Hochschild cochains. It is a sheaf of DGLA under the Gerstenhaber bracket and the Hochschild differential  $\delta$ . The canonical flat connection on  $\mathcal{J}_X$  induces one, also denoted  $\nabla^{\mathrm{can}}$ , on  $\overline{\mathcal{C}}^\bullet(\mathcal{J}_X)[1]$ . The flat connection  $\nabla^{\mathrm{can}}$  commutes with the differential  $\delta$  and acts by derivations of the Gerstenhaber bracket. Therefore, the de Rham complex  $\mathrm{DR}(\overline{\mathcal{C}}^\bullet(\mathcal{J}_X)[1]) := \Omega_X^\bullet \otimes \overline{\mathcal{C}}^\bullet(\mathcal{J}_X)[1]$  equipped with the differential  $\nabla^{\mathrm{can}} + \delta$  and the Lie bracket induced by the Gerstenhaber bracket is a sheaf of DGLA on  $X$  giving rise to the DGLA  $\mathfrak{g}(\mathcal{J}_X)$  of global sections.

The sheaf of abelian Lie algebras  $\mathcal{J}_X/\mathcal{O}_X$  acts by derivations of degree  $-1$  on the graded Lie algebra  $\overline{\mathcal{C}}^\bullet(\mathcal{J}_X)[1]$  via the adjoint action. Moreover, this action commutes with the Hochschild differential. Therefore, the (abelian) graded Lie algebra  $\Omega_X^\bullet \otimes \mathcal{J}_X/\mathcal{O}_X$  acts by derivations on the graded Lie algebra  $\Omega_X^\bullet \otimes \overline{\mathcal{C}}^\bullet(\mathcal{J}_X)[1]$ . We denote the action of the form  $\omega \in \Omega_X^\bullet \otimes \mathcal{J}_X/\mathcal{O}_X$  by  $\iota_\omega$ . Consider now the subsheaf of closed forms  $(\Omega_X^\bullet \otimes \mathcal{J}_X/\mathcal{O}_X)^{\mathrm{cl}}$  which is by definition the kernel of  $\nabla^{\mathrm{can}}$ .  $(\Omega_X^k \otimes \mathcal{J}_X/\mathcal{O}_X)^{\mathrm{cl}}$  acts by derivations of degree  $k - 1$  and this action commutes with the differential  $\nabla^{\mathrm{can}} + \delta$ . Therefore, for  $\omega \in \Gamma(X; (\Omega^2 \otimes \mathcal{J}_X/\mathcal{O}_X)^{\mathrm{cl}})$  one can define the  $\omega$ -twist  $\mathfrak{g}(\mathcal{J}_X)_\omega$  as the DGLA with the same underlying graded Lie algebra structure as  $\mathfrak{g}(\mathcal{J}_X)$  and the differential given by  $\nabla^{\mathrm{can}} + \delta + \iota_\omega$ . The isomorphism class of this DGLA depends only on the cohomology class of  $\omega$  in  $H^2(\Gamma(X; \Omega_X^\bullet \otimes \mathcal{J}_X/\mathcal{O}_X), \nabla^{\mathrm{can}})$ .

More precisely, for  $\beta \in \Gamma(X; \Omega_X^1 \otimes \mathcal{J}_X/\mathcal{O}_X)$  the DGLA  $\mathfrak{g}_{\mathrm{DR}}(\mathcal{J}_X)_\omega$  and  $\mathfrak{g}_{\mathrm{DR}}(\mathcal{J}_X)_{\omega + \nabla^{\mathrm{can}}\beta}$  are canonically isomorphic with the isomorphism depending only on the equivalence class  $\beta + \mathrm{Im}(\nabla^{\mathrm{can}})$ .

As we remarked before a twisted form  $\mathcal{S}$  of  $\mathcal{O}_X$  is determined up to equivalence by its class in  $H^2(X; \mathcal{O}^\times)$ . The composition  $\mathcal{O}^\times \rightarrow \mathcal{O}^\times/\mathbb{C}^\times \xrightarrow{\log} \mathcal{O}/\mathbb{C} \xrightarrow{j^\infty} \mathrm{DR}(\mathcal{J}/\mathcal{O})$  induces the map  $H^2(X; \mathcal{O}^\times) \rightarrow H^2(X; \mathrm{DR}(\mathcal{J}/\mathcal{O})) \cong H^2(\Gamma(X; \Omega_X^\bullet \otimes \mathcal{J}_X/\mathcal{O}_X), \nabla^{\mathrm{can}})$ . We denote by  $[\mathcal{S}] \in H^2(\Gamma(X; \Omega_X^\bullet \otimes \mathcal{J}_X/\mathcal{O}_X), \nabla^{\mathrm{can}})$  the image of the class of  $\mathcal{S}$ . By the remarks above we have the well-defined up to a canonical isomorphism DGLA  $\mathfrak{g}_{\mathrm{DR}}(\mathcal{J}_X)_{[\mathcal{S}]}$ .

The rest of this paper is organized as follows. In the Section 2 we review some preliminary facts. In the Section 3 we review the construction of Deligne 2-groupoid, its relation with the deformation theory and its cosimplicial analogues. In the Section 4 we review the notion of algebroid stacks. Next we define matrix algebras associated with a descent datum in the Section 5. In the Section 6 we define the deformations of algebroid stacks and relate them to the cosimplicial DGLA of Hochschild cochains on matrix algebras. In the Section 7 we establish quasiisomorphism of the DGLA controlling the deformations of twisted forms of  $\mathcal{O}_X$  with a simpler cosimplicial DGLA. Finally, the proof the main result of this paper, Theorem 1, is given in the Section 8

## 2. Preliminaries

### 2.1. Simplicial notions.

**2.1.1. The category of simplices.** For  $n = 0, 1, 2, \dots$  we denote by  $[n]$  the category with objects  $0, \dots, n$  freely generated by the graph

$$0 \rightarrow 1 \rightarrow \dots \rightarrow n .$$

For  $0 \leq p \leq q \leq n$  we denote by  $(pq)$  the unique morphism  $p \rightarrow q$ .

We denote by  $\Delta$  the full subcategory of  $\mathbf{Cat}$  with objects the categories  $[n]$  for  $n = 0, 1, 2, \dots$

For  $0 \leq i \leq n + 1$  we denote by  $\partial_i = \partial_i^n : [n] \rightarrow [n + 1]$  the  $i^{\text{th}}$  face map, i.e. the unique map whose image does not contain the object  $i \in [n + 1]$ .

For  $0 \leq i \leq n - 1$  we denote by  $s_i = s_i^n : [n] \rightarrow [n - 1]$  the  $i^{\text{th}}$  degeneracy map, i.e. the unique surjective map such that  $s_i(i) = s_i(i + 1)$ .

**2.1.2. Simplicial and cosimplicial objects.** Suppose that  $\mathcal{C}$  is a category. By definition, a *simplicial object in  $\mathcal{C}$*  (respectively, a *cosimplicial object in  $\mathcal{C}$* ) is a functor  $\Delta^{op} \rightarrow \mathcal{C}$  (respectively, a functor  $\Delta \rightarrow \mathcal{C}$ ). Morphisms of (co)simplicial objects are natural transformations of functors.

For a simplicial (respectively, cosimplicial) object  $F$  we denote the object  $F([n]) \in \mathcal{C}$  by  $F_n$  (respectively,  $F^n$ ).

**2.2. Cosimplicial vector spaces.** Let  $V^\bullet$  be a cosimplicial vector space. We denote by  $C^\bullet(V)$  the associated complex with component  $C^n(V) = V^n$  and the differential  $\partial^n : C^n(V) \rightarrow C^{n+1}(V)$  defined by  $\partial^n = \sum_i (-1)^i \partial_i^n$ , where  $\partial_i^n$  is the map induced by the  $i^{\text{th}}$  face map  $[n] \rightarrow [n + 1]$ . We denote cohomology of this complex by  $H^\bullet(V)$ .

The complex  $C^\bullet(V)$  contains a normalized subcomplex  $\overline{C}^\bullet(V)$ . Here  $\overline{C}^n(V) = \{V \in V^n \mid s_i^n v = 0\}$ , where  $s_i^n : [n] \rightarrow [n - 1]$  is the  $i^{\text{th}}$  degeneracy map. Recall that the inclusion  $\overline{C}^\bullet(V) \rightarrow C^\bullet(V)$  is a quasiisomorphism.

Starting from a cosimplicial vector space  $V^\bullet$  one can construct a new cosimplicial vector space  $\widehat{V}^\bullet$  as follows. For every  $\lambda : [n] \rightarrow \Delta$  set  $\widehat{V}^\lambda = V^{\lambda(n)}$ . Suppose given another simplex  $\mu : [m] \rightarrow \Delta$  and morphism  $\phi : [m] \rightarrow [n]$  such that  $\mu = \lambda \circ \phi$  i.e.  $\phi$  is a morphism of simplices  $\mu \rightarrow \lambda$ . The morphism  $(0n)$  factors uniquely into  $0 \rightarrow \phi(0) \rightarrow \phi(m) \rightarrow n$ , which, under  $\lambda$ , gives the factorization of  $\lambda(0n) : \lambda(0) \rightarrow \lambda(n)$  (in  $\Delta$ ) into

$$\lambda(0) \xrightarrow{f} \mu(0) \xrightarrow{g} \mu(m) \xrightarrow{h} \lambda(n), \quad (2.1)$$

where  $g = \mu(0m)$ . The map  $\mu(m) \rightarrow \lambda(n)$  induces the map

$$\phi_* : \widehat{V}^\mu \rightarrow \widehat{V}^\lambda \quad (2.2)$$

Set now  $\widehat{V}^n = \prod_{[n] \xrightarrow{\lambda} \Delta} \widehat{V}^\lambda$ . The maps (2.2) endow  $\widehat{V}^\bullet$  with the structure of a cosimplicial vector space. We then have the following well-known result:

**Lemma 2.1.**

$$H^\bullet(V) \cong H^\bullet(\widehat{V}) \quad (2.3)$$

*Proof.* We construct morphisms of complexes inducing the isomorphisms in cohomology. We will use the following notations. If  $f \in \widehat{V}^n$  and  $\lambda : [n] \rightarrow \Delta$  we will denote by  $f(\lambda) \in \widehat{V}^\lambda$  its component in  $\widehat{V}^\lambda$ . For  $\lambda : [n] \rightarrow \Delta$  we denote by  $\lambda|_{[j]l} : [l-j] \rightarrow \Delta$  its truncation:  $\lambda|_{[j]l}(i) = \lambda(i+j)$ ,  $\lambda|_{[j]l}(i, k) = \lambda((i+j)(k+j))$ . For  $\lambda_1 : [n_1] \rightarrow \Delta$  and  $\lambda_2 : [n_2] \rightarrow \Delta$  with  $\lambda_1(n_1) = \lambda_2(0)$  define their concatenation  $\Lambda = \lambda_1 * \lambda_2 : [n_1 + n_2] \rightarrow \Delta$  by the following formulas.

$$\Lambda(i) = \begin{cases} \lambda_1(i) & \text{if } i \leq n_1 \\ \lambda_2(i - n_1) & \text{if } i \geq n_1 \end{cases}$$

$$\Lambda(ik) = \begin{cases} \lambda_1(ik) & \text{if } i, k \leq n_1 \\ \lambda_2((i - n_1)(k - n_1)) & \text{if } i, k \geq n_1 \\ \lambda_2(0(k - n_1)) \circ \lambda_1(in_1) & \text{if } i \leq n_1 \leq k \end{cases}$$

This operation is associative. Finally we will identify in our notations  $\lambda : [1] \rightarrow \Delta$  with the morphism  $\lambda(01)$  in  $\Delta$

The morphism  $C^\bullet(V) \rightarrow C^\bullet(\widehat{V})$  is constructed as follows. Let  $\lambda : [n] \rightarrow \Delta$  be a simplex in  $\Delta$  and define  $\lambda_k$  by  $\lambda(k) = [\lambda_k]$ ,  $k = 0, 1, \dots, n$ . Let  $\Upsilon(\lambda) : [n] \rightarrow \lambda(n)$  be a morphism in  $\Delta$  defined by

$$(\Upsilon(\lambda))(k) = \lambda(kn)(\lambda_k) \quad (2.4)$$

Then define the map  $\iota : V^\bullet \rightarrow \widehat{V}^\bullet$  by the formula

$$(\iota(v))(\lambda) = \Upsilon(\lambda)_* v \text{ for } v \in V^n$$

This is a map of cosimplicial vector spaces, and therefore it induces a morphism of complexes.

The morphism  $\pi : C^\bullet(\widehat{V}) \rightarrow C^\bullet(V)$  is defined by the formula

$$\pi(f) = (-1)^{\frac{n(n+1)}{2}} \sum_{0 \leq i_k \leq k+1} (-1)^{i_0 + \dots + i_{n-1}} f(\partial_{i_0}^0 * \partial_{i_1}^1 * \dots * \partial_{i_{n-1}}^{n-1}) \text{ for } f \in \widehat{V}^n$$

when  $n > 0$ , and  $\pi(f)$  is  $V^0$  component of  $f$  if  $n = 0$ .

The morphism  $\iota \circ \pi$  is homotopic to  $\text{Id}$  with the homotopy  $h : C^\bullet(\widehat{V}) \rightarrow C^{\bullet-1}(\widehat{V})$  given by the formula

$$hf(\lambda) = \sum_{j=0}^{n-1} \sum_{0 \leq i_k \leq k+1} (-1)^{i_0 + \dots + i_{j-1}} f(\partial_{i_0}^0 * \dots * \partial_{i_{j-1}}^{j-1} * \Upsilon(\lambda|_{[0j]}) * \lambda|_{[j \ (n-1)]})$$

for  $f \in \widehat{V}^n$  when  $n > 0$ , and  $h(f) = 0$  if  $n = 0$ .

The composition  $\pi \circ \iota : C^\bullet(V) \rightarrow C^\bullet(V)$  preserves the normalized subcomplex  $\overline{C}^\bullet(V)$  and acts by identity on it. Therefore  $\pi \circ \iota$  induces the identity map on cohomology. It follows that  $\pi$  and  $\iota$  are quasiisomorphisms inverse to each other.  $\square$

**2.3. Covers.** A cover (open cover) of a space  $X$  is a collection  $\mathcal{U}$  of open subsets of  $X$  such that  $\bigcup_{U \in \mathcal{U}} U = X$ .

**2.3.1. The nerve of a cover.** Let  $N_0\mathcal{U} = \coprod_{U \in \mathcal{U}} U$ . There is a canonical augmentation map

$$\epsilon_0 : N_0\mathcal{U} \xrightarrow{\coprod_{U \in \mathcal{U}} (U \hookrightarrow X)} X$$

Let

$$N_p\mathcal{U} = N_0\mathcal{U} \times_X \dots \times_X N_0\mathcal{U},$$

the  $(p+1)$ -fold fiber product.

The assignment  $N\mathcal{U} : \Delta \ni [p] \mapsto N_p\mathcal{U}$  extends to a simplicial space called *the nerve of the cover*  $\mathcal{U}$ . The effect of the face map  $\partial_i^n$  (respectively, the degeneracy map  $s_i^n$ ) will be denoted by  $d^i = d_n^i$  (respectively,  $\varsigma_i = \varsigma_i^n$ ) and is given by the projection *along* the  $i^{\text{th}}$  factor (respectively, the diagonal embedding on the  $i^{\text{th}}$  factor). Therefore for every morphism  $f : [p] \rightarrow [q]$  in  $\Delta$  we have a morphism  $N_q\mathcal{U} \rightarrow N_p\mathcal{U}$  which we denote by  $f^*$ . We will denote by  $f_*$  the operation  $(f^*)^*$  of pull-back along  $f^*$ ; if  $\mathcal{F}$  is a sheaf on  $N_p\mathcal{U}$  then  $f_*\mathcal{F}$  is a sheaf on  $N_q\mathcal{U}$ .

For  $0 \leq i \leq n+1$  we denote by  $\text{pr}_i^n : N_n\mathcal{U} \rightarrow N_0\mathcal{U}$  the projection onto the  $i^{\text{th}}$  factor. For  $0 \leq j \leq m$ ,  $0 \leq i_j \leq n$  the map  $\text{pr}_{i_0} \times \dots \times \text{pr}_{i_m} : N_n\mathcal{U} \rightarrow (N_0\mathcal{U})^m$  can be factored uniquely as a composition of a map  $N_n\mathcal{U} \rightarrow N_m\mathcal{U}$  and the canonical imbedding  $N_m\mathcal{U} \rightarrow (N_0\mathcal{U})^m$ . We denote this map  $N_n\mathcal{U} \rightarrow N_m\mathcal{U}$  by  $\text{pr}_{i_0 \dots i_m}^n$ .

The augmentation map  $\epsilon_0$  extends to a morphism  $\epsilon : N\mathcal{U} \rightarrow X$  where the latter is regarded as a constant simplicial space. Its component of degree  $n$   $\epsilon_n : N_n\mathcal{U} \rightarrow X$  is given by the formula  $\epsilon_n = \epsilon_0 \circ \text{pr}_i^n$ . Here  $0 \leq i \leq n+1$  is arbitrary.

**2.3.2. Čech complex.** Let  $\mathcal{F}$  be a sheaf of abelian groups on  $X$ . One defines a cosimplicial group  $\check{C}^\bullet(\mathcal{U}, \mathcal{F}) = \Gamma(N_\bullet \mathcal{U}; \epsilon^* \mathcal{F})$ , with the cosimplicial structure induced by the simplicial structure of  $N\mathcal{U}$ . The associated complex is the Čech complex of the cover  $\mathcal{U}$  with coefficients in  $\mathcal{F}$ . The differential  $\check{d}$  in this complex is given by  $\sum (-1)^i (d^i)^*$ .

**2.3.3. Refinement.** Suppose that  $\mathcal{U}$  and  $\mathcal{V}$  are two covers of  $X$ . A morphism of covers  $\rho : \mathcal{U} \rightarrow \mathcal{V}$  is a map of sets  $\rho : \mathcal{U} \rightarrow \mathcal{V}$  with the property  $U \subseteq \rho(U)$  for all  $U \in \mathcal{U}$ .

A morphism  $\rho : \mathcal{U} \rightarrow \mathcal{V}$  induces the map  $N\rho : N\mathcal{U} \rightarrow N\mathcal{V}$  of simplicial spaces which commutes with respective augmentations to  $X$ . The map  $N_0\rho$  is determined by the commutativity of

$$\begin{array}{ccc} U & \longrightarrow & N_0\mathcal{U} \\ \downarrow & & \downarrow N_0\rho \\ \rho(U) & \longrightarrow & N_0\mathcal{V} \end{array}$$

It is clear that the map  $N_0\rho$  commutes with the respective augmentations (i.e. is a map of spaces over  $X$ ) and, consequently induces maps  $N_n\rho = (N_0\rho)^{\times_{X^{n+1}}}$  which commute with all structure maps.

**2.3.4. The category of covers.** Let  $\text{Cov}(X)_0$  denote the set of open covers of  $X$ . For  $\mathcal{U}, \mathcal{V} \in \text{Cov}(X)_0$  we denote by  $\text{Cov}(X)_1(\mathcal{U}, \mathcal{V})$  the set of morphisms  $\mathcal{U} \rightarrow \mathcal{V}$ . Let  $\text{Cov}(X)$  denote the category with objects  $\text{Cov}(X)_0$  and morphisms  $\text{Cov}(X)_1$ . The construction of 2.3.1 is a functor

$$N : \text{Cov}(X) \rightarrow \text{Top}^{\Delta^{op}}/X .$$

## 3. Deligne 2-groupoid and its cosimplicial analogues

In this section we begin by recalling the definition of Deligne 2-groupoid and its relation with the deformation theory. We then describe the cosimplicial analogues of Deligne 2-groupoids and establish some of their properties.

**3.1. Deligne 2-groupoid.** In this subsection we review the construction of Deligne 2-groupoid of a nilpotent differential graded algebra (DGLA). We follow [11, 10] and references therein.

Suppose that  $\mathfrak{g}$  is a nilpotent DGLA such that  $\mathfrak{g}^i = 0$  for  $i < -1$ .

A Maurer-Cartan element of  $\mathfrak{g}$  is an element  $\gamma \in \mathfrak{g}^1$  satisfying

$$d\gamma + \frac{1}{2}[\gamma, \gamma] = 0. \quad (3.1)$$

We denote by  $\text{MC}^2(\mathfrak{g})_0$  the set of Maurer-Cartan elements of  $\mathfrak{g}$ .

The unipotent group  $\exp \mathfrak{g}^0$  acts on the set of Maurer-Cartan elements of  $\mathfrak{g}$  by the gauge equivalences. This action is given by the formula

$$(\exp X) \cdot \gamma = \gamma - \sum_{i=0}^{\infty} \frac{(\operatorname{ad} X)^i}{(i+1)!} (dX + [\gamma, X])$$

If  $\exp X$  is a gauge equivalence between two Maurer-Cartan elements  $\gamma_1$  and  $\gamma_2 = (\exp X) \cdot \gamma_1$  then

$$d + \operatorname{ad} \gamma_2 = \operatorname{Ad} \exp X (d + \operatorname{ad} \gamma_1). \quad (3.2)$$

We denote by  $\operatorname{MC}^2(\mathfrak{g})_1(\gamma_1, \gamma_2)$  the set of gauge equivalences between  $\gamma_1, \gamma_2$ . The composition

$$\operatorname{MC}^2(\mathfrak{g})_1(\gamma_2, \gamma_3) \times \operatorname{MC}^2(\mathfrak{g})_1(\gamma_1, \gamma_2) \rightarrow \operatorname{MC}^2(\mathfrak{g})_1(\gamma_1, \gamma_3)$$

is given by the product in the group  $\exp \mathfrak{g}^0$ .

If  $\gamma \in \operatorname{MC}^2(\mathfrak{g})_0$  we can define a Lie bracket  $[\cdot, \cdot]_\gamma$  on  $\mathfrak{g}^{-1}$  by

$$[a, b]_\gamma = [a, db + [\gamma, b]]. \quad (3.3)$$

With this bracket  $\mathfrak{g}^{-1}$  becomes a nilpotent Lie algebra. We denote by  $\exp_\gamma \mathfrak{g}^{-1}$  the corresponding unipotent group, and by  $\exp_\gamma$  the corresponding exponential map  $\mathfrak{g}^{-1} \rightarrow \exp_\gamma \mathfrak{g}^{-1}$ . If  $\gamma_1, \gamma_2$  are two Maurer-Cartan elements, then the group  $\exp_\gamma \mathfrak{g}^{-1}$  acts on  $\operatorname{MC}^2(\mathfrak{g})_1(\gamma_1, \gamma_2)$ . Let  $\exp_\gamma t \in \exp_\gamma \mathfrak{g}^{-1}$  and let  $\exp X \in \operatorname{MC}^2(\mathfrak{g})_1(\gamma_1, \gamma_2)$ . Then

$$(\exp_\gamma t) \cdot (\exp X) = \exp(dt + [\gamma, t]) \exp X \in \exp \mathfrak{g}^0$$

Such an element  $\exp_\gamma t$  is called a 2-morphism between  $\exp X$  and  $(\exp t) \cdot (\exp X)$ .

We denote by  $\operatorname{MC}^2(\mathfrak{g})_2(\exp X, \exp Y)$  the set of 2-morphisms between  $\exp X$  and  $\exp Y$ . This set is endowed with a vertical composition given by the product in the group  $\exp_\gamma \mathfrak{g}^{-1}$ .

Let  $\gamma_1, \gamma_2, \gamma_3 \in \operatorname{MC}^2(\mathfrak{g})_0$ . Let  $\exp X_{12}, \exp Y_{12} \in \operatorname{MC}^2(\mathfrak{g})_1(\gamma_1, \gamma_2)$  and  $\exp X_{23}, \exp Y_{23} \in \operatorname{MC}^2(\mathfrak{g})_1(\gamma_2, \gamma_3)$ . Then one defines the horizontal composition

$$\begin{aligned} \otimes : \operatorname{MC}^2(\mathfrak{g})_2(\exp X_{23}, \exp Y_{23}) \times \operatorname{MC}^2(\mathfrak{g})_2(\exp X_{12}, \exp Y_{12}) \rightarrow \\ \operatorname{MC}^2(\mathfrak{g})_2(\exp X_{23} \exp X_{12}, \exp X_{23} \exp Y_{12}) \end{aligned}$$

as follows. Let  $\exp_{\gamma_2} t_{12} \in \operatorname{MC}^2(\mathfrak{g})_2(\exp X_{12}, \exp Y_{12})$ ,  $\exp_{\gamma_3} t_{23} \in \operatorname{MC}^2(\mathfrak{g})_2(\exp X_{23}, \exp Y_{23})$ . Then

$$\exp_{\gamma_3} t_{23} \otimes \exp_{\gamma_2} t_{12} = \exp_{\gamma_3} t_{23} \exp_{\gamma_3} (e^{\operatorname{ad} X_{23}}(t_{12}))$$

To summarize, the data described above forms a 2-groupoid which we denote by  $\operatorname{MC}^2(\mathfrak{g})$  as follows:

1. the set of objects is  $\operatorname{MC}^2(\mathfrak{g})_0$
2. the groupoid of morphisms  $\operatorname{MC}^2(\mathfrak{g})(\gamma_1, \gamma_2)$ ,  $\gamma_i \in \operatorname{MC}^2(\mathfrak{g})_0$  consists of:

- objects i.e. 1-morphisms in  $\mathrm{MC}^2(\mathfrak{g})$  are given by  $\mathrm{MC}^2(\mathfrak{g})_1(\gamma_1, \gamma_2)$  – the gauge transformations between  $\gamma_1$  and  $\gamma_2$ .
- morphisms between  $\exp X, \exp Y \in \mathrm{MC}^2(\mathfrak{g})_1(\gamma_1, \gamma_2)$  are given by  $\mathrm{MC}^2(\mathfrak{g})_2(\exp X, \exp Y)$ .

A morphism of nilpotent DGLA  $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$  induces a functor  $\phi : \mathrm{MC}^2(\mathfrak{g}) \rightarrow \mathrm{MC}^2(\mathfrak{h})$ .

We have the following important result ([12], [11] and references therein).

**Theorem 3.1.** *Suppose that  $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$  is a quasi-isomorphism of DGLA and let  $\mathfrak{m}$  be a nilpotent commutative ring. Then the induced map  $\phi : \mathrm{MC}^2(\mathfrak{g} \otimes \mathfrak{m}) \rightarrow \mathrm{MC}^2(\mathfrak{h} \otimes \mathfrak{m})$  is an equivalence of 2-groupoids.*

**3.2. Deformations and Deligne 2-groupoid.** Let  $k$  be an algebraically closed field of characteristic zero.

**3.2.1. Hochschild cochains.** Suppose that  $A$  is a  $k$ -vector space. The  $k$ -vector space  $C^n(A)$  of Hochschild cochains of degree  $n \geq 0$  is defined by

$$C^n(A) := \mathrm{Hom}_k(A^{\otimes n}, A) .$$

The graded vector space  $\mathfrak{g}(A) := C^\bullet(A)[1]$  has a canonical structure of a graded Lie algebra under the Gerstenhaber bracket denoted by  $[ , ]$  below. Namely,  $C^\bullet(A)[1]$  is canonically isomorphic to the (graded) Lie algebra of derivations of the free associative co-algebra generated by  $A[1]$ .

Suppose in addition that  $A$  is equipped with a bilinear operation  $\mu : A \otimes A \rightarrow A$ , i.e.  $\mu \in C^2(A) = \mathfrak{g}^1(A)$ . The condition  $[\mu, \mu] = 0$  is equivalent to the associativity of  $\mu$ .

Suppose that  $A$  is an associative  $k$ -algebra with the product  $\mu$ . For  $a \in \mathfrak{g}(A)$  let  $\delta(a) = [\mu, a]$ . Thus,  $\delta$  is a derivation of the graded Lie algebra  $\mathfrak{g}(A)$ . The associativity of  $\mu$  implies that  $\delta^2 = 0$ , i.e.  $\delta$  defines a differential on  $\mathfrak{g}(A)$  called the Hochschild differential.

For a unital algebra the subspace of *normalized cochains*  $\overline{C}^n(A) \subset C^n(A)$  is defined by

$$\overline{C}^n(A) := \mathrm{Hom}_k((A/k \cdot 1)^{\otimes n}, A) .$$

The subspace  $\overline{C}^\bullet(A)[1]$  is closed under the Gerstenhaber bracket and the action of the Hochschild differential and the inclusion  $\overline{C}^\bullet(A)[1] \hookrightarrow C^\bullet(A)[1]$  is a quasi-isomorphism of DGLA.

Suppose in addition that  $R$  is a commutative Artin  $k$ -algebra with the nilpotent maximal ideal  $\mathfrak{m}_R$ . The DGLA  $\mathfrak{g}(A) \otimes_k \mathfrak{m}_R$  is nilpotent and satisfies  $\mathfrak{g}^i(A) \otimes_k \mathfrak{m}_R = 0$  for  $i < -1$ . Therefore, the Deligne 2-groupoid  $\mathrm{MC}^2(\mathfrak{g}(A) \otimes_k \mathfrak{m}_R)$  is defined. Moreover, it is clear that the assignment  $R \mapsto \mathrm{MC}^2(\mathfrak{g}(A) \otimes_k \mathfrak{m}_R)$  extends to a functor on the category of commutative Artin algebras.

**3.2.2. Star products.** Suppose that  $A$  is an associative unital  $k$ -algebra. Let  $m$  denote the product on  $A$ .

Let  $R$  be a commutative Artin  $k$ -algebra with maximal ideal  $\mathfrak{m}_R$ . There is a canonical isomorphism  $R/\mathfrak{m}_R \cong k$ .

A  $(R)$ -star product on  $A$  is an associative  $R$ -bilinear product on  $A \otimes_k R$  such that the canonical isomorphism of  $k$ -vector spaces  $(A \otimes_k R) \otimes_R k \cong A$  is an isomorphism of algebras. Thus, a star product is an  $R$ -deformation of  $A$ .

The 2-category of  $R$ -star products on  $A$ , denoted  $\text{Def}(A)(R)$ , is defined as the subcategory of the 2-category  $\text{Alg}_R^2$  of  $R$ -algebras (see 4.1.1) with

- Objects:  $R$ -star products on  $A$ ,
- 1-morphisms  $\phi : m_1 \rightarrow m_2$  between the star products  $\mu_i$  those  $R$ -algebra homomorphisms  $\phi : (A \otimes_k R, m_1) \rightarrow (A \otimes_k R, m_2)$  which reduce to the identity map modulo  $\mathfrak{m}_R$ , i.e.  $\phi \otimes_R k = \text{Id}_A$
- 2-morphisms  $b : \phi \rightarrow \psi$ , where  $\phi, \psi : m_1 \rightarrow m_2$  are two 1-morphisms, are elements  $b \in 1 + A \otimes_k \mathfrak{m}_R \subset A \otimes_k R$  such that  $m_2(\phi(a), b) = m_2(b, \psi(a))$  for all  $a \in A \otimes_k R$ .

It follows easily from the above definition and the nilpotency of  $\mathfrak{m}_R$  that  $\text{Def}(A)(R)$  is a 2-groupoid.

Note that  $\text{Def}(A)(R)$  is non-empty: it contains the trivial deformation, i.e. the star product, still denoted  $m$ , which is the  $R$ -bilinear extension of the product on  $A$ .

It is clear that the assignment  $R \mapsto \text{Def}(A)(R)$  extends to a functor on the category of commutative Artin  $k$ -algebras.

**3.2.3. Star products and the Deligne 2-groupoid.** We continue in notations introduced above. In particular, we are considering an associative unital  $k$ -algebra  $A$ . The product  $m \in C^2(A)$  determines a cochain, still denoted  $m \in \mathfrak{g}^1(A) \otimes_k R$ , hence the Hochschild differential  $\delta = [m, ]$  in  $\mathfrak{g}(A) \otimes_k R$  for any commutative Artin  $k$ -algebra  $R$ .

Suppose that  $m'$  is an  $R$ -star product on  $A$ . Since  $\mu(m') := m' - m = 0 \pmod{\mathfrak{m}_R}$  we have  $\mu(m') \in \mathfrak{g}^1(A) \otimes_k \mathfrak{m}_R$ . Moreover, the associativity of  $m'$  implies that  $\mu(m')$  satisfies the Maurer-Cartan equation, i.e.  $\mu(m') \in \text{MC}^2(\mathfrak{g}(A) \otimes_k \mathfrak{m}_R)_0$ .

It is easy to see that the assignment  $m' \mapsto \mu(m')$  extends to a functor

$$\text{Def}(A)(R) \rightarrow \text{MC}^2(\mathfrak{g}(A) \otimes_k \mathfrak{m}_R) . \quad (3.4)$$

The following proposition is well-known (cf. [9, 11, 10]).

**Proposition 3.2.** *The functor (3.4) is an isomorphism of 2-groupoids.*

**3.2.4. Star products on sheaves of algebras.** The above considerations generalize to sheaves of algebras in a straightforward way.

Suppose that  $\mathcal{A}$  is a sheaf of  $k$ -algebras on a space  $X$ . Let  $m : \mathcal{A} \otimes_k \mathcal{A} \rightarrow \mathcal{A}$  denote the product.

An  $R$ -star product on  $\mathcal{A}$  is a structure of a sheaf of an associative algebras on  $\mathcal{A} \otimes_k R$  which reduces to  $\mu$  modulo the maximal ideal  $\mathfrak{m}_R$ . The 2-category (groupoid) of  $R$  star products on  $\mathcal{A}$ , denoted  $\text{Def}(\mathcal{A})(R)$  is defined just as in the case of algebras; we leave the details to the reader.

The sheaf of Hochschild cochains of degree  $n$  is defined by

$$C^n(\mathcal{A}) := \underline{\text{Hom}}(\mathcal{A}^{\otimes n}, \mathcal{A}).$$

We have the sheaf of DGLA  $\mathfrak{g}(A) := C^\bullet(\mathcal{A})[1]$ , hence the nilpotent DGLA  $\Gamma(X; \mathfrak{g}(\mathcal{A}) \otimes_k \mathfrak{m}_R)$  for every commutative Artin  $k$ -algebra  $R$  concentrated in degrees  $\geq -1$ . Therefore, the 2-groupoid  $\text{MC}^2(\Gamma(X; \mathfrak{g}(\mathcal{A}) \otimes_k \mathfrak{m}_R))$  is defined.

The canonical functor  $\text{Def}(\mathcal{A})(R) \rightarrow \text{MC}^2(\Gamma(X; \mathfrak{g}(\mathcal{A}) \otimes_k \mathfrak{m}_R))$  defined just as in the case of algebras is an isomorphism of 2-groupoids.

**3.3.  $\mathfrak{G}$ -stacks.** Suppose that  $\mathfrak{G}: [n] \rightarrow \mathfrak{G}^n$  is a cosimplicial DGLA. We assume that each  $\mathfrak{G}^n$  is a nilpotent DGLA. We denote its component of degree  $i$  by  $\mathfrak{G}^{n,i}$  and assume that  $\mathfrak{G}^{n,i} = 0$  for  $i < -1$ .

**Definition 3.3.** A  $\mathfrak{G}$ -stack is a triple  $\gamma = (\gamma^0, \gamma^1, \gamma^2)$ , where

- $\gamma^0 \in \text{MC}^2(\mathfrak{G}^0)_0$ ,
- $\gamma^1 \in \text{MC}^2(\mathfrak{G}^1)_1(\partial_0^0 \gamma^0, \partial_1^0 \gamma^0)$ ,  
satisfying the condition
$$s_0^1 \gamma^1 = \text{Id}$$
- $\gamma^2 \in \text{MC}^2(\mathfrak{G}^2)_2(\partial_2^1(\gamma^1) \circ \partial_0^1(\gamma^1), \partial_1^1(\gamma^1))$

satisfying the conditions

$$\begin{aligned} \partial_2^2 \gamma_2 \circ (\text{Id} \otimes \partial_0^2 \gamma_2) &= \partial_1^2 \gamma_2 \circ (\partial_3^2 \gamma_2 \otimes \text{Id}) \\ s_0^2 \gamma^2 &= s_1^2 \gamma^2 = \text{Id} \end{aligned} \tag{3.5}$$

Let  $\text{Stack}(\mathfrak{G})_0$  denote the set of  $\mathfrak{G}$ -stacks.

**Definition 3.4.** For  $\gamma_1, \gamma_2 \in \text{Stack}(\mathfrak{G})_0$  a 1-morphism  $\mathfrak{J} : \gamma_1 \rightarrow \gamma_2$  is a pair  $\mathfrak{J} = (\mathfrak{J}^1, \mathfrak{J}^2)$ , where  $\mathfrak{J}^1 \in \text{MC}^2(\mathfrak{G}^0)_1(\gamma_1^0, \gamma_2^0)$ ,  $\mathfrak{J}^2 \in \text{MC}^2(\mathfrak{G}^1)_2(\gamma_2^1 \circ \partial_0^0(\mathfrak{J}^1), \partial_1^0(\mathfrak{J}^1) \circ \gamma_1^1)$ , satisfying

$$\begin{aligned} (\text{Id} \otimes \gamma_1^2) \circ (\partial_2^1 \mathfrak{J}^2 \otimes \text{Id}) \circ (\text{Id} \otimes \partial_0^1 \mathfrak{J}^2) &= \partial_1^1 \mathfrak{J}^2 \circ (\gamma_2^2 \otimes \text{Id}) \\ s_0^1 \mathfrak{J}^2 &= \text{Id} \end{aligned} \tag{3.6}$$

Let  $\text{Stack}(\mathfrak{G})_1(\gamma_1, \gamma_2)$  denote the set of 1-morphisms  $\gamma_1 \rightarrow \gamma_2$

Composition of 1-morphisms  $\mathfrak{J} : \gamma_1 \rightarrow \gamma_2$  and  $\mathfrak{T} : \gamma_2 \rightarrow \gamma_3$  is given by  $(\mathfrak{T}^1 \circ \mathfrak{J}^1, (\mathfrak{T}^2 \otimes \text{Id}) \circ (\text{Id} \otimes \mathfrak{T}^2))$ .

**Definition 3.5.** For  $\mathfrak{J}_1, \mathfrak{J}_2 \in \text{Stack}(\mathfrak{G})_1(\gamma_1, \gamma_2)$  a 2-morphism  $\phi : \mathfrak{J}_1 \rightarrow \mathfrak{J}_2$  is a 2-morphism  $\phi \in \text{MC}^2(\mathfrak{G}^0)_2(\mathfrak{J}_1^1, \mathfrak{J}_2^1)$  which satisfies

$$\mathfrak{J}_2^2 \circ (\text{Id} \otimes \partial_0^0 \phi) = (\partial_1^0 \phi \otimes \text{Id}) \circ \mathfrak{J}_1^2 \tag{3.7}$$

Let  $\text{Stack}(\mathfrak{G})_2(\mathfrak{J}_1, \mathfrak{J}_2)$  denote the set of 2-morphisms.

Compositions of 2-morphisms are given by the compositions in  $\text{MC}^2(\mathfrak{G}^0)_2$ .

For  $\gamma_1, \gamma_2 \in \text{Stack}(\mathfrak{G})_0$ , we have the groupoid  $\text{Stack}(\mathfrak{G})(\gamma_1, \gamma_2)$  with the set of objects  $\text{Stack}(\mathfrak{G})_1(\gamma_1, \gamma_2)$  and the set of morphisms  $\text{Stack}(\mathfrak{G})_2(\mathfrak{J}_1, \mathfrak{J}_2)$  under vertical composition.

Every morphism  $\theta$  of cosimplicial DGLA induces in an obvious manner a functor  $\theta_* : \text{Stack}(\mathfrak{G} \otimes \mathfrak{m}) \rightarrow \text{Stack}(\mathfrak{H} \otimes \mathfrak{m})$

We have the following cosimplicial analogue of the Theorem 3.1:

**Theorem 3.6.** *Suppose that  $\theta : \mathfrak{G} \rightarrow \mathfrak{H}$  is a quasi-isomorphism of cosimplicial DGLA and  $\mathfrak{m}$  is a commutative nilpotent ring. Then the induced map  $\theta_* : \text{Stack}(\mathfrak{G} \otimes \mathfrak{m}) \rightarrow \text{Stack}(\mathfrak{H} \otimes \mathfrak{m})$  is an equivalence.*

*Proof.* The proof can be obtained by applying the Theorem 3.1 repeatedly.

Let  $\gamma_1, \gamma_2 \in \text{Stack}(\mathfrak{G} \otimes \mathfrak{m})_0$ , and let  $\mathfrak{J}_1, \mathfrak{J}_2$  be two 1-morphisms between  $\gamma_1$  and  $\gamma_2$ . Note that by the Theorem 3.1  $\theta_* : \text{MC}^2(\mathfrak{G}^0 \otimes \mathfrak{m})_2(\mathfrak{J}_1^1, \mathfrak{J}_2^1) \rightarrow \text{MC}^2(\mathfrak{H}^0 \otimes \mathfrak{m})_2(\theta_* \mathfrak{J}_1^1, \theta_* \mathfrak{J}_2^1)$  is a bijection. Injectivity of the map  $\theta_* : \text{Stack}(\mathfrak{G} \otimes \mathfrak{m})_2(\mathfrak{J}_1, \mathfrak{J}_2) \rightarrow \text{Stack}(\mathfrak{H} \otimes \mathfrak{m})_2(\theta_* \mathfrak{J}_1, \theta_* \mathfrak{J}_2)$  follows immediately. For the surjectivity, notice that an element of  $\text{Stack}(\mathfrak{H} \otimes \mathfrak{m})_2(\theta_* \mathfrak{J}_1, \theta_* \mathfrak{J}_2)$  is necessarily given by  $\theta_* \phi$  for some  $\phi \in \text{MC}^2(\mathfrak{G}^0 \otimes \mathfrak{m})(\mathfrak{J}_1^1, \mathfrak{J}_2^1)_2$  and the following identity is satisfied in  $\text{MC}^2(\mathfrak{H}^1 \otimes \mathfrak{m})(\theta_*(\gamma_2^1 \circ \partial_0^0 \mathfrak{J}_1^1), \theta_*(\partial_1^0 \mathfrak{J}_2^1 \circ \gamma_1^1))_2$ :  $\theta_*(\mathfrak{J}_2^2 \circ (\partial_0^1 \phi \otimes \text{Id})) = \theta_*((\text{Id} \otimes \partial_1^1 \phi) \circ \mathfrak{J}_1^2)$ . Since  $\theta_* : \text{MC}^2(\mathfrak{G}^1 \otimes \mathfrak{m})(\gamma_2^1 \circ \partial_0^0 \mathfrak{J}_1^1, \partial_1^0 \mathfrak{J}_2^1 \circ \gamma_1^1)_2 \rightarrow \text{MC}^2(\mathfrak{H}^1 \otimes \mathfrak{m})(\theta_*(\gamma_2^1 \circ \partial_0^0 \mathfrak{J}_1^1), \theta_*(\partial_1^0 \mathfrak{J}_2^1 \circ \gamma_1^1))_2$  is bijective, and in particular injective,  $\mathfrak{J}_2^2 \circ (\partial_0^1 \phi \otimes \text{Id}) = (\text{Id} \otimes \partial_1^1 \phi) \circ \mathfrak{J}_1^2$ , and  $\phi$  defines an element in  $\text{Stack}(\mathfrak{G} \otimes \mathfrak{m})_2(\mathfrak{J}_1, \mathfrak{J}_2)$ .

Next, let  $\gamma_1, \gamma_2 \in \text{Stack}(\mathfrak{G} \otimes \mathfrak{m})_0$ , and let  $\mathfrak{J}$  be a 1-morphisms between  $\theta_* \gamma_1$  and  $\theta_* \gamma_2$ . We show that there exists  $\mathfrak{T} \in \text{Stack}(\mathfrak{G} \otimes \mathfrak{m})_1(\gamma_1, \gamma_2)$  such that  $\theta_* \mathfrak{T}$  is isomorphic to  $\mathfrak{J}$ . Indeed, by Theorem 3.1, there exists  $\mathfrak{T}^1 \in \text{MC}^2(\mathfrak{G}^0 \otimes \mathfrak{m})_1(\gamma_1, \gamma_2)$  such that  $\text{MC}^2(\mathfrak{H}^0 \otimes \mathfrak{m})_2(\theta_* \mathfrak{T}^1, \mathfrak{J}) \neq \emptyset$ . Let  $\phi \in \text{MC}^2(\mathfrak{H}^0 \otimes \mathfrak{m})_2(\theta_* \mathfrak{T}^1, \mathfrak{J})$ . Define  $\psi \in \text{MC}^2(\mathfrak{H}^1 \otimes \mathfrak{m})_2(\theta_*(\gamma_2^1 \circ \partial_0^0 \mathfrak{T}^1), \theta_*(\partial_1^0 \mathfrak{T}^1 \circ \gamma_1^1))$  by  $\psi = (\partial_1^0 \phi \otimes \text{Id})^{-1} \circ \mathfrak{J}^2 \circ (\text{Id} \otimes \partial_0^0 \phi)$ . It is easy to verify that the following identities holds:

$$\begin{aligned} (\text{Id} \otimes \theta_* \gamma_1^2) \circ (\partial_2^1 \psi \otimes \text{Id}) \circ (\text{Id} \otimes \partial_0^1 \psi) &= \partial_1^1 \psi \circ (\theta_* \gamma_2^2 \otimes \text{Id}) \\ s_0^1 \psi &= \text{Id} \end{aligned}$$

By bijectivity of  $\theta_*$  on  $\text{MC}^2$  there exists a unique  $\mathfrak{T}^2 \in \text{MC}^2(\mathfrak{H}^1 \otimes \mathfrak{m})_2(\gamma_2^1 \circ \partial_0^0 \mathfrak{T}^1, \partial_1^0 \mathfrak{T}^1 \circ \gamma_1^1)$  such that  $\theta_* \mathfrak{T}^2 = \psi$ . Moreover, as before, injectivity of  $\theta_*$  implies that the conditions (3.6) are satisfied. Therefore  $\mathfrak{T} = (\mathfrak{T}^1, \mathfrak{T}^2)$  defines a 1-morphism  $\gamma_1 \rightarrow \gamma_2$  and  $\phi$  is a 2-morphism  $\theta_* \mathfrak{T} \rightarrow \mathfrak{J}$ .

Now, let  $\gamma \in \text{Stack}(\mathfrak{H} \otimes \mathfrak{m})_0$ . We construct  $\lambda \in \text{Stack}(\mathfrak{G} \otimes \mathfrak{m})_0$  such that  $\text{Stack}(\mathfrak{H} \otimes \mathfrak{m})_1(\theta_* \lambda, \gamma) \neq \emptyset$ . Indeed, by the Theorem 3.1 there exists  $\lambda^0 \in \text{MC}^2(\mathfrak{G}^0 \otimes \mathfrak{m})_0$  such that  $\text{MC}^2(\mathfrak{H}^0 \otimes \mathfrak{m})_1(\theta_* \lambda^0, \gamma^0) \neq \emptyset$ . Let  $\mathfrak{J}^1 \in \text{MC}^2(\mathfrak{H}^0 \otimes \mathfrak{m})_1(\theta_* \lambda, \gamma)$ . Applying Theorem 3.1 again we obtain that there exists  $\mu \in \text{MC}^2(\mathfrak{G}^1 \otimes \mathfrak{m})_1(\partial_0^0 \lambda^0, \partial_1^0 \lambda^0)$  such that there exists  $\phi \in \text{MC}^2(\mathfrak{H}^1 \otimes \mathfrak{m})_2(\gamma^1 \circ \partial_0^0 \mathfrak{J}^1, \partial_1^0 \mathfrak{J}^1 \circ \theta_* \mu)$ . Then  $s_0^1 \phi$  is then a 2-morphism  $\mathfrak{J}^1 \rightarrow \mathfrak{J}^1 \circ (s_0^1 \mu)$ , which induces a 2-morphism  $\psi : (s_0^1 \mu)^{-1} \rightarrow \text{Id}$ .

Set now  $\lambda^1 = \mu \circ (\partial_0^0 (s_0^1 \mu))^{-1} \in \text{MC}^2(\mathfrak{G}^1 \otimes \mathfrak{m})_1(\partial_0^0 \lambda^0, \partial_1^0 \lambda^0)$ ,  $\mathfrak{J}^2 = \phi \circ (\partial_0^0 \psi)^{-1} \in \text{MC}^2(\mathfrak{H}^1 \otimes \mathfrak{m})_2(\gamma^1 \circ \partial_0^0 \mathfrak{J}^1, \partial_1^0 \mathfrak{J}^1 \circ \theta_* \lambda^1)$ . It is easy to see that  $s_0^1 \lambda^1 = \text{Id}$ ,  $s_0^1 \mathfrak{J}^2 = \text{Id}$ .

We then conclude that there exists a unique  $\lambda^2$  such that

$$(\mathrm{Id} \otimes \theta_* \lambda^2) \circ (\partial_2^1 \mathfrak{J}^2 \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \partial_0^1 \mathfrak{J}^2) = \partial_1^1 \mathfrak{J}^2 \circ (\gamma^2 \otimes \mathrm{Id}).$$

Such a  $\lambda^2$  necessarily satisfies the conditions

$$\begin{aligned} \partial_2^2 \lambda^2 \circ (\mathrm{Id} \otimes \partial_0^2 \lambda^2) &= \partial_1^2 \lambda^2 \circ (\partial_3^2 \lambda^2 \otimes \mathrm{Id}). \\ s_0^2 \lambda^2 &= s_1^2 \lambda^2 = \mathrm{Id} \end{aligned}$$

Therefore  $\lambda = (\lambda^0, \lambda^1, \lambda^2) \in \mathrm{Stack}(\mathfrak{G} \otimes \mathfrak{m})$ , and  $\mathfrak{J} = (\mathfrak{J}^1, \mathfrak{J}^2)$  defines a 1-morphism  $\theta_* \lambda \rightarrow \gamma$ .  $\square$

### 3.4. Acyclicity and strictness.

**Definition 3.7.** A  $\mathfrak{G}$ -stack  $(\gamma^0, \gamma^1, \gamma^2)$  is called *strict* if  $\partial_0^0 \gamma^0 = \partial_1^0 \gamma^0$ ,  $\gamma_1 = \mathrm{Id}$  and  $\gamma_2 = \mathrm{Id}$ .

Let  $\mathrm{Stack}_{Str}(\mathfrak{G})_0$  denote the subset of strict  $\mathfrak{G}$ -stacks.

**Lemma 3.8.**  $\mathrm{Stack}_{Str}(\mathfrak{G})_0 = \mathrm{MC}^2(\ker(\mathfrak{G}^0 \rightrightarrows \mathfrak{G}^1))_0$

**Definition 3.9.** For  $\gamma_1, \gamma_2 \in \mathrm{Stack}_{Str}(\mathfrak{G})_0$  a 1-morphism  $\mathfrak{J} = (\mathfrak{J}^1, \mathfrak{J}^2) \in \mathrm{Stack}(\mathfrak{G})_1(\gamma_1, \gamma_2)$  is called *strict* if  $\partial_0^0(\mathfrak{J}^1) = \partial_1^0(\mathfrak{J}^1)$  and  $\mathfrak{J}^2 = \mathrm{Id}$ .

For  $\gamma_1, \gamma_2 \in \mathrm{Stack}_{Str}(\mathfrak{G})_0$  we denote by  $\mathrm{Stack}_{Str}(\mathfrak{G})_1(\gamma_1, \gamma_2)$  the subset of strict morphisms.

**Lemma 3.10.** For  $\gamma_1, \gamma_2 \in \mathrm{Stack}_{Str}(\mathfrak{G})_0$   $\mathrm{Stack}_{Str}(\mathfrak{G})_1(\gamma_1, \gamma_2) = \mathrm{MC}^2(\ker(\mathfrak{G}^0 \rightrightarrows \mathfrak{G}^1))_1$

For  $\gamma_1, \gamma_2 \in \mathrm{Stack}_{Str}(\mathfrak{G})_0$  let  $\mathrm{Stack}_{Str}(\mathfrak{G})(\gamma_1, \gamma_2)$  denote the full subcategory of  $\mathrm{Stack}(\mathfrak{G})(\gamma_1, \gamma_2)$  with objects  $\mathrm{Stack}_{Str}(\mathfrak{G})_1(\gamma_1, \gamma_2)$ .

Thus, we have the 2-groupoids  $\mathrm{Stack}(\mathfrak{G})$  and  $\mathrm{Stack}_{Str}(\mathfrak{G})$  and an embedding of the latter into the former which is fully faithful on the respective groupoids of 1-morphisms.

**Lemma 3.11.**  $\mathrm{Stack}_{Str}(\mathfrak{G}) = \mathrm{MC}^2(\ker(\mathfrak{G}^0 \rightrightarrows \mathfrak{G}^1))$

Suppose that  $\mathfrak{G}$  is a cosimplicial DGLA. For each  $n$  and  $i$  we have the vector space  $\mathfrak{G}^{n,i}$ , namely the degree  $i$  component of  $\mathfrak{G}^n$ . The assignment  $n \mapsto \mathfrak{G}^{n,i}$  is a cosimplicial vector space  $\mathfrak{G}^{\bullet,i}$ .

We will be considering the following acyclicity condition on the cosimplicial DGLA  $\mathfrak{G}$ :

$$\text{for all } i \in \mathbb{Z}, H^p(\mathfrak{G}^{\bullet,i}) = 0 \text{ for } p \neq 0 \quad (3.8)$$

**Theorem 3.12.** *Suppose that  $\mathfrak{G}$  is a cosimplicial DGLA which satisfies the condition (3.8), and  $\mathfrak{m}$  a commutative nilpotent ring. Then, the functor  $\iota : \mathrm{Stack}_{Str}(\mathfrak{G} \otimes \mathfrak{m}) \rightarrow \mathrm{Stack}(\mathfrak{G} \otimes \mathfrak{m})$  is an equivalence.*

*Proof.* As we noted before, it is immediate from the definitions that if  $\gamma_1, \gamma_2 \in \text{Stack}_{Str}(\mathfrak{G} \otimes \mathfrak{m})_0$ , and  $\mathfrak{J}_1, \mathfrak{J}_2 \in \text{Stack}_{Str}(\mathfrak{G} \otimes \mathfrak{m})_1(\gamma_1, \gamma_2)$ , then  $\iota : \text{Stack}_{Str}(\mathfrak{G} \otimes \mathfrak{m})_2(\mathfrak{J}_1, \mathfrak{J}_2) \rightarrow \text{Stack}_{Str}(\mathfrak{G} \otimes \mathfrak{m})_2(\mathfrak{J}_1, \mathfrak{J}_2)$  is a bijection.

Suppose now that  $\gamma_1, \gamma_2 \in \text{Stack}_{Str}(\mathfrak{G} \otimes \mathfrak{m})_0$ ,  $\mathfrak{J} = (\mathfrak{J}^1, \mathfrak{J}^2) \in \text{Stack}(\mathfrak{G} \otimes \mathfrak{m})_1(\gamma_1, \gamma_2)$ . We show that then there exists  $\mathfrak{T} \in \text{Stack}_{Str}(\mathfrak{G})_1(\gamma_1, \gamma_2)$  and a 2-morphism  $\phi : \mathfrak{J} \rightarrow \mathfrak{T}$ . Let  $\mathfrak{J}^2 = \exp_{\partial_0^0 \gamma_2^0} g$ ,  $g \in (\mathfrak{G}^1 \otimes \mathfrak{m})$ . Then  $\partial_0^1 g - \partial_1^1 g + \partial_2^1 g = 0 \pmod{\mathfrak{m}^2}$ . By the acyclicity condition there exists  $a \in (\mathfrak{G}^0 \otimes \mathfrak{m})$  such that  $\partial_0^0 a - \partial_1^0 a = g \pmod{\mathfrak{m}^2}$ . Set  $\phi_1 = \exp_{\gamma_2^0} a$ . Define then  $\mathfrak{T}_1 = (\phi_1 \cdot \mathfrak{J}^1, (\partial_1^0 \phi_1 \otimes \text{Id}) \circ \mathfrak{J}^2 \circ (\text{Id} \otimes \partial_0^0 \phi_1)^{-1}) \in \text{Stack}(\mathfrak{G} \otimes \mathfrak{m})_1(\gamma_1, \gamma_2)$ . Note that  $\phi_1$  defines a 2-morphism  $\mathfrak{J} \rightarrow \mathfrak{T}_1$ . Note also that  $\mathfrak{T}_1^2 \in \exp_{\partial_0^0 \gamma_2^0}(\mathfrak{G} \otimes \mathfrak{m}^2)$ . Proceeding inductively one constructs a sequence  $\mathfrak{T}_k \in \text{Stack}(\mathfrak{G} \otimes \mathfrak{m})_1(\gamma_1, \gamma_2)$  such that  $\mathfrak{T}_k^2 \in \exp(\mathfrak{G} \otimes \mathfrak{m}^{k+1})$  and 2-morphisms  $\phi_k : \mathfrak{J} \rightarrow \mathfrak{T}_k$ ,  $\phi_{k+1} = \phi_k \pmod{\mathfrak{m}^k}$ . Since  $\mathfrak{m}$  is nilpotent, for  $k$  large enough  $\mathfrak{T}_k \in \text{Stack}_{Str}(\mathfrak{G})_1(\gamma_1, \gamma_2)$ .

Assume now that  $\gamma \in \text{Stack}(\mathfrak{G} \otimes \mathfrak{m})_0$ . We will construct  $\lambda \in \text{Stack}_{Str}(\mathfrak{G} \otimes \mathfrak{m})_0$  and a 1-morphism  $\mathfrak{J} : \gamma \rightarrow \lambda$ .

We begin by constructing  $\mu \in \text{Stack}(\mathfrak{G} \otimes \mathfrak{m})_0$  such that  $\mu^2 = \text{Id}$  and a 1-morphism  $\mathfrak{T} : \gamma \rightarrow \mu$ . We have:  $\gamma^2 = \exp_{\partial_1^1 \partial_1^0 \gamma^0} c$ ,  $c \in (\mathfrak{G}^2 \otimes \mathfrak{m})$ . In view of the equations (3.5)  $c$  satisfies the following identities:

$$\begin{aligned} \partial_0^2 c - \partial_1^2 c + \partial_2^2 c - \partial_3^2 c &= 0 \pmod{\mathfrak{m}^2} \\ s_0^2 c &= s_1^2 c = 0 \end{aligned} \tag{3.9}$$

By the acyclicity of the normalized complex we can find  $b \in \mathfrak{G}^1 \otimes \mathfrak{m}$  such that  $\partial_0^1 b - \partial_1^1 b + \partial_2^1 b = c \pmod{\mathfrak{m}^2}$ ,  $s_0^1 b = 0$ . Let  $\phi_1 = \exp_{\partial_1^0 \gamma^0}(-b)$ . Define then  $\mu_1 = (\mu_1^0, \mu_1^1, \mu_1^2)$  where  $\mu_1^0 = \gamma^0$ ,  $\mu_1^1 = \phi_1 \cdot \gamma^1$ , and  $\mu_1^2$  is such that

$$(\text{Id} \otimes \gamma^2) \circ (\partial_2^1 \phi_1 \otimes \text{Id}) \circ (\text{Id} \otimes \partial_0^1 \phi_1) = \partial_1^1 \phi_1 \circ (\mu_1^2 \otimes \text{Id})$$

Note that  $\mu_1^2 = \text{Id} \pmod{\mathfrak{m}^2}$  and  $(\text{Id}, \phi_1)$  is a 1-morphism  $\gamma \rightarrow \mu_1$ . As before we can construct a sequence  $\mu_k$  such that  $\mu_k^2 = \text{Id} \pmod{\mathfrak{m}^{k+1}}$ , and 1-morphisms  $(\text{Id}, \phi_k) : \gamma \rightarrow \mu_k$ ,  $\phi_{k+1} = \phi_k \pmod{\mathfrak{m}^k}$ . As before we conclude that this gives the desired construction of  $\mu$ . The rest of the proof, i.e. the construction of  $\lambda$  is completely analogous.  $\square$

**Corollary 3.13.** *Suppose that  $\mathfrak{G}$  is a cosimplicial DGLA which satisfies the condition (3.8). Then there is a canonical equivalence:*

$$\text{Stack}(\mathfrak{G} \otimes \mathfrak{m}) \cong \text{MC}^2(\ker(\mathfrak{G}^0 \rightrightarrows \mathfrak{G}^1) \otimes \mathfrak{m})$$

*Proof.* Combine Lemma 3.11 with Theorem 3.12.  $\square$

## 4. Algebroid stacks

In this section we review the notions of algebroid stack and twisted form. We also define the notion of descent datum and relate it with algebroid stacks.

## 4.1. Algebroids and algebroid stacks.

**4.1.1. Algebroids.** For a category  $\mathcal{C}$  we denote by  $i\mathcal{C}$  the subcategory of isomorphisms in  $\mathcal{C}$ ; equivalently,  $i\mathcal{C}$  is the maximal subgroupoid in  $\mathcal{C}$ .

Suppose that  $R$  is a commutative  $k$ -algebra.

**Definition 4.1.** An  $R$ -algebroid is a *nonempty*  $R$ -linear category  $\mathcal{C}$  such that the groupoid  $i\mathcal{C}$  is connected

Let  $\text{Alg}_R$  denote the 2-category of  $R$ -algebroids (full 2-subcategory of the 2-category of  $R$ -linear categories).

Suppose that  $A$  is an  $R$ -algebra. The  $R$ -linear category with one object and morphisms  $A$  is an  $R$ -algebroid denoted  $A^+$ .

Suppose that  $\mathcal{C}$  is an  $R$ -algebroid and  $L$  is an object of  $\mathcal{C}$ . Let  $A = \text{End}_{\mathcal{C}}(L)$ . The functor  $A^+ \rightarrow \mathcal{C}$  which sends the unique object of  $A^+$  to  $L$  is an equivalence.

Let  $\text{Alg}_R^2$  denote the 2-category of with

- objects  $R$ -algebras
- 1-morphisms homomorphism of  $R$ -algebras
- 2-morphisms  $\phi \rightarrow \psi$ , where  $\phi, \psi : A \rightarrow B$  are two 1-morphisms are elements  $b \in B$  such that  $\phi(a) \cdot b = b \cdot \psi(a)$  for all  $a \in A$ .

It is clear that the 1- and the 2- morphisms in  $\text{Alg}_R^2$  as defined above induce 1- and 2-morphisms of the corresponding algebroids under the assignment  $A \mapsto A^+$ . The structure of a 2-category on  $\text{Alg}_R^2$  (i.e. composition of 1- and 2- morphisms) is determined by the requirement that the assignment  $A \mapsto A^+$  extends to an embedding  $(\bullet)^+ : \text{Alg}_R^2 \rightarrow \text{Alg}_R$ .

Suppose that  $R \rightarrow S$  is a morphism of commutative  $k$ -algebras. The assignment  $A \rightarrow A \otimes_R S$  extends to a functor  $(\bullet) \otimes_R S : \text{Alg}_R^2 \rightarrow \text{Alg}_S^2$ .

**4.1.2. Algebroid stacks.** We refer the reader to [1] and [20] for basic definitions. We will use the notion of fibered category interchangeably with that of a pseudo-functor. A *prestack*  $\mathcal{C}$  on a space  $X$  is a category fibered over the category of open subsets of  $X$ , equivalently, a pseudo-functor  $U \mapsto \mathcal{C}(U)$ , satisfying the following additional requirement. For an open subset  $U$  of  $X$  and two objects  $A, B \in \mathcal{C}(U)$  we have the presheaf  $\underline{\text{Hom}}_{\mathcal{C}}(A, B)$  on  $U$  defined by  $U \supseteq V \mapsto \text{Hom}_{\mathcal{C}(V)}(A|_V, B|_V)$ . The fibered category  $\mathcal{C}$  is a prestack if for any  $U, A, B \in \mathcal{C}(U)$ , the presheaf  $\underline{\text{Hom}}_{\mathcal{C}}(A, B)$  is a sheaf. A prestack is a *stack* if, in addition, it satisfies the condition of effective descent for objects. For a prestack  $\mathcal{C}$  we denote the associated stack by  $\tilde{\mathcal{C}}$ .

**Definition 4.2.** A stack in  $R$ -linear categories  $\mathcal{C}$  on  $X$  is an  $R$ -algebroid stack if it is locally nonempty and locally connected, i.e. satisfies

1. any point  $x \in X$  has a neighborhood  $U$  such that  $\mathcal{C}(U)$  is nonempty;
2. for any  $U \subseteq X, x \in U, A, B \in \mathcal{C}(U)$  there exists a neighborhood  $V \subseteq U$  of  $x$  and an isomorphism  $A|_V \cong B|_V$ .

*Remark 4.3.* Equivalently, the stack associated to the substack of isomorphisms  $\widetilde{i\mathcal{C}}$  is a gerbe.

*Example 4.4.* Suppose that  $\mathcal{A}$  is a sheaf of  $R$ -algebras on  $X$ . The assignment  $X \supseteq U \mapsto \mathcal{A}(U)^+$  extends in an obvious way to a prestack in  $R$ -algebroids denoted  $\mathcal{A}^+$ . The associated stack  $\widetilde{\mathcal{A}^+}$  is canonically equivalent to the stack of locally free  $\mathcal{A}^{op}$ -modules of rank one. The canonical morphism  $\mathcal{A}^+ \rightarrow \widetilde{\mathcal{A}^+}$  sends the unique (locally defined) object of  $\mathcal{A}^+$  to the free module of rank one.

1-morphisms and 2-morphisms of  $R$ -algebroid stacks are those of stacks in  $R$ -linear categories. We denote the 2-category of  $R$ -algebroid stacks by  $\text{AlgStack}_R(X)$ .

## 4.2. Descent data.

### 4.2.1. Convolution data.

**Definition 4.5.** An  $R$ -linear convolution datum is a triple  $(\mathcal{U}, \mathcal{A}_{01}, \mathcal{A}_{012})$  consisting of:

- a cover  $\mathcal{U} \in \text{Cov}(X)$
- a sheaf  $\mathcal{A}_{01}$  of  $R$ -modules  $\mathcal{A}_{01}$  on  $N_1\mathcal{U}$
- a morphism

$$\mathcal{A}_{012} : (\text{pr}_{01}^2)^* \mathcal{A}_{01} \otimes_R (\text{pr}_{12}^2)^* \mathcal{A}_{01} \rightarrow (\text{pr}_{02}^2)^* \mathcal{A}_{01} \quad (4.1)$$

of  $R$ -modules

subject to the associativity condition expressed by the commutativity of the diagram

$$\begin{array}{ccc} (\text{pr}_{01}^3)^* \mathcal{A}_{01} \otimes_R (\text{pr}_{12}^3)^* \mathcal{A}_{01} \otimes_R (\text{pr}_{23}^3)^* \mathcal{A}_{01} & \xrightarrow{(\text{pr}_{012}^3)^*(\mathcal{A}_{012}) \otimes \text{Id}} & (\text{pr}_{02}^3)^* \mathcal{A}_{01} \otimes_R (\text{pr}_{23}^3)^* \mathcal{A}_{01} \\ \text{Id} \otimes (\text{pr}_{123}^3)^*(\mathcal{A}_{012}) \downarrow & & \downarrow (\text{pr}_{023}^3)^*(\mathcal{A}_{012}) \\ (\text{pr}_{01}^3)^* \mathcal{A}_{01} \otimes_R (\text{pr}_{13}^3)^* \mathcal{A}_{01} & \xrightarrow{(\text{pr}_{013}^3)^*(\mathcal{A}_{012})} & (\text{pr}_{03}^3)^* \mathcal{A}_{01} \end{array}$$

For a convolution datum  $(\mathcal{U}, \mathcal{A}_{01}, \mathcal{A}_{012})$  we denote by  $\underline{\mathcal{A}}$  the pair  $(\mathcal{A}_{01}, \mathcal{A}_{012})$  and abbreviate the convolution datum by  $(\mathcal{U}, \underline{\mathcal{A}})$ .

For a convolution datum  $(\mathcal{U}, \underline{\mathcal{A}})$  let

- $\mathcal{A} := (\text{pr}_{00}^0)^* \mathcal{A}_{01}$ ;  $\mathcal{A}$  is a sheaf of  $R$ -modules on  $N_0\mathcal{U}$
- $\mathcal{A}_i^p := (\text{pr}_i^p)^* \mathcal{A}$ ; thus for every  $p$  we get sheaves  $\mathcal{A}_i^p$ ,  $0 \leq i \leq p$  on  $N_p\mathcal{U}$ .

The identities  $\text{pr}_{01}^0 \circ \text{pr}_{000}^0 = \text{pr}_{12}^0 \circ \text{pr}_{000}^0 = \text{pr}_{02}^0 \circ \text{pr}_{000}^0 = \text{pr}_{00}^0$  imply that the pull-back of  $\mathcal{A}_{012}$  to  $N_0\mathcal{U}$  by  $\text{pr}_{000}^0$  gives the pairing

$$(\text{pr}_{000}^0)^*(\mathcal{A}_{012}) : \mathcal{A} \otimes_R \mathcal{A} \rightarrow \mathcal{A}. \quad (4.2)$$

The associativity condition implies that the pairing (4.2) endows  $\mathcal{A}$  with a structure of a sheaf of associative  $R$ -algebras on  $N_0\mathcal{U}$ .

The sheaf  $\mathcal{A}_i^p$  is endowed with the associative  $R$ -algebra structure induced by that on  $\mathcal{A}$ . We denote by  $\mathcal{A}_{ii}^p$  the  $\mathcal{A}_i^p \otimes_R (\mathcal{A}_i^p)^{op}$ -module  $\mathcal{A}_i^p$ , with the module structure given by the left and right multiplication.

The identities  $\text{pr}_{01}^2 \circ \text{pr}_{001}^1 = \text{pr}_{00}^0 \circ \text{pr}_0^1$ ,  $\text{pr}_{12}^2 \circ \text{pr}_{001}^1 = \text{pr}_{02}^2 \circ \text{pr}_{001}^1 = \text{Id}$  imply that the pull-back of  $\mathcal{A}_{012}$  to  $N_1\mathcal{U}$  by  $\text{pr}_{001}^1$  gives the pairing

$$(\text{pr}_{001}^1)^*(\mathcal{A}_{012}) : \mathcal{A}_0^1 \otimes_R \mathcal{A}_{01} \rightarrow \mathcal{A}_{01} \quad (4.3)$$

The associativity condition implies that the pairing (4.3) endows  $\mathcal{A}_{01}$  with a structure of a  $\mathcal{A}_0^1$ -module. Similarly, the pull-back of  $\mathcal{A}_{012}$  to  $N_1\mathcal{U}$  by  $\text{pr}_{011}^1$  endows  $\mathcal{A}_{01}$  with a structure of a  $(\mathcal{A}_1^1)^{op}$ -module. Together, the two module structures define a structure of a  $\mathcal{A}_0^1 \otimes_R (\mathcal{A}_1^1)^{op}$ -module on  $\mathcal{A}_{01}$ .

The map (4.1) factors through the map

$$(\text{pr}_{01}^2)^* \mathcal{A}_{01} \otimes_{\mathcal{A}_1^2} (\text{pr}_{12}^2)^* \mathcal{A}_{01} \rightarrow (\text{pr}_{02}^2)^* \mathcal{A}_{01} \quad (4.4)$$

**Definition 4.6.** A *unit* for a convolution datum  $\underline{\mathcal{A}}$  is a morphism of  $R$ -modules

$$\mathbf{1} : R \rightarrow \mathcal{A}$$

such that the compositions

$$\mathcal{A}_{01} \xrightarrow{\mathbf{1} \otimes \text{Id}} \mathcal{A}_0^1 \otimes_R \mathcal{A}_{01} \xrightarrow{(\text{pr}_{001}^1)^*(\mathcal{A}_{012})} \mathcal{A}_{01}$$

and

$$\mathcal{A}_{01} \xrightarrow{\text{Id} \otimes \mathbf{1}} \mathcal{A}_{01} \otimes_R \mathcal{A}_1^1 \xrightarrow{(\text{pr}_{011}^1)^*(\mathcal{A}_{012})} \mathcal{A}_{01}$$

are equal to the respective identity morphisms.

#### 4.2.2. Descent data.

**Definition 4.7.** A *descent datum* on  $X$  is an  $R$ -linear convolution datum  $(\mathcal{U}, \underline{\mathcal{A}})$  on  $X$  with a unit which satisfies the following additional conditions:

1.  $\mathcal{A}_{01}$  is locally free of rank one as a  $\mathcal{A}_0^1$ -module and as a  $(\mathcal{A}_1^1)^{op}$ -module;
2. the map (4.4) is an isomorphism.

**4.2.3. 1-morphisms.** Suppose given convolution data  $(\mathcal{U}, \underline{\mathcal{A}})$  and  $(\mathcal{U}, \underline{\mathcal{B}})$  as in Definition 4.5.

**Definition 4.8.** A *1-morphism of convolution data*

$$\underline{\phi} : (\mathcal{U}, \underline{\mathcal{A}}) \rightarrow (\mathcal{U}, \underline{\mathcal{B}})$$

is a morphism of  $R$ -modules  $\phi_{01} : \mathcal{A}_{01} \rightarrow \mathcal{B}_{01}$  such that the diagram

$$\begin{array}{ccc} (\mathrm{pr}_{01}^2)^* \mathcal{A}_{01} \otimes_R (\mathrm{pr}_{12}^2)^* \mathcal{A}_{01} & \xrightarrow{\mathcal{A}_{012}} & (\mathrm{pr}_{02}^2)^* \mathcal{A}_{01} \\ (\mathrm{pr}_{01}^2)^* (\phi_{01}) \otimes (\mathrm{pr}_{12}^2)^* (\phi_{01}) \downarrow & & \downarrow (\mathrm{pr}_{02}^2)^* (\phi_{01}) \\ (\mathrm{pr}_{01}^2)^* \mathcal{B}_{01} \otimes_R (\mathrm{pr}_{12}^2)^* \mathcal{B}_{01} & \xrightarrow{\mathcal{B}_{012}} & (\mathrm{pr}_{02}^2)^* \mathcal{B}_{01} \end{array} \quad (4.5)$$

is commutative.

The 1-morphism  $\underline{\phi}$  induces a morphism of  $R$ -algebras  $\phi := (\mathrm{pr}_{00}^0)^* (\phi_{01}) : \mathcal{A} \rightarrow \mathcal{B}$  on  $N_0\mathcal{U}$  as well as morphisms  $\phi_i^p := (\mathrm{pr}_i^p)^* (\phi) : \mathcal{A}_i^p \rightarrow \mathcal{B}_i^p$  on  $N_p\mathcal{U}$ . The morphism  $\phi_{01}$  is compatible with the morphism of algebras  $\phi_0 \otimes \phi_1^{op} : \mathcal{A}_0^1 \otimes_R (\mathcal{A}_1^1)^{op} \rightarrow \mathcal{B}_0^1 \otimes_R (\mathcal{B}_1^1)^{op}$  and the respective module structures.

A 1-morphism of descent data is a 1-morphism of the underlying convolution data which preserves respective units.

**4.2.4. 2-morphisms.** Suppose that we are given descent data  $(\mathcal{U}, \underline{\mathcal{A}})$  and  $(\mathcal{U}, \underline{\mathcal{B}})$  as in 4.2.2 and two 1-morphisms

$$\underline{\phi}, \underline{\psi} : (\mathcal{U}, \underline{\mathcal{A}}) \rightarrow (\mathcal{U}, \underline{\mathcal{B}})$$

as in 4.2.3.

A 2-morphism

$$\underline{b} : \underline{\phi} \rightarrow \underline{\psi}$$

is a section  $b \in \Gamma(N_0\mathcal{U}; \mathcal{B})$  such that the diagram

$$\begin{array}{ccc} \mathcal{A}_{01} & \xrightarrow{b \otimes \phi_{01}} & \mathcal{B}_0^1 \otimes_R \mathcal{B}_{01} \\ \psi_{01} \otimes b \downarrow & & \downarrow \\ \mathcal{B}_{01} \otimes_R \mathcal{B}_1^1 & \longrightarrow & \mathcal{B}_{01} \end{array}$$

is commutative.

**4.2.5. The 2-category of descent data.** Fix a cover  $\mathcal{U}$  of  $X$ .

Suppose that we are given descent data  $(\mathcal{U}, \underline{\mathcal{A}})$ ,  $(\mathcal{U}, \underline{\mathcal{B}})$ ,  $(\mathcal{U}, \underline{\mathcal{C}})$  and 1-morphisms  $\underline{\phi} : (\mathcal{U}, \underline{\mathcal{A}}) \rightarrow (\mathcal{U}, \underline{\mathcal{B}})$  and  $\underline{\psi} : (\mathcal{U}, \underline{\mathcal{B}}) \rightarrow (\mathcal{U}, \underline{\mathcal{C}})$ . The map  $\psi_{01} \circ \phi_{01} : \mathcal{A}_{01} \rightarrow \mathcal{C}_{01}$  is a 1-morphism of descent data  $\underline{\psi} \circ \underline{\phi} : (\mathcal{U}, \underline{\mathcal{A}}) \rightarrow (\mathcal{U}, \underline{\mathcal{C}})$ , the *composition of  $\underline{\phi}$  and  $\underline{\psi}$* .

Suppose that  $\underline{\phi}^{(i)} : (\mathcal{U}, \underline{\mathcal{A}}) \rightarrow (\mathcal{U}, \underline{\mathcal{B}})$ ,  $i = 1, 2, 3$ , are 1-morphisms and  $\underline{b}^{(j)} : \underline{\phi}^{(j)} \rightarrow \underline{\phi}^{(j+1)}$ ,  $j = 1, 2$ , are 2-morphisms. The section  $b^{(2)} \cdot b^{(1)} \in \Gamma(N_0\mathcal{U}; \mathcal{B})$  defines a 2-morphism, denoted  $\underline{b}^{(2)} \underline{b}^{(1)} : \underline{\phi}^{(1)} \rightarrow \underline{\phi}^{(3)}$ , the *vertical composition of  $\underline{b}^{(1)}$  and  $\underline{b}^{(2)}$* .

Suppose that  $\underline{\phi}^{(i)} : (\mathcal{U}, \underline{\mathcal{A}}) \rightarrow (\mathcal{U}, \underline{\mathcal{B}})$ ,  $\underline{\psi}^{(i)} : (\mathcal{U}, \underline{\mathcal{B}}) \rightarrow (\mathcal{U}, \underline{\mathcal{C}})$ ,  $i = 1, 2$ , are 1-morphisms and  $\underline{b} : \underline{\phi}^{(1)} \rightarrow \underline{\phi}^{(2)}$ ,  $\underline{c} : \underline{\psi}^{(1)} \rightarrow \underline{\psi}^{(2)}$  are 2-morphisms. The section  $c \cdot \psi^{(1)}(b) \in \Gamma(N_0\mathcal{U}; \mathcal{C})$  defines a 2-morphism, denoted  $\underline{c} \otimes \underline{b}$ , the *horizontal composition of  $\underline{b}$  and  $\underline{c}$* .

We leave it to the reader to check that with the compositions defined above descent data, 1-morphisms and 2-morphisms form a 2-category, denoted  $\mathrm{Desc}_R(\mathcal{U})$ .

**4.2.6. Fibered category of descent data.** Suppose that  $\rho : \mathcal{V} \rightarrow \mathcal{U}$  is a morphism of covers and  $(\mathcal{U}, \underline{\mathcal{A}})$  is a descent datum. Let  $\mathcal{A}_{01}^\rho = (N_1\rho)^*\mathcal{A}_{01}$ ,  $\mathcal{A}_{012}^\rho = (N_2\rho)^*(\mathcal{A}_{012})$ . Then,  $(\mathcal{V}, \underline{\mathcal{A}}^\rho)$  is a descent datum. The assignment  $(\mathcal{U}, \underline{\mathcal{A}}) \mapsto (\mathcal{V}, \underline{\mathcal{A}}^\rho)$  extends to a functor, denoted  $\rho^* : \text{Desc}_R(\mathcal{U}) \rightarrow \text{Desc}_R(\mathcal{V})$ .

The assignment  $\text{Cov}(X)^{op} \ni \mathcal{U} \rightarrow \text{Desc}_R(\mathcal{U})$ ,  $\rho \rightarrow \rho^*$  is (pseudo-)functor. Let  $\text{Desc}_R(X)$  denote the corresponding 2-category fibered in  $R$ -linear 2-categories over  $\text{Cov}(X)$  with object pairs  $(\mathcal{U}, \underline{\mathcal{A}})$  with  $\mathcal{U} \in \text{Cov}(X)$  and  $(\mathcal{U}, \underline{\mathcal{A}}) \in \text{Desc}_R(\mathcal{U})$ ; a morphism  $(\mathcal{U}', \underline{\mathcal{A}}') \rightarrow (\mathcal{U}, \underline{\mathcal{A}})$  in  $\text{Desc}_R(X)$  is a pair  $(\rho, \underline{\phi})$ , where  $\rho : \mathcal{U}' \rightarrow \mathcal{U}$  is a morphism in  $\text{Cov}(X)$  and  $\underline{\phi} : (\mathcal{U}', \underline{\mathcal{A}}') \rightarrow \rho^*(\mathcal{U}, \underline{\mathcal{A}}) = (\mathcal{U}', \underline{\mathcal{A}}^\rho)$ .

### 4.3. Trivializations.

#### 4.3.1. Definition of a trivialization.

**Definition 4.9.** A *trivialization* of an algebroid stack  $\mathcal{C}$  on  $X$  is an object in  $\mathcal{C}(X)$ .

Suppose that  $\mathcal{C}$  is an algebroid stack on  $X$  and  $L \in \mathcal{C}(X)$  is a trivialization. The object  $L$  determines a morphism  $\underline{\text{End}}_{\mathcal{C}}(L)^+ \rightarrow \mathcal{C}$ .

**Lemma 4.10.** *The induced morphism  $\widetilde{\underline{\text{End}}_{\mathcal{C}}(L)^+} \rightarrow \mathcal{C}$  is an equivalence.*

*Remark 4.11.* Suppose that  $\mathcal{C}$  is an  $R$ -algebroid stack on  $X$ . Then, there exists a cover  $\mathcal{U}$  of  $X$  such that the stack  $\epsilon_0^*\mathcal{C}$  admits a trivialization.

**4.3.2. The 2-category of trivializations.** Let  $\text{Triv}_R(X)$  denote the 2-category with

- objects the triples  $(\mathcal{C}, \mathcal{U}, L)$  where  $\mathcal{C}$  is an  $R$ -algebroid stack on  $X$ ,  $\mathcal{U}$  is an open cover of  $X$  such that  $\epsilon_0^*\mathcal{C}(N_0\mathcal{U})$  is nonempty and  $L$  is a trivialization of  $\epsilon_0^*\mathcal{C}$ .
- 1-morphisms  $(\mathcal{C}', \mathcal{U}', L) \rightarrow (\mathcal{C}, \mathcal{U}, L)$  are pairs  $(F, \rho)$  where  $\rho : \mathcal{U}' \rightarrow \mathcal{U}$  is a morphism of covers,  $F : \mathcal{C}' \rightarrow \mathcal{C}$  is a functor such that  $(N_0\rho)^*F(L') = L$
- 2-morphisms  $(F, \rho) \rightarrow (G, \rho)$ , where  $(F, \rho), (G, \rho) : (\mathcal{C}', \mathcal{U}', L) \rightarrow (\mathcal{C}, \mathcal{U}, L)$ , are the morphisms of functors  $F \rightarrow G$ .

The assignment  $(\mathcal{C}, \mathcal{U}, L) \mapsto \mathcal{C}$  extends in an obvious way to a functor  $\text{Triv}_R(X) \rightarrow \text{AlgStack}_R(X)$ .

The assignment  $(\mathcal{C}, \mathcal{U}, L) \mapsto \mathcal{U}$  extends to a functor  $\text{Triv}_R(X) \rightarrow \text{Cov}(X)$  making  $\text{Triv}_R(X)$  a fibered 2-category over  $\text{Cov}(X)$ . For  $\mathcal{U} \in \text{Cov}(X)$  we denote the fiber over  $\mathcal{U}$  by  $\text{Triv}_R(X)(\mathcal{U})$ .

**4.3.3. Algebroid stacks from descent data.** Consider  $(\mathcal{U}, \underline{\mathcal{A}}) \in \text{Desc}_R(\mathcal{U})$ .

The sheaf of algebras  $\mathcal{A}$  on  $N_0\mathcal{U}$  gives rise to the algebroid stack  $\widetilde{\mathcal{A}^+}$ . The sheaf  $\mathcal{A}_{01}$  defines an equivalence

$$\phi_{01} := (\bullet) \otimes_{\mathcal{A}_0^1} \mathcal{A}_{01} : (\text{pr}_0^1)^*\widetilde{\mathcal{A}^+} \rightarrow (\text{pr}_1^1)^*\widetilde{\mathcal{A}^+} .$$

The convolution map  $\mathcal{A}_{012}$  defines an isomorphism of functors

$$\phi_{012} : (\mathbf{pr}_{01}^2)^*(\phi_{01}) \circ (\mathbf{pr}_{12}^2)^*(\phi_{01}) \rightarrow (\mathbf{pr}_{02}^2)^*(\phi_{01}) .$$

We leave it to the reader to verify that the triple  $(\widetilde{\mathcal{A}}^+, \phi_{01}, \phi_{012})$  constitutes a descent datum for an algebroid stack on  $X$  which we denote by  $\mathbf{St}(\mathcal{U}, \underline{\mathcal{A}})$ .

By construction there is a canonical equivalence  $\widetilde{\mathcal{A}}^+ \rightarrow \epsilon_0^* \mathbf{St}(\mathcal{U}, \underline{\mathcal{A}})$  which endows  $\epsilon_0^* \mathbf{St}(\mathcal{U}, \underline{\mathcal{A}})$  with a canonical trivialization  $\mathbb{1}$ .

The assignment  $(\mathcal{U}, \underline{\mathcal{A}}) \mapsto (\mathbf{St}(\mathcal{U}, \underline{\mathcal{A}}), \mathcal{U}, \mathbb{1})$  extends to a cartesian functor

$$\mathbf{St} : \text{Desc}_R(X) \rightarrow \text{Triv}_R(X) .$$

**4.3.4. Descent data from trivializations.** Consider  $(\mathcal{C}, \mathcal{U}, L) \in \text{Triv}_R(X)$ . Since  $\epsilon_0 \circ \mathbf{pr}_0^1 = \epsilon_0 \circ \mathbf{pr}_1^1 = \epsilon_1$  we have canonical identifications  $(\mathbf{pr}_0^1)^* \epsilon_0^* \mathcal{C} \cong (\mathbf{pr}_1^1)^* \epsilon_0^* \mathcal{C} \cong \epsilon_1^* \mathcal{C}$ . The object  $L \in \epsilon_0^* \mathcal{C}(N_0 \mathcal{U})$  gives rise to the objects  $(\mathbf{pr}_0^1)^* L$  and  $(\mathbf{pr}_1^1)^* L$  in  $\epsilon_1^* \mathcal{C}(N_0 \mathcal{U})$ . Let  $\mathcal{A}_0 = \underline{\text{Hom}}_{\epsilon_1^* \mathcal{C}}((\mathbf{pr}_1^1)^* L, (\mathbf{pr}_0^1)^* L)$ . Thus,  $\mathcal{A}_0$  is a sheaf of  $R$ -modules on  $N_1 \mathcal{U}$ .

The object  $L \in \epsilon_0^* \mathcal{C}(N_0 \mathcal{U})$  gives rise to the objects  $(\mathbf{pr}_0^2)^* L$ ,  $(\mathbf{pr}_1^2)^* L$  and  $(\mathbf{pr}_2^2)^* L$  in  $\epsilon_2^* \mathcal{C}(N_2 \mathcal{U})$ . There are canonical isomorphisms  $(\mathbf{pr}_{ij}^2)^* \mathcal{A}_0 \cong \underline{\text{Hom}}_{\epsilon_2^* \mathcal{C}}((\mathbf{pr}_i^2)^* L, (\mathbf{pr}_j^2)^* L)$ . The composition of morphisms

$$\underline{\text{Hom}}_{\epsilon_2^* \mathcal{C}}((\mathbf{pr}_1^2)^* L, (\mathbf{pr}_0^2)^* L) \otimes_R \underline{\text{Hom}}_{\epsilon_2^* \mathcal{C}}((\mathbf{pr}_2^2)^* L, (\mathbf{pr}_1^2)^* L) \rightarrow \underline{\text{Hom}}_{\epsilon_2^* \mathcal{C}}((\mathbf{pr}_2^2)^* L, (\mathbf{pr}_0^2)^* L)$$

gives rise to the map

$$\mathcal{A}_{012} : (\mathbf{pr}_{01}^2)^* \mathcal{A}_{01} \otimes_R (\mathbf{pr}_{12}^2)^* \mathcal{A}_{01} \rightarrow (\mathbf{pr}_{02}^2)^* \mathcal{A}_{02}$$

Since  $\mathbf{pr}_i^1 \circ \mathbf{pr}_{00}^0 = \text{Id}$  there is a canonical isomorphism  $\mathcal{A} := (\mathbf{pr}_{00}^0)^* \mathcal{A}_{01} \cong \underline{\text{End}}(L)$  which supplies  $\mathcal{A}$  with the unit section  $\mathbf{1} : R \xrightarrow{\mathbf{1}} \underline{\text{End}}(L) \rightarrow \mathcal{A}$ .

The pair  $(\mathcal{U}, \underline{\mathcal{A}})$ , together with the section  $\mathbf{1}$  is a decent datum which we denote  $\text{dd}(\mathcal{C}, \mathcal{U}, L)$ .

The assignment  $(\mathcal{U}, \underline{\mathcal{A}}) \mapsto \text{dd}(\mathcal{C}, \mathcal{U}, L)$  extends to a cartesian functor

$$\text{dd} : \text{Triv}_R(X) \rightarrow \text{Desc}_R(X) .$$

**Lemma 4.12.** *The functors  $\mathbf{St}$  and  $\text{dd}$  are mutually quasi-inverse equivalences.*

**4.4. Base change.** For an  $R$ -linear category  $\mathcal{C}$  and homomorphism of algebras  $R \rightarrow S$  we denote by  $\mathcal{C} \otimes_R S$  the category with the same objects as  $\mathcal{C}$  and morphisms defined by  $\text{Hom}_{\mathcal{C} \otimes_R S}(A, B) = \text{Hom}_{\mathcal{C}}(A, B) \otimes_R S$ .

For an  $R$ -algebra  $A$  the categories  $(A \otimes_R S)^+$  and  $A^+ \otimes_R S$  are canonically isomorphic.

For a prestack  $\mathcal{C}$  in  $R$ -linear categories we denote by  $\mathcal{C} \otimes_R S$  the prestack associated to the fibered category  $U \mapsto \mathcal{C}(U) \otimes_R S$ .

For  $U \subseteq X$ ,  $A, B \in \mathcal{C}(U)$ , there is an isomorphism of sheaves  $\underline{\text{Hom}}_{\mathcal{C} \otimes_R S}(A, B) = \underline{\text{Hom}}_{\mathcal{C}}(A, B) \otimes_R S$ .

**Lemma 4.13.** *Suppose that  $\mathcal{A}$  is a sheaf of  $R$ -algebras and  $\mathcal{C}$  is an  $R$ -algebroid stack.*

1.  $(\widetilde{\mathcal{A}^+ \otimes_R S})^\sim$  is an algebroid stack equivalent to  $(\widetilde{\mathcal{A} \otimes_R S})^+$ .
2.  $\widetilde{\mathcal{C} \otimes_R S}$  is an algebroid stack.

*Proof.* Suppose that  $\mathcal{A}$  is a sheaf of  $R$ -algebras. There is a canonical isomorphism of prestacks  $(\mathcal{A} \otimes_R S)^+ \cong \mathcal{A}^+ \otimes_R S$  which induces the canonical equivalence  $(\widetilde{\mathcal{A} \otimes_R S})^+ \cong \widetilde{\mathcal{A}^+ \otimes_R S}$ .

The canonical functor  $\mathcal{A}^+ \rightarrow \widetilde{\mathcal{A}^+}$  induces the functor  $\mathcal{A}^+ \otimes_R S \rightarrow \widetilde{\mathcal{A}^+ \otimes_R S}$ , hence the functor  $\widetilde{\mathcal{A}^+ \otimes_R S} \rightarrow (\widetilde{\mathcal{A}^+ \otimes_R S})^\sim$ .

The map  $\mathcal{A} \rightarrow \mathcal{A} \otimes_R S$  induces the functor  $\widetilde{\mathcal{A}^+} \rightarrow (\widetilde{\mathcal{A} \otimes_R S})^+$  which factors through the functor  $\widetilde{\mathcal{A}^+ \otimes_R S} \rightarrow (\widetilde{\mathcal{A} \otimes_R S})^+$ . From this we obtain the functor  $(\widetilde{\mathcal{A}^+ \otimes_R S})^\sim \rightarrow (\widetilde{\mathcal{A} \otimes_R S})^+$ .

We leave it to the reader to check that the two constructions are mutually inverse equivalences. It follows that  $(\widetilde{\mathcal{A}^+ \otimes_R S})^\sim$  is an algebroid stack equivalent to  $(\widetilde{\mathcal{A} \otimes_R S})^+$ .

Suppose that  $\mathcal{C}$  is an  $R$ -algebroid stack. Let  $\mathcal{U}$  be a cover such that  $\epsilon_0^* \mathcal{C}(N_0 \mathcal{U})$  is nonempty. Let  $L$  be an object in  $\epsilon_0^* \mathcal{C}(N_0 \mathcal{U})$ ; put  $\mathcal{A} := \underline{\text{End}}_{\epsilon_0^* \mathcal{C}}(L)$ . The equivalence  $\widetilde{\mathcal{A}^+} \rightarrow \epsilon_0^* \mathcal{C}$  induces the equivalence  $(\widetilde{\mathcal{A}^+ \otimes_R S})^\sim \rightarrow (\epsilon_0^* \mathcal{C} \otimes_R S)^\sim$ . Since the former is an algebroid stack so is the latter. There is a canonical equivalence  $(\epsilon_0^* \mathcal{C} \otimes_R S)^\sim \cong \epsilon_0^*(\widetilde{\mathcal{C} \otimes_R S})$ ; since the former is an algebroid stack so is the latter. Since the property of being an algebroid stack is local, the stack  $\widetilde{\mathcal{C} \otimes_R S}$  is an algebroid stack.  $\square$

**4.5. Twisted forms.** Suppose that  $\mathcal{A}$  is a sheaf of  $R$ -algebras on  $X$ . We will call an  $R$ -algebroid stack locally equivalent to  $\widetilde{\mathcal{A}^{op}}$  a *twisted form* of  $\mathcal{A}$ .

Suppose that  $\mathcal{S}$  is twisted form of  $\mathcal{O}_X$ . Then, the substack  $i\mathcal{S}$  is an  $\mathcal{O}_X^\times$ -gerbe. The assignment  $\mathcal{S} \mapsto i\mathcal{S}$  extends to an equivalence between the 2-groupoid of twisted forms of  $\mathcal{O}_X$  (a subcategory of  $\text{AlgStack}_{\mathbb{C}}(X)$ ) and the 2-groupoid of  $\mathcal{O}_X^\times$ -gerbes.

Let  $\mathcal{S}$  be a twisted form of  $\mathcal{O}_X$ . Then for any  $U \subseteq X$ ,  $A \in \mathcal{S}(U)$  the canonical map  $\mathcal{O}_U \rightarrow \underline{\text{End}}_{\mathcal{S}}(A)$  is an isomorphism. Consequently, if  $\mathcal{U}$  is a cover of  $X$  and  $L$  is a trivialization of  $\epsilon_0^* \mathcal{S}$ , then there is a canonical isomorphism of sheaves of algebras  $\mathcal{O}_{N_0 \mathcal{U}} \rightarrow \underline{\text{End}}_{\epsilon_0^* \mathcal{S}}(L)$ .

Conversely, suppose that  $(\mathcal{U}, \underline{\mathcal{A}})$  is a  $\mathbb{C}$ -descent datum. If the sheaf of algebras  $\mathcal{A}$  is isomorphic to  $\mathcal{O}_{N_0 \mathcal{U}}$  then such an isomorphism is unique since the latter has no non-trivial automorphisms. Thus, we may and will identify  $\mathcal{A}$  with  $\mathcal{O}_{N_0 \mathcal{U}}$ . Hence,  $\mathcal{A}_{01}$  is a line bundle on  $N_1 \mathcal{U}$  and the convolution map  $\mathcal{A}_{012}$  is a morphism of line bundles. The stack which corresponds to  $(\mathcal{U}, \underline{\mathcal{A}})$  (as in 4.3.3) is a twisted form of  $\mathcal{O}_X$ .

Isomorphism classes of twisted forms of  $\mathcal{O}_X$  are classified by  $H^2(X; \mathcal{O}_X^\times)$ . We recall the construction presently. Suppose that the twisted form  $\mathcal{S}$  of  $\mathcal{O}_X$  is represented by the descent datum  $(\mathcal{U}, \mathcal{A})$ . Assume in addition that the line bundle  $\mathcal{A}_{01}$  on  $N_1\mathcal{U}$  is trivialized. Then we can consider  $\mathcal{A}_{012}$  as an element in  $\Gamma(N_2\mathcal{U}; \mathcal{O}^\times)$ . The associativity condition implies that  $\mathcal{A}_{012}$  is a cocycle in  $\check{C}^2(\mathcal{U}; \mathcal{O}^\times)$ . The class of this cocycle in  $\check{H}^2(\mathcal{U}; \mathcal{O}^\times)$  does not depend on the choice of trivializations of the line bundle  $\mathcal{A}_{01}$  and yields a class in  $H^2(X; \mathcal{O}_X^\times)$ .

We can write this class using the de Rham complex for jets. We refer to the Section 7.1 for the notations and a brief review. The composition  $\mathcal{O}^\times \rightarrow \mathcal{O}^\times / \mathbb{C}^\times \xrightarrow{\log} \mathcal{O} / \mathbb{C} \xrightarrow{j^\infty} \mathrm{DR}(\mathcal{J} / \mathcal{O})$  induces the map  $H^2(X; \mathcal{O}^\times) \rightarrow H^2(X; \mathrm{DR}(\mathcal{J} / \mathcal{O})) \cong H^2(\Gamma(X; \Omega_X^\bullet \otimes \mathcal{J}_X / \mathcal{O}_X), \nabla^{can})$ . Here the latter isomorphism follows from the fact that the sheaf  $\Omega_X^\bullet \otimes \mathcal{J}_X / \mathcal{O}_X$  is soft. We denote by  $[S] \in H^2(\Gamma(X; \Omega_X^\bullet \otimes \mathcal{J}_X / \mathcal{O}_X), \nabla^{can})$  the image of the class of  $\mathcal{S}$  in  $H^2(X; \mathcal{O}^\times)$ . In the Lemma 7.13 (see also Lemma 7.15) we will construct an explicit representative for  $[S]$ .

## 5. DGLA of local cochains on matrix algebras

In this section we define matrix algebras from a descent datum and use them to construct a cosimplicial DGLA of local cochains. We also establish the acyclicity of this cosimplicial DGLA.

### 5.1. Definition of matrix algebras.

**5.1.1. Matrix entries.** Suppose that  $(\mathcal{U}, \mathcal{A})$  is an  $R$ -descent datum. Let  $\mathcal{A}_{10} := \tau^* \mathcal{A}_{01}$ , where  $\tau = \mathrm{pr}_{10}^1 : N_1\mathcal{U} \rightarrow N_1\mathcal{U}$  is the transposition of the factors. The pairings  $(\mathrm{pr}_{100}^1)^*(\mathcal{A}_{012}) : \mathcal{A}_{10} \otimes_R \mathcal{A}_0^1 \rightarrow \mathcal{A}_{10}$  and  $(\mathrm{pr}_{110}^1)^*(\mathcal{A}_{012}) : \mathcal{A}_1^1 \otimes_R \mathcal{A}_{10} \rightarrow \mathcal{A}_{10}$  of sheaves on  $N_1\mathcal{U}$  endow  $\mathcal{A}_{10}$  with a structure of a  $\mathcal{A}_1^1 \otimes (\mathcal{A}_0^1)^{op}$ -module.

The identities  $\mathrm{pr}_{01}^2 \circ \mathrm{pr}_{010}^1 = \mathrm{Id}$ ,  $\mathrm{pr}_{12}^2 \circ \mathrm{pr}_{010}^1 = \tau$  and  $\mathrm{pr}_{02}^2 \circ \mathrm{pr}_{010}^1 = \mathrm{pr}_{00}^0 \circ \mathrm{pr}_0^1$  imply that the pull-back of  $\mathcal{A}_{012}$  by  $\mathrm{pr}_{010}^1$  gives the pairing

$$(\mathrm{pr}_{010}^1)^*(\mathcal{A}_{012}) : \mathcal{A}_{01} \otimes_R \mathcal{A}_{10} \rightarrow \mathcal{A}_{00}^1 \quad (5.1)$$

which is a morphism of  $\mathcal{A}_0^1 \otimes_K (\mathcal{A}_0^1)^{op}$ -modules. Similarly, we have the pairing

$$(\mathrm{pr}_{101}^1)^*(\mathcal{A}_{012}) : \mathcal{A}_{10} \otimes_R \mathcal{A}_{01} \rightarrow \mathcal{A}_{11}^1 \quad (5.2)$$

The pairings (4.2), (4.3), (5.1) and (5.2),  $\mathcal{A}_{ij} \otimes_R \mathcal{A}_{jk} \rightarrow \mathcal{A}_{ik}$  are morphisms of  $\mathcal{A}_i^1 \otimes_R (\mathcal{A}_k^1)^{op}$ -modules which, as a consequence of associativity, factor through maps

$$\mathcal{A}_{ij} \otimes_{\mathcal{A}_j^1} \mathcal{A}_{jk} \rightarrow \mathcal{A}_{ik} \quad (5.3)$$

induced by  $\mathcal{A}_{ijk} = (\mathrm{pr}_{ijk}^1)^*(\mathcal{A}_{012})$ ; here  $i, j, k = 0, 1$ . Define now for every  $p \geq 0$  the sheaves  $\mathcal{A}_{ij}^p$ ,  $0 \leq i, j \leq p$ , on  $N_p\mathcal{U}$  by  $\mathcal{A}_{ij}^p = (\mathrm{pr}_{ij}^p)^* \mathcal{A}_{01}$ . Define also  $\mathcal{A}_{ijk}^p = (\mathrm{pr}_{ijk}^p)^*(\mathcal{A}_{012})$ . We immediately obtain for every  $p$  the morphisms

$$\mathcal{A}_{ijk}^p : \mathcal{A}_{ij}^p \otimes_{\mathcal{A}_j^p} \mathcal{A}_{jk}^p \rightarrow \mathcal{A}_{ik}^p \quad (5.4)$$

**5.1.2. Matrix algebras.** Let  $\mathrm{Mat}(\underline{\mathcal{A}})^0 = \mathcal{A}$ ; thus,  $\mathrm{Mat}(\underline{\mathcal{A}})^0$  is a sheaf of algebras on  $N_0\mathcal{U}$ . For  $p = 1, 2, \dots$  let  $\mathrm{Mat}(\underline{\mathcal{A}})^p$  denote the sheaf on  $N_p\mathcal{U}$  defined by

$$\mathrm{Mat}(\underline{\mathcal{A}})^p = \bigoplus_{i,j=0}^p \mathcal{A}_{ij}^p$$

The maps (5.4) define the pairing

$$\mathrm{Mat}(\underline{\mathcal{A}})^p \otimes \mathrm{Mat}(\underline{\mathcal{A}})^p \rightarrow \mathrm{Mat}(\underline{\mathcal{A}})^p$$

which endows the sheaf  $\mathrm{Mat}(\underline{\mathcal{A}})^p$  with a structure of an associative algebra by virtue of the associativity condition. The unit section 1 is given by  $1 = \sum_{i=0}^p 1_{ii}$ , where  $1_{ii}$  is the image of the unit section of  $\mathcal{A}_{ii}^p$ .

**5.1.3. Combinatorial restriction.** The algebras  $\mathrm{Mat}(\underline{\mathcal{A}})^p$ ,  $p = 0, 1, \dots$ , do not form a cosimplicial sheaf of algebras on  $N\mathcal{U}$  in the usual sense. They are, however, related by *combinatorial restriction* which we describe presently.

For a morphism  $f : [p] \rightarrow [q]$  in  $\Delta$  define a sheaf on  $N_q\mathcal{U}$  by

$$f^\# \mathrm{Mat}(\underline{\mathcal{A}})^q = \bigoplus_{i,j=0}^p \mathcal{A}_{f(i)f(j)}^q.$$

Note that  $f^\# \mathrm{Mat}(\underline{\mathcal{A}})^q$  inherits a structure of an algebra.

Recall from the Section 2.3.1 that the morphism  $f$  induces the map  $f^* : N_q\mathcal{U} \rightarrow N_p\mathcal{U}$  and that  $f_*$  denotes the pull-back along  $f^*$ . We will also use  $f_*$  to denote the canonical isomorphism of algebras

$$f_* : f_* \mathrm{Mat}(\underline{\mathcal{A}})^p \rightarrow f^\# \mathrm{Mat}(\underline{\mathcal{A}})^q \quad (5.5)$$

induced by the isomorphisms  $f_* \mathcal{A}_{ij}^p \cong \mathcal{A}_{f(i)f(j)}^q$ .

**5.1.4. Refinement.** Suppose that  $\rho : \mathcal{V} \rightarrow \mathcal{U}$  is a morphism of covers. For  $p = 0, 1, \dots$  there is a natural isomorphism

$$(N_p\rho)^* \mathrm{Mat}(\underline{\mathcal{A}})^p \cong \mathrm{Mat}(\underline{\mathcal{A}}^p)^p$$

of sheaves of algebras on  $N_p\mathcal{V}$ . The above isomorphisms are obviously compatible with combinatorial restriction.

## 5.2. Local cochains on matrix algebras.

**5.2.1. Local cochains.** A *sheaf of matrix algebras* is a sheaf of algebras  $\mathcal{B}$  together with a decomposition

$$\mathcal{B} = \bigoplus_{i,j=0}^p \mathcal{B}_{ij}$$

as a sheaf of vector spaces which satisfies  $\mathcal{B}_{ij} \cdot \mathcal{B}_{jk} = \mathcal{B}_{ik}$ .

To a matrix algebra  $\mathcal{B}$  one can associate to the DGLA of *local cochains* defined as follows. For  $n = 0$  let  $C^0(\mathcal{B})^{loc} = \bigoplus \mathcal{B}_{ii} \subset \mathcal{B} = C^0(\mathcal{B})$ . For  $n > 0$  let  $C^n(\mathcal{B})^{loc}$  denote the subsheaf of  $C^n(\mathcal{B})$  whose stalks consist of multilinear maps  $D$  such that for any collection of  $s_{i_k j_k} \in \mathcal{B}_{i_k j_k}$

1.  $D(s_{i_1 j_1} \otimes \cdots \otimes s_{i_n j_n}) = 0$  unless  $j_k = i_{k+1}$  for all  $k = 1, \dots, n-1$
2.  $D(s_{i_0 i_1} \otimes s_{i_1 i_2} \otimes \cdots \otimes s_{i_{n-1} i_n}) \in \mathcal{B}_{i_0 i_n}$

For  $I = (i_0, \dots, i_n) \in [p]^{\times n+1}$  let

$$C^I(\mathcal{B})^{loc} := \underline{\text{Hom}}_k(\bigotimes_{j=0}^{n-1} \mathcal{B}_{i_j i_{j+1}}, \mathcal{B}_{i_0 i_n}).$$

The restriction maps along the embeddings  $\bigotimes_{j=0}^{n-1} \mathcal{B}_{i_j i_{j+1}} \hookrightarrow \mathcal{B}^{\otimes n}$  induce an isomorphism  $C^n(\mathcal{B})^{loc} \rightarrow \bigoplus_{I \in [p]^{\times n+1}} C^I(\mathcal{B})^{loc}$ .

The sheaf  $C^\bullet(\mathcal{B})^{loc}[1]$  is a subDGLA of  $C^\bullet(\mathcal{B})[1]$  and the inclusion map is a quasi-isomorphism.

For a matrix algebra  $\mathcal{B}$  on  $X$  we denote by  $\text{Def}(\mathcal{B})^{loc}(R)$  the subgroupoid of  $\text{Def}(\mathcal{B})(R)$  with objects  $R$ -star products which respect the decomposition given by  $(\mathcal{B} \otimes_k R)_{ij} = \mathcal{B}_{ij} \otimes_k R$  and 1- and 2-morphisms defined accordingly. The composition

$$\text{Def}(\mathcal{B})^{loc}(R) \rightarrow \text{Def}(\mathcal{B})(R) \rightarrow \text{MC}^2(\Gamma(X; C^\bullet(\mathcal{B})[1]) \otimes_k \mathfrak{m}_R)$$

takes values in  $\text{MC}^2(\Gamma(X; C^\bullet(\mathcal{B})^{loc}[1]) \otimes_k \mathfrak{m}_R)$  and establishes an isomorphism of 2-groupoids  $\text{Def}(\mathcal{B})^{loc}(R) \cong \text{MC}^2(\Gamma(X; C^\bullet(\mathcal{B})^{loc}[1]) \otimes_k \mathfrak{m}_R)$ .

**5.2.2. Combinatorial restriction of local cochains.** Suppose given a matrix algebra  $\mathcal{B} = \bigoplus_{i,j=0}^q \mathcal{B}_{ij}$  is a sheaf of matrix  $k$ -algebras.

The DGLA  $C^\bullet(\mathcal{B})^{loc}[1]$  has additional variance not exhibited by  $C^\bullet(\mathcal{B})[1]$ . Namely, for  $f : [p] \rightarrow [q]$  – a morphism in  $\Delta$  – there is a natural map of DGLA

$$f^\# : C^\bullet(\mathcal{B})^{loc}[1] \rightarrow C^\bullet(f^\# \mathcal{B})^{loc}[1] \quad (5.6)$$

defined as follows. Let  $f^\#{}^{ij} : (f^\# \mathcal{B})_{ij} \rightarrow \mathcal{B}_{f(i)f(j)}$  denote the tautological isomorphism. For each multi-index  $I = (i_0, \dots, i_n) \in [p]^{\times n+1}$  let

$$f^\#{}^I := \bigotimes_{j=0}^{n-1} f^\#{}^{i_j i_{j+1}} : \bigotimes_{j=0}^{n-1} \mathcal{B}_{i_j i_{j+1}} \rightarrow \bigotimes_{i=0}^{n-1} \mathcal{B}_{f(i)f(i+1)}.$$

Let  $f^\#{}^n := \bigoplus_{I \in \Sigma_p^{\times n+1}} f^\#{}^I$ . The map (5.6) is defined as restriction along  $f^\#{}^n$ .

**Lemma 5.1.** *The map (5.6) is a morphism of DGLA*

$$f^\# : C^\bullet(\mathcal{B})^{loc}[1] \rightarrow C^\bullet(f^\# \mathcal{B})^{loc}[1].$$

It follows that combinatorial restriction of local cochains induces the functor  $\mathrm{MC}^2(f^\sharp) : \mathrm{MC}^2(\Gamma(X; C^\bullet(\mathcal{B})^{\mathrm{loc}}[1]) \otimes_k \mathfrak{m}_R) \rightarrow \mathrm{MC}^2(\Gamma(X; C^\bullet(f^\sharp \mathcal{B})^{\mathrm{loc}}[1]) \otimes_k \mathfrak{m}_R)$ .

Combinatorial restriction with respect to  $f$  induces the functor

$$f^\sharp : \mathrm{Def}(\mathcal{B})^{\mathrm{loc}}(R) \rightarrow \mathrm{Def}(f^\sharp \mathcal{B})^{\mathrm{loc}}(R) .$$

It is clear that the diagram

$$\begin{array}{ccc} \mathrm{Def}(\mathcal{B})^{\mathrm{loc}}(R) & \xrightarrow{f^\sharp} & \mathrm{Def}(f^\sharp \mathcal{B})^{\mathrm{loc}}(R) \\ \downarrow & & \downarrow \\ \mathrm{MC}^2(\Gamma(X; C^\bullet(\mathcal{B})^{\mathrm{loc}}[1]) \otimes_k \mathfrak{m}_R) & \xrightarrow{\mathrm{MC}^2(f^\sharp)} & \mathrm{MC}^2(\Gamma(X; C^\bullet(f^\sharp \mathcal{B})^{\mathrm{loc}}[1]) \otimes_k \mathfrak{m}_R) \end{array}$$

**5.2.3. Cosimplicial DGLA from descent datum .** Suppose that  $(\mathcal{U}, \underline{\mathcal{A}})$  is a descent datum for a twisted sheaf of algebras as in 4.2.2. Then, for each  $p = 0, 1, \dots$  we have the matrix algebra  $\mathrm{Mat}(\underline{\mathcal{A}})^p$  as defined in 5.1, and therefore the DGLA of local cochains  $C^\bullet(\mathrm{Mat}(\underline{\mathcal{A}})^p)^{\mathrm{loc}}[1]$  defined in 5.2.1. For each morphism  $f : [p] \rightarrow [q]$  there is a morphism of DGLA

$$f^\sharp : C^\bullet(\mathrm{Mat}(\underline{\mathcal{A}})^q)^{\mathrm{loc}}[1] \rightarrow C^\bullet(f^\sharp \mathrm{Mat}(\underline{\mathcal{A}})^q)^{\mathrm{loc}}[1]$$

and an isomorphism of DGLA

$$C^\bullet(f^\sharp \mathrm{Mat}(\underline{\mathcal{A}})^q)^{\mathrm{loc}}[1] \cong f_* C^\bullet(\mathrm{Mat}(\underline{\mathcal{A}})^p)^{\mathrm{loc}}[1]$$

induced by the isomorphism  $f_* : f_* \mathrm{Mat}(\underline{\mathcal{A}})^p \rightarrow f^\sharp \mathrm{Mat}(\underline{\mathcal{A}})^q$  from the equation (5.5). These induce the morphisms of the DGLA of global sections

$$\begin{array}{ccc} \Gamma(N_q \mathcal{U}; C^\bullet(\mathrm{Mat}(\underline{\mathcal{A}})^q)^{\mathrm{loc}}[1]) & & \Gamma(N_p \mathcal{U}; C^\bullet(\mathrm{Mat}(\underline{\mathcal{A}})^p)^{\mathrm{loc}}[1]) \\ & \searrow f^\sharp & \swarrow f_* \\ & \Gamma(N_q \mathcal{U}; f_* C^\bullet(\mathrm{Mat}(\underline{\mathcal{A}})^p)^{\mathrm{loc}}[1]) & \end{array}$$

For  $\lambda : [n] \rightarrow \Delta$  let

$$\mathfrak{G}(\underline{\mathcal{A}})^\lambda = \Gamma(N_{\lambda(n)} \mathcal{U}; \lambda(0n)_* C^\bullet(\mathrm{Mat}(\underline{\mathcal{A}})^{\lambda(0)})^{\mathrm{loc}}[1])$$

Suppose given another simplex  $\mu : [m] \rightarrow \Delta$  and morphism  $\phi : [m] \rightarrow [n]$  such that  $\mu = \lambda \circ \phi$  (i.e.  $\phi$  is a morphism of simplices  $\mu \rightarrow \lambda$ ). The morphism  $(0n)$  factors uniquely into  $0 \rightarrow \phi(0) \rightarrow \phi(m) \rightarrow n$ , which, under  $\lambda$ , gives the factorization of  $\lambda(0n) : \lambda(0) \rightarrow \lambda(n)$  (in  $\Delta$ ) into

$$\lambda(0) \xrightarrow{f} \mu(0) \xrightarrow{g} \mu(m) \xrightarrow{h} \lambda(n) , \quad (5.7)$$

where  $g = \mu(0m)$ . The map

$$\phi_* : \mathfrak{G}(\underline{\mathcal{A}})^\mu \rightarrow \mathfrak{G}(\underline{\mathcal{A}})^\lambda$$

is the composition

$$\begin{aligned} \Gamma(N_{\mu(m)}\mathcal{U}; g_* C^\bullet(\mathbf{Mat}(\underline{\mathcal{A}})^{\mu(0)})^{loc}[1]) &\xrightarrow{h_*} \\ \Gamma(N_{\lambda(n)}\mathcal{U}; h_* g_* C^\bullet(\mathbf{Mat}(\underline{\mathcal{A}})^{\mu(0)})^{loc}[1]) &\xrightarrow{f^\sharp} \\ \Gamma(N_{\lambda(n)}\mathcal{U}; h_* g_* f_* C^\bullet(\mathbf{Mat}(\underline{\mathcal{A}})^{\lambda(0)})^{loc}[1]) & \end{aligned}$$

Suppose given yet another simplex,  $\nu : [l] \rightarrow \Delta$  and a morphism of simplices  $\psi : \nu \rightarrow \mu$ , i.e. a morphism  $\psi : [l] \rightarrow [m]$  such that  $\nu = \mu \circ \psi$ . Then, the composition  $\phi_* \circ \psi_* : \mathfrak{G}(\underline{\mathcal{A}})^\nu \rightarrow \mathfrak{G}(\underline{\mathcal{A}})^\lambda$  coincides with the map  $(\phi \circ \psi)_*$ .

For  $n = 0, 1, 2, \dots$  let

$$\mathfrak{G}(\underline{\mathcal{A}})^n = \coprod_{[n] \xrightarrow{\lambda} \Delta} \mathfrak{G}(\underline{\mathcal{A}})^\lambda \quad (5.8)$$

A morphism  $\phi : [m] \rightarrow [n]$  in  $\Delta$  induces the map of DGLA  $\phi_* : \mathfrak{G}(\underline{\mathcal{A}})^m \rightarrow \mathfrak{G}(\underline{\mathcal{A}})^n$ . The assignment  $\Delta \ni [n] \mapsto \mathfrak{G}(\underline{\mathcal{A}})^n$ ,  $\phi \mapsto \phi_*$  defines the cosimplicial DGLA denoted  $\mathfrak{G}(\underline{\mathcal{A}})$ .

### 5.3. Acyclicity.

**Theorem 5.2.** *The cosimplicial DGLA  $\mathfrak{G}(\underline{\mathcal{A}})$  is acyclic, i.e. it satisfies the condition (3.8).*

The rest of the section is devoted to the proof of the Theorem 5.2. We fix a degree of Hochschild cochains  $k$ .

For  $\lambda : [n] \rightarrow \Delta$  let  $\mathfrak{c}^\lambda = \Gamma(N_{\lambda(n)}\mathcal{U}; \lambda(0n)_* C^k(\mathbf{Mat}(\underline{\mathcal{A}})^{\lambda(0)})^{loc})$ . For a morphism  $\phi : \mu \rightarrow \lambda$  we have the map  $\phi_* : \mathfrak{c}^\mu \rightarrow \mathfrak{c}^\lambda$  defined as in 5.2.3.

Let  $(\mathfrak{C}^\bullet, \partial)$  denote the corresponding cochain complex whose definition we recall below. For  $n = 0, 1, \dots$  let  $\mathfrak{C}^n = \prod_{[n] \xrightarrow{\lambda} \Delta} \mathfrak{c}^\lambda$ . The differential  $\partial^n : \mathfrak{C}^n \rightarrow \mathfrak{C}^{n+1}$  is

defined by the formula  $\partial^n = \sum_{i=0}^{n+1} (-1)^i (\partial_i^n)_*$ .

**5.3.1. Decomposition of local cochains.** As was noted in 5.2.1, for  $n, q = 0, 1, \dots$  there is a direct sum decomposition

$$C^k(\mathbf{Mat}(\underline{\mathcal{A}})^q)^{loc} = \bigoplus_{I \in [q]^{k+1}} C^I(\mathbf{Mat}(\underline{\mathcal{A}})^q)^{loc}. \quad (5.9)$$

In what follows we will interpret a multi-index  $I = (i_0, \dots, i_n) \in [q]^{k+1}$  as a map  $I : \{0, \dots, k\} \rightarrow [q]$ . For  $I$  as above let  $s(I) = |\text{Im}(I)| - 1$ . The map  $I$  factors uniquely into the composition

$$\{0, \dots, k\} \xrightarrow{I'} [s(I)] \xrightarrow{m(I)} [q]$$

where the second map is a morphism in  $\Delta$  (i.e. is order preserving). Then, the isomorphisms  $m(I)_* \mathcal{A}_{I'(i)I'(j)} \cong \mathcal{A}_{I(i)I(j)}$  induce the isomorphism

$$m(I)_* C^{I'}(\mathbf{Mat}(\underline{\mathcal{A}})^{s(I)})^{loc} \rightarrow C^I(\mathbf{Mat}(\underline{\mathcal{A}})^q)^{loc}$$

Therefore, the decomposition (5.9) may be rewritten as follows:

$$C^k(\mathbf{Mat}(\underline{\mathcal{A}})^q)^{loc} = \bigoplus_e \bigoplus_I e_* C^I(\mathbf{Mat}(\underline{\mathcal{A}})^p)^{loc} \quad (5.10)$$

where the summation is over *injective* (monotone) maps  $e : [s(e)] \rightarrow [q]$  and *surjective* maps  $I : \{0, \dots, k\} \rightarrow [s(e)]$ . Note that, for  $e, I$  as above, there is an isomorphism

$$e_* C^I(\mathbf{Mat}(\underline{\mathcal{A}})^p)^{loc} \cong C^{e \circ I}(e^\# \mathbf{Mat}(\underline{\mathcal{A}})^{s(e)})^{loc}$$

**5.3.2. Filtrations.** Let  $F^\bullet C^k(\mathbf{Mat}(\underline{\mathcal{A}})^q)^{loc}$  denote the *decreasing* filtration defined by

$$F^s C^k(\mathbf{Mat}(\underline{\mathcal{A}})^q)^{loc} = \bigoplus_{e: s(e) \geq s} \bigoplus_I e_* C^I(\mathbf{Mat}(\underline{\mathcal{A}})^p)^{loc}$$

Note that  $F^s C^k(\mathbf{Mat}(\underline{\mathcal{A}})^q)^{loc} = 0$  for  $s > k$  and  $Gr^s C^k(\mathbf{Mat}(\underline{\mathcal{A}})^q)^{loc} = 0$  for  $s < 0$ .

The filtration  $F^\bullet C^k(\mathbf{Mat}(\underline{\mathcal{A}})^{\lambda(0)})^{loc}$  induces the filtration  $F^\bullet \mathfrak{c}^\lambda$ , hence the filtration  $F^\bullet \mathfrak{C}^\bullet$  with  $F^s \mathfrak{c}^\lambda = \Gamma(N_{\lambda(n)} \mathcal{U}; \lambda(0n)_* F^s C^k(\mathbf{Mat}(\underline{\mathcal{A}})^{\lambda(0)})^{loc})$ ,  $F^s \mathfrak{C}^n = \prod_{[n] \xrightarrow{\lambda} \Delta} F^s \mathfrak{c}^\lambda$ .

The following result then is an easy consequence of the definitions:

**Lemma 5.3.** *For  $\phi : \mu \rightarrow \lambda$  the induced map  $\phi_* : \mathfrak{c}^\mu \rightarrow \mathfrak{c}^\lambda$  preserves filtration.*

**Corollary 5.4.** *The differential  $\partial^n$  preserves the filtration.*

**Proposition 5.5.** *For any  $s$  the complex  $Gr^s(\mathfrak{C}^\bullet, \partial)$  is acyclic in non-zero degrees.*

*Proof.* Use the following notation: for a simplex  $\mu : [m] \rightarrow \Delta$  and an arrow  $e : [s] \rightarrow [\mu(0)]$  the simplex  $\mu^e : [m+1] \rightarrow \Delta$  is defined by  $\mu^e(0) = [s]$ ,  $\mu^e(01) = e$ , and  $\mu^e(i) = \mu(i-1)$ ,  $\mu^e(i, j) = \mu(i-1, j-1)$  for  $i > 0$ .

For  $D = (D_\lambda) \in F^s \mathfrak{C}^n$ ,  $D_\lambda \in \mathfrak{c}^\lambda$  and  $\mu : [n-1] \rightarrow \Delta$  let

$$h^n(D)_\mu = \sum_{I, e} e_* D_{\mu^e}^I$$

where  $e : [s] \rightarrow \mu(0)$  is an injective map,  $I : \{0, \dots, k\} \rightarrow [s]$  is a surjection, and  $D_{\mu^e}^I$  is the  $I$ -component of  $D_{\mu^e}$  in the sense of the decomposition (5.9). Put  $h^n(D) = (h^n(D)_\mu) \in \mathfrak{C}^{n-1}$ . It is clear that  $h^n(D) \in F^s \mathfrak{C}^{n-1}$ . Thus the assignment  $D \mapsto h^n(D)$  defines a filtered map  $h^n : \mathfrak{C}^n \rightarrow \mathfrak{C}^{n-1}$  for all  $n \geq 1$ . Hence, for any  $s$  we have the map  $Gr^s h^n : Gr^s \mathfrak{C}^n \rightarrow Gr^s \mathfrak{C}^{n-1}$ .

Next, we calculate the effect of the face maps  $\partial_i^{n-1} : [n-1] \rightarrow [n]$ . First of all, note that, for  $1 \leq i \leq n-1$ ,  $(\partial_i^{n-1})_* = \text{Id}$ . The effect of  $(\partial_0^{n-1})_*$  is the map on global sections induced by

$$\lambda(01)^\# : \lambda(1n)_* C^k(\mathbf{Mat}(\underline{\mathcal{A}})^{\lambda(1)})^{loc} \rightarrow \lambda(0n)_* C^k(\mathbf{Mat}(\underline{\mathcal{A}})^{\lambda(0)})^{loc}$$

and  $(\partial_n^{n-1})_* = \lambda(n-1, n)_*$ . Thus, for  $D = (D_\lambda) \in F^s \mathfrak{C}^n$ , we have:

$$((\partial_i^{n-1})_* h^n(D))_\lambda = (\partial_i^{n-1})_* \sum_{I,e} e_* D_{(\lambda \circ \partial_i^{n-1})^e}^I = \begin{cases} \lambda(01)^\# \sum_{I,e} e_* D_{(\lambda \circ \partial_0^{n-1})^e}^I & \text{if } i = 0 \\ \sum_{I,e} e_* D_{(\lambda \circ \partial_i^{n-1})^e}^I & \text{if } 1 \leq i \leq n-1 \\ \lambda(n-1, n)_* \sum_{I,e} e_* D_{(\lambda \circ \partial_n^{n-1})^e}^I & \text{if } i = n \end{cases}$$

On the other hand,

$$h^{n+1}((\partial_j^n)_* D)_\lambda = \sum_{I,e} e_* ((\partial_j^n)_* D)_{\lambda^e}^I = \begin{cases} \sum_{I,e} e_* e^\# D_\lambda^I & \text{if } j = 0 \\ \sum_{I,e} e_* D_{(\lambda \circ \partial_0^{n-1})^{\lambda(01) \circ e}}^I & \text{if } j = 1 \\ \sum_{I,e} e_* D_{(\lambda \circ \partial_{j-1}^{n-1})^e}^I & \text{if } 2 \leq j \leq n \\ \sum_{I,e} \lambda(n-1, n)_* e_* D_{(\lambda \circ \partial_n^{n-1})^e}^I & \text{if } j = n+1 \end{cases}$$

Note that  $\sum_{I,e} e_* e^\# D_\lambda^I = D_\lambda \pmod{F^{s+1} \mathfrak{C}^\lambda}$ , i.e. the map  $h^{n+1} \circ \partial_0^n$  induces the identity map on  $Gr^s \mathfrak{C}^\lambda$  for all  $\lambda$ , hence the identity map on  $Gr^s \mathfrak{C}^n$ .

The identities  $(\partial_i^{n-1} h^n(D))_\lambda = h^{n+1}((\partial_{i+1}^n)_* D)_\lambda \pmod{F^s \mathfrak{C}^\lambda}$  hold for  $0 \leq i \leq n$ .

For  $n = 0, 1, \dots$  let  $\partial^n = \sum_i (-1) \partial_i^n$ . The above identities show that, for  $n > 0$ ,  $h^{n+1} \circ \partial^n + \partial^{n-1} \circ h^n = \text{Id}$ .  $\square$

*Proof of Theorem 5.2.* Consider for a fixed  $k$  the complex of Hochschild degree  $k$  cochains  $\mathfrak{C}^\bullet$ . It is shown in Proposition 5.5 that this complex admits a finite filtration  $F^\bullet \mathfrak{C}^\bullet$  such that  $Gr \mathfrak{C}^\bullet$  is acyclic in positive degrees. Therefore  $\mathfrak{C}^\bullet$  is also acyclic in positive degree. As this holds for every  $k$ , the condition (3.8) is satisfied.  $\square$

## 6. Deformations of algebroid stacks

In this section we define a 2-groupoid of deformations of an algebroid stack. We also define 2-groupoids of deformations and star products of a descent datum and relate it with the 2-groupoid  $\mathfrak{G}$ -stacks of an appropriate cosimplicial DGLA.

### 6.1. Deformations of linear stacks.

**Definition 6.1.** Let  $\mathcal{B}$  be a prestack on  $X$  in  $R$ -linear categories. We say that  $\mathcal{B}$  is *flat* if for any  $U \subseteq X$ ,  $A, B \in \mathcal{B}(U)$  the sheaf  $\underline{\text{Hom}}_{\mathcal{B}}(A, B)$  is flat (as a sheaf of  $R$ -modules).

Suppose now that  $\mathcal{C}$  is a stack in  $k$ -linear categories on  $X$  and  $R$  is a commutative Artin  $k$ -algebra. We denote by  $\text{Def}(\mathcal{C})(R)$  the 2-category with

- objects: pairs  $(\mathcal{B}, \varpi)$ , where  $\mathcal{B}$  is a stack in  $R$ -linear categories flat over  $R$  and  $\varpi : \mathcal{B} \otimes_R k \rightarrow \mathcal{C}$  is an equivalence of stacks in  $k$ -linear categories

- 1-morphisms: a 1-morphism  $(\mathcal{B}^{(1)}, \varpi^{(1)}) \rightarrow (\mathcal{B}^{(2)}, \varpi^{(2)})$  is a pair  $(F, \theta)$  where  $F : \mathcal{B}^{(1)} \rightarrow \mathcal{B}^{(2)}$  is a  $R$ -linear functor and  $\theta : \varpi^{(2)} \circ (F \otimes_R k) \rightarrow \varpi^{(1)}$  is an isomorphism of functors
- 2-morphisms: a 2-morphism  $(F', \theta') \rightarrow (F'', \theta'')$  is a morphism of  $R$ -linear functors  $\kappa : F' \rightarrow F''$  such that  $\theta'' \circ (\text{Id}_{\varpi^{(2)}} \otimes (\kappa \otimes_R k)) = \theta'$

The 2-category  $\text{Def}(\mathcal{C})(R)$  is a 2-groupoid.

**Lemma 6.2.** *Suppose that  $\mathcal{B}$  is a flat  $R$ -linear stack on  $X$  such that  $\widetilde{\mathcal{B} \otimes_R k}$  is an algebroid stack. Then,  $\mathcal{B}$  is an algebroid stack.*

*Proof.* Let  $x \in X$ . Suppose that for any neighborhood  $U$  of  $x$  the category  $\mathcal{B}(U)$  is empty. Then, the same is true about  $\widetilde{\mathcal{B} \otimes_R k}(U)$  which contradicts the assumption that  $\widetilde{\mathcal{B} \otimes_R k}$  is an algebroid stack. Therefore,  $\mathcal{B}$  is locally nonempty.

Suppose that  $U$  is an open subset and  $A, B$  are two objects in  $\mathcal{B}(U)$ . Let  $\bar{A}$  and  $\bar{B}$  be their respective images in  $\widetilde{\mathcal{B} \otimes_R k}(U)$ . We have:  $\text{Hom}_{\widetilde{\mathcal{B} \otimes_R k}(U)}(\bar{A}, \bar{B}) = \Gamma(U; \underline{\text{Hom}}_{\mathcal{B}}(A, B) \otimes_R k)$ . Replacing  $U$  by a subset we may assume that there is an isomorphism  $\bar{\phi} : \bar{A} \rightarrow \bar{B}$ .

The short exact sequence

$$0 \rightarrow \mathfrak{m}_R \rightarrow R \rightarrow k \rightarrow 0$$

gives rise to the sequence

$$0 \rightarrow \underline{\text{Hom}}_{\mathcal{B}}(A, B) \otimes_R \mathfrak{m}_R \rightarrow \underline{\text{Hom}}_{\mathcal{B}}(A, B) \rightarrow \underline{\text{Hom}}_{\mathcal{B}}(A, B) \otimes_R k \rightarrow 0$$

of sheaves on  $U$  which is exact due to flatness of  $\underline{\text{Hom}}_{\mathcal{B}}(A, B)$ . The surjectivity of the map  $\underline{\text{Hom}}_{\mathcal{B}}(A, B) \rightarrow \underline{\text{Hom}}_{\mathcal{B}}(A, B) \otimes_R k$  implies that for any  $x \in U$  there exists a neighborhood  $x \in V \subseteq U$  and  $\phi \in \Gamma(V; \underline{\text{Hom}}_{\mathcal{B}}(A|_V, B|_V)) = \text{Hom}_{\mathcal{B}(V)}(A|_V, B|_V)$  such that  $\bar{\phi}|_V$  is the image of  $\phi$ . Since  $\bar{\phi}$  is an isomorphism and  $\mathfrak{m}_R$  is nilpotent it follows that  $\phi$  is an isomorphism.  $\square$

**6.2. Deformations of descent data.** Suppose that  $(\mathcal{U}, \underline{\mathcal{A}})$  is a  $k$ -descent datum and  $R$  is a commutative Artin  $k$ -algebra.

We denote by  $\text{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R)$  the 2-category with

- objects:  $R$ -deformations of  $(\mathcal{U}, \underline{\mathcal{A}})$ ; such a gadget is a flat  $R$ -descent datum  $(\mathcal{U}, \underline{\mathcal{B}})$  together with an isomorphism of  $k$ -descent data  $\pi : (\mathcal{U}, \underline{\mathcal{B}} \otimes_R k) \rightarrow (\mathcal{U}, \underline{\mathcal{A}})$
- 1-morphisms: a 1-morphism of deformations  $(\mathcal{U}, \underline{\mathcal{B}}^{(1)}, \pi^{(1)}) \rightarrow (\mathcal{U}, \underline{\mathcal{B}}^{(2)}, \pi^{(2)})$  is a pair  $(\underline{\phi}, \underline{a})$ , where  $\underline{\phi} : (\mathcal{U}, \underline{\mathcal{B}}^{(1)}) \rightarrow (\mathcal{U}, \underline{\mathcal{B}}^{(2)})$  is a 1-morphism of  $R$ -descent data and  $\underline{a} : \pi^{(2)} \circ (\underline{\phi} \otimes_R k) \rightarrow \pi^{(1)}$  is 2-isomorphism
- 2-morphisms: a 2-morphism  $(\underline{\phi}', \underline{a}') \rightarrow (\underline{\phi}'', \underline{a}'')$  is a 2-morphism  $\underline{b} : \underline{\phi}' \rightarrow \underline{\phi}''$  such that  $\underline{a}'' \circ (\text{Id}_{\pi^{(2)}} \otimes (\underline{b} \otimes_R k)) = \underline{a}'$

Suppose that  $(\underline{\phi}, \underline{a})$  is a 1-morphism. It is immediate from the definition above that the morphism of  $k$ -descent data  $\underline{\phi} \otimes_R k$  is an isomorphism. Since  $R$  is an extension of  $k$  by a nilpotent ideal the morphism  $\underline{\phi}$  is an isomorphism. Similarly, any 2-morphism is an isomorphism, i.e.  $\text{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R)$  is a 2-groupoid.

The assignment  $R \mapsto \text{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R)$  is fibered 2-category in 2-groupoids over the category of commutative Artin  $k$ -algebras ([3]).

**6.2.1. Star products.** A  $(R)$ -star product on  $(\mathcal{U}, \underline{\mathcal{A}})$  is a deformation  $(\mathcal{U}, \underline{\mathcal{B}}, \pi)$  of  $(\mathcal{U}, \underline{\mathcal{A}})$  such that  $\mathcal{B}_{01} = \mathcal{A}_{01} \otimes_k R$  and  $\pi : \mathcal{B}_{01} \otimes_R k \rightarrow \mathcal{A}_{01}$  is the canonical isomorphism. In other words, a star product is a structure of an  $R$ -descent datum on  $(\mathcal{U}, \underline{\mathcal{A}} \otimes_k R)$  such that the natural map

$$(\mathcal{U}, \underline{\mathcal{A}} \otimes_k R) \rightarrow (\mathcal{U}, \underline{\mathcal{A}})$$

is a morphism of such.

We denote by  $\text{Def}(\mathcal{U}, \underline{\mathcal{A}})(R)$  the full 2-subcategory of  $\text{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R)$  whose objects are star products.

The assignment  $R \mapsto \text{Def}(\mathcal{U}, \underline{\mathcal{A}})(R)$  is fibered 2-category in 2-groupoids over the category of commutative Artin  $k$ -algebras ([3]) and the inclusions  $\text{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) \rightarrow \text{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R)$  extend to a morphism of fibered 2-categories.

**Proposition 6.3.** *Suppose that  $(\mathcal{U}, \underline{\mathcal{A}})$  is a  $\mathbb{C}$ -linear descent datum with  $\mathcal{A} = \mathcal{O}_{N_0\mathcal{U}}$ . Then, the embedding  $\text{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) \rightarrow \text{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R)$  is an equivalence.*

**6.2.2. Deformations and  $\mathfrak{G}$ -stacks.** Suppose that  $(\mathcal{U}, \underline{\mathcal{A}})$  is a  $k$ -descent datum and  $(\mathcal{U}, \underline{\mathcal{B}})$  is an  $R$ -star product on  $(\mathcal{U}, \underline{\mathcal{A}})$ . Then, for every  $p = 0, 1, \dots$  the matrix algebra  $\text{Mat}(\underline{\mathcal{B}})^p$  is a flat  $R$ -deformation of the matrix algebra  $\text{Mat}(\underline{\mathcal{A}})^p$ . The identification  $\mathcal{B}_{01} = \mathcal{A}_{01} \otimes_k R$  gives rise to the identification  $\text{Mat}(\underline{\mathcal{B}})^p = \text{Mat}(\underline{\mathcal{A}})^p \otimes_k R$  of the underlying sheaves of  $R$ -modules. Using this identification we obtain the Maurer-Cartan element  $\mu^p \in \Gamma(N_p\mathcal{U}; C^2(\text{Mat}(\underline{\mathcal{A}})^p)^{loc} \otimes_k \mathfrak{m}_R)$ . Moreover, the equation (5.5) implies that for a morphism  $f : [p] \rightarrow [q]$  in  $\Delta$  we have  $f_*\mu^p = f^\# \mu^q$ . Therefore the collection  $\mu^p$  defines an element in  $\text{Stack}_{str}(\mathfrak{G}(\underline{\mathcal{A}}) \otimes_k \mathfrak{m}_R)_0$ . The considerations of 5.2.1 and 5.2.2 imply that this construction extends to an isomorphism of 2-groupoids

$$\text{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) \rightarrow \text{Stack}_{str}(\mathfrak{G}(\underline{\mathcal{A}}) \otimes_k \mathfrak{m}_R) \quad (6.1)$$

Combining (6.1) with the embedding

$$\text{Stack}_{str}(\mathfrak{G}(\underline{\mathcal{A}}) \otimes_k \mathfrak{m}_R) \rightarrow \text{Stack}(\mathfrak{G}(\underline{\mathcal{A}}) \otimes_k \mathfrak{m}_R) \quad (6.2)$$

we obtain the functor

$$\text{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) \rightarrow \text{Stack}(\mathfrak{G}(\underline{\mathcal{A}}) \otimes_k \mathfrak{m}_R) \quad (6.3)$$

The naturality properties of (6.3) with respect to base change imply that (6.3) extends to morphism of functors on the category of commutative Artin algebras.

Combining this with the results of Theorems 5.2 and 3.12 implies the following:

**Proposition 6.4.** *The functor (6.3) is an equivalence.*

*Proof.* By Theorem 5.2 the DGLA  $\mathfrak{G}(\mathcal{A}) \otimes_k \mathfrak{m}_R$  satisfies the assumptions of Theorem 3.12. The latter says that the inclusion (6.2) is an equivalence. Since (6.1) is an isomorphism, the composition (6.3) is an equivalence as claimed.  $\square$

## 7. Jets

In this section we use constructions involving the infinite jets to simplify the cosimplicial DGLA governing the deformations of a descent datum.

**7.1. Infinite jets of a vector bundle.** Let  $M$  be a smooth manifold, and  $\mathcal{E}$  a locally-free  $\mathcal{O}_M$ -module of finite rank.

Let  $\pi_i : M \times M \rightarrow M$ ,  $i = 1, 2$  denote the projection on the  $i^{\text{th}}$ . Denote by  $\Delta_M : M \rightarrow M \times M$  the diagonal embedding and let  $\Delta_M^* : \mathcal{O}_{M \times M} \rightarrow \mathcal{O}_M$  be the induced map. Let  $\mathcal{I}_M := \ker(\Delta_M^*)$ .

Let

$$\mathcal{J}^k(\mathcal{E}) := (\pi_1)_* \left( \mathcal{O}_{M \times M} / \mathcal{I}_M^{k+1} \otimes_{\pi_2^{-1} \mathcal{O}_M} \pi_2^{-1} \mathcal{E} \right),$$

$\mathcal{J}_M^k := \mathcal{J}^k(\mathcal{O}_M)$ . It is clear from the above definition that  $\mathcal{J}_M^k$  is, in a natural way, a sheaf of commutative algebras and  $\mathcal{J}^k(\mathcal{E})$  is a sheaf of  $\mathcal{J}_M^k$ -modules. If moreover  $\mathcal{E}$  is a sheaf of algebras,  $\mathcal{J}^k(\mathcal{E})$  will canonically be a sheaf of algebras as well. We regard  $\mathcal{J}^k(\mathcal{E})$  as  $\mathcal{O}_M$ -modules via the pull-back map  $\pi_1^* : \mathcal{O}_M \rightarrow (\pi_1)_* \mathcal{O}_{M \times M}$

For  $0 \leq k \leq l$  the inclusion  $\mathcal{I}_M^{l+1} \rightarrow \mathcal{I}_M^{k+1}$  induces the surjective map  $\mathcal{J}^l(\mathcal{E}) \rightarrow \mathcal{J}^k(\mathcal{E})$ . The sheaves  $\mathcal{J}^k(\mathcal{E})$ ,  $k = 0, 1, \dots$  together with the maps just defined form an inverse system. Define  $\mathcal{J}(\mathcal{E}) := \varprojlim \mathcal{J}^k(\mathcal{E})$ . Thus,  $\mathcal{J}(\mathcal{E})$  carries a natural topology.

We denote by  $p_{\mathcal{E}} : \mathcal{J}(\mathcal{E}) \rightarrow \mathcal{E}$  the canonical projection. In the case when  $\mathcal{E} = \mathcal{O}_M$  we denote by  $p$  the corresponding projection  $p : \mathcal{J}_M \rightarrow \mathcal{O}_M$ . By  $j^k : \mathcal{E} \rightarrow \mathcal{J}^k(\mathcal{E})$  we denote the map  $e \mapsto 1 \otimes e$ , and  $j^\infty := \varprojlim j^k$ . In the case  $\mathcal{E} = \mathcal{O}_M$  we also have the canonical embedding  $\mathcal{O}_M \rightarrow \mathcal{J}_M$  given by  $f \mapsto f \cdot j^\infty(1)$ .

Let

$$d_1 : \mathcal{O}_{M \times M} \otimes_{\pi_2^* \mathcal{O}_M} \pi_2^{-1} \mathcal{E} \rightarrow \pi_1^{-1} \Omega_M^1 \otimes_{\pi_1^{-1} \mathcal{O}_M} \mathcal{O}_{M \times M} \otimes_{\pi_2^{-1} \mathcal{O}_M} \pi_2^{-1} \mathcal{E}$$

denote the exterior derivative along the first factor. It satisfies

$$d_1(\mathcal{I}_M^{k+1} \otimes_{\pi_2^{-1} \mathcal{O}_M} \pi_2^{-1} \mathcal{E}) \subset \pi_1^{-1} \Omega_M^1 \otimes_{\pi_1^{-1} \mathcal{O}_M} \mathcal{I}_M^k \otimes_{\pi_2^{-1} \mathcal{O}_M} \pi_2^{-1} \mathcal{E}$$

for each  $k$  and, therefore, induces the map

$$d_1^{(k)} : \mathcal{J}^k(\mathcal{E}) \rightarrow \Omega_{M/\mathcal{P}}^1 \otimes_{\mathcal{O}_M} \mathcal{J}^{k-1}(\mathcal{E})$$

The maps  $d_1^{(k)}$  for different values of  $k$  are compatible with the maps  $\mathcal{J}^l(\mathcal{E}) \rightarrow \mathcal{J}^k(\mathcal{E})$  giving rise to the *canonical flat connection*

$$\nabla^{can} : \mathcal{J}(\mathcal{E}) \rightarrow \Omega_M^1 \otimes \mathcal{J}(\mathcal{E})$$

Here and below we use notation  $(\bullet) \otimes \mathcal{J}(\mathcal{E})$  for  $\lim_{\leftarrow} (\bullet) \otimes_{\mathcal{O}_M} \mathcal{J}^k(\mathcal{E})$ .

Since  $\nabla^{can}$  is flat we obtain the complex of sheaves  $\mathbf{DR}(\mathcal{J}(\mathcal{E})) = (\Omega_M^\bullet \otimes \mathcal{J}(\mathcal{E}), \nabla^{can})$ . When  $\mathcal{E} = \mathcal{O}_M$  embedding  $\mathcal{O}_M \rightarrow \mathcal{J}_M$  induces embedding of de Rham complex  $\mathbf{DR}(\mathcal{O}) = (\Omega_M^\bullet, d)$  into  $\mathbf{DR}(\mathcal{J})$ . We denote the quotient by  $\mathbf{DR}(\mathcal{J}/\mathcal{O})$ . All the complexes above are complexes of soft sheaves. We have the following:

**Proposition 7.1.** *The (hyper)cohomology  $H^i(M, \mathbf{DR}(\mathcal{J}(\mathcal{E}))) \cong H^i(\Gamma(M; \Omega_M \otimes \mathcal{J}(\mathcal{E})), \nabla^{can})$  is 0 if  $i > 0$ . The map  $j^\infty : \mathcal{E} \rightarrow \mathcal{J}(\mathcal{E})$  induces the isomorphism between  $\Gamma(\mathcal{E})$  and  $H^0(M, \mathbf{DR}(\mathcal{J}(\mathcal{E}))) \cong H^0(\Gamma(M; \Omega_M \otimes \mathcal{J}(\mathcal{E})), \nabla^{can})$*

**7.2. Jets of line bundles.** Let, as before,  $M$  be a smooth manifold,  $\mathcal{J}_M$  be the sheaf of infinite jets of smooth functions on  $M$  and  $p : \mathcal{J}_M \rightarrow \mathcal{O}_M$  be the canonical projection. Set  $\mathcal{J}_{M,0} = \ker p$ . Note that  $\mathcal{J}_{M,0}$  is a sheaf of  $\mathcal{O}_M$  modules and therefore is soft.

Suppose now that  $\mathcal{L}$  is a line bundle on  $M$ . Let  $\mathbf{Isom}_0(\mathcal{L} \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}))$  denote the sheaf of local  $\mathcal{J}_M$ -module isomorphisms  $\mathcal{L} \otimes \mathcal{J}_M \rightarrow \mathcal{J}(\mathcal{L})$  such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{L} \otimes \mathcal{J}_M & \longrightarrow & \mathcal{J}(\mathcal{L}) \\ & \searrow \text{Id} \otimes p & \swarrow p_{\mathcal{L}} \\ & & \mathcal{L} \end{array}$$

It is easy to see that the canonical map  $\mathcal{J}_M \rightarrow \mathbf{End}_{\mathcal{J}_M}(\mathcal{L} \otimes \mathcal{J}_M)$  is an isomorphism. For  $\phi \in \mathcal{J}_{M,0}$  the exponential series  $\exp(\phi)$  converges. The composition

$$\mathcal{J}_{M,0} \xrightarrow{\exp} \mathcal{J}_M \rightarrow \mathbf{End}_{\mathcal{J}_M}(\mathcal{L} \otimes \mathcal{J}_M)$$

defines an isomorphism of sheaves of groups

$$\exp : \mathcal{J}_{M,0} \rightarrow \mathbf{Aut}_0(\mathcal{L} \otimes \mathcal{J}_M),$$

where  $\mathbf{Aut}_0(\mathcal{L} \otimes \mathcal{J}_M)$  is the sheaf of groups of (locally defined)  $\mathcal{J}_M$ -linear automorphisms of  $\mathcal{L} \otimes \mathcal{J}_M$  making the diagram

$$\begin{array}{ccc} \mathcal{L} \otimes \mathcal{J}_M & \longrightarrow & \mathcal{L} \otimes \mathcal{J}_M \\ & \searrow \text{Id} \otimes p & \swarrow \text{Id} \otimes p \\ & & \mathcal{L} \end{array}$$

commutative.

**Lemma 7.2.** *The sheaf  $\mathbf{Isom}_0(\mathcal{L} \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}))$  is a torsor under the sheaf of groups  $\exp \mathcal{J}_{M,0}$ .*

*Proof.* Since  $\mathcal{L}$  is locally trivial, both  $\mathcal{J}(\mathcal{L})$  and  $\mathcal{L} \otimes \mathcal{J}_M$  are locally isomorphic to  $\mathcal{J}_M$ . Therefore the sheaf  $\mathbf{Isom}_0(\mathcal{L} \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}))$  is locally non-empty, hence a torsor.  $\square$

**Corollary 7.3.** *The torsor  $\underline{\text{Isom}}_0(\mathcal{L} \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}))$  is trivial, i.e.  $\text{Isom}_0(\mathcal{L} \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L})) := \Gamma(M; \underline{\text{Isom}}_0(\mathcal{L} \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}))) \neq \emptyset$ .*

*Proof.* Since the sheaf of groups  $\mathcal{J}_{M,0}$  is soft we have  $H^1(M, \mathcal{J}_{M,0}) = 0$  ([8], Lemme 22, cf. also [6], Proposition 4.1.7). Therefore every  $\mathcal{J}_{M,0}$ -torsor is trivial.  $\square$

**Corollary 7.4.** *The set  $\text{Isom}_0(\mathcal{L} \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}))$  is an affine space with the underlying vector space  $\Gamma(M; \mathcal{J}_{M,0})$ .*

Let  $\mathcal{L}_1$  and  $\mathcal{L}_2$  be two line bundles, and  $f : \mathcal{L}_1 \rightarrow \mathcal{L}_2$  an isomorphism. Then  $f$  induces a map  $\text{Isom}_0(\mathcal{L}_2 \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}_2)) \rightarrow \text{Isom}_0(\mathcal{L}_1 \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}_1))$  which we denote by  $\text{Ad } f$ :

$$\text{Ad } f(\sigma) = (j^\infty(f))^{-1} \circ \sigma \circ (f \otimes \text{Id}).$$

Let  $\mathcal{L}$  be a line bundle on  $M$  and  $f : N \rightarrow M$  is a smooth map. Then there is a pull-back map  $f^* : \text{Isom}_0(\mathcal{L} \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L})) \rightarrow \text{Isom}_0(f^*\mathcal{L} \otimes \mathcal{J}_N, \mathcal{J}(f^*\mathcal{L}))$ .

If  $\mathcal{L}_1, \mathcal{L}_2$  are two line bundles, and  $\sigma_i \in \text{Isom}_0(\mathcal{L}_i \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}_i))$ ,  $i = 1, 2$ . Then we denote by  $\sigma_1 \otimes \sigma_2$  the induced element of  $\text{Isom}_0((\mathcal{L}_1 \otimes \mathcal{L}_2) \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}_1 \otimes \mathcal{L}_2))$ .

For a line bundle  $\mathcal{L}$  let  $\mathcal{L}^*$  be its dual. Then given  $\sigma \in \text{Isom}_0(\mathcal{L} \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}))$  there exists a unique  $\sigma^* \in \text{Isom}_0(\mathcal{L}^* \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}^*))$  such that  $\sigma^* \otimes \sigma = \text{Id}$ .

For any bundle  $E$   $\mathcal{J}(E)$  has a canonical flat connection which we denote  $\nabla^{\text{can}}$ . A choice of  $\sigma \in \text{Isom}_0(\mathcal{L} \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}))$  induces the flat connection  $\sigma^{-1} \circ \nabla_{\mathcal{L}}^{\text{can}} \circ \sigma$  on  $\mathcal{L} \otimes \mathcal{J}_M$ .

Let  $\nabla$  be a connection on  $\mathcal{L}$  with the curvature  $\theta$ . It gives rise to the connection  $\nabla \otimes \text{Id} + \text{Id} \otimes \nabla^{\text{can}}$  on  $\mathcal{L} \otimes \mathcal{J}_M$ .

**Lemma 7.5.** *1. Choose  $\sigma, \nabla$  as above. Then the difference*

$$F(\sigma, \nabla) = \sigma^{-1} \circ \nabla_{\mathcal{L}}^{\text{can}} \circ \sigma - (\nabla \otimes \text{Id} + \text{Id} \otimes \nabla^{\text{can}}) \quad (7.1)$$

*is an element of  $\Omega^1 \otimes \underline{\text{End}}_{\mathcal{J}_M}(\mathcal{L} \otimes \mathcal{J}_M) \cong \Omega^1 \otimes \mathcal{J}_M$ .*

*2. Moreover,  $F$  satisfies*

$$\nabla^{\text{can}} F(\sigma, \nabla) + \theta = 0 \quad (7.2)$$

*Proof.* We leave the verification of the first claim to the reader. The flatness of  $\sigma^{-1} \circ \nabla_{\mathcal{L}}^{\text{can}} \circ \sigma$  implies the second claim.  $\square$

The following properties of our construction are immediate

**Lemma 7.6.** *1. Let  $\mathcal{L}_1$  and  $\mathcal{L}_2$  be two line bundles, and  $f : \mathcal{L}_1 \rightarrow \mathcal{L}_2$  an isomorphism. Let  $\nabla$  be a connection on  $\mathcal{L}_2$  and  $\sigma \in \text{Isom}_0(\mathcal{L}_2 \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}_2))$ . Then*

$$F(\sigma, \nabla) = F(\text{Ad } f(\sigma), \text{Ad } f(\nabla))$$

*2. Let  $\mathcal{L}$  be a line bundle on  $M$ ,  $\nabla$  a connection on  $\mathcal{L}$  and  $\sigma \in \text{Isom}_0(\mathcal{L} \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}))$ . Let  $f : N \rightarrow M$  be a smooth map. Then*

$$f^* F(\sigma, \nabla) = F(f^*\sigma, f^*\nabla)$$

3. Let  $\mathcal{L}$  be a line bundle on  $M$ ,  $\nabla$  a connection on  $\mathcal{L}$  and  $\sigma \in \text{Isom}_0(\mathcal{L} \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}))$ . Let  $\phi \in \Gamma(M; \mathcal{J}_{M,0})$ . Then

$$F(\phi \cdot \sigma, \nabla) = F(\sigma, \nabla) + \nabla^{can} \phi$$

4. Let  $\mathcal{L}_1, \mathcal{L}_2$  be two line bundles with connections  $\nabla_1$  and  $\nabla_2$  respectively, and let  $\sigma_i \in \text{Isom}_0(\mathcal{L}_i \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}_i))$ ,  $i = 1, 2$ . Then

$$F(\sigma_1 \otimes \sigma_2, \nabla_1 \otimes \text{Id} + \text{Id} \otimes \nabla_2) = F(\sigma_1, \nabla_1) + F(\sigma_2, \nabla_2)$$

**7.3. DGLAs of infinite jets.** Suppose that  $(\mathcal{U}, \underline{\mathcal{A}})$  is a descent datum representing a twisted form of  $\mathcal{O}_X$ . Thus, we have the matrix algebra  $\text{Mat}(\underline{\mathcal{A}})$  and the cosimplicial DGLA  $\mathfrak{G}(\underline{\mathcal{A}})$  of local  $\mathbb{C}$ -linear Hochschild cochains.

The descent datum  $(\mathcal{U}, \underline{\mathcal{A}})$  gives rise to the descent datum  $(\mathcal{U}, \mathcal{J}(\underline{\mathcal{A}}))$ ,  $\mathcal{J}(\underline{\mathcal{A}}) = (\mathcal{J}(\mathcal{A}), \mathcal{J}(\mathcal{A}_{01}), j^\infty(\mathcal{A}_{012}))$ , representing a twisted form of  $\mathcal{J}_X$ , hence to the matrix algebra  $\text{Mat}(\mathcal{J}(\underline{\mathcal{A}}))$  and the corresponding cosimplicial DGLA  $\mathfrak{G}(\mathcal{J}(\underline{\mathcal{A}}))$  of local  $\mathcal{O}$ -linear continuous Hochschild cochains.

The canonical flat connection  $\nabla^{can}$  on  $\mathcal{J}(\mathcal{A})$  induces the flat connection, still denoted  $\nabla^{can}$  on  $\text{Mat}(\mathcal{J}(\underline{\mathcal{A}}))^p$  for each  $p$ ; the product on  $\text{Mat}(\mathcal{J}(\underline{\mathcal{A}}))^p$  is horizontal with respect to  $\nabla^{can}$ . The flat connection  $\nabla^{can}$  induces the flat connection, still denoted  $\nabla^{can}$  on  $C^\bullet(\text{Mat}(\mathcal{J}(\underline{\mathcal{A}}))^p)^{loc}[1]$  which acts by derivations of the Gerstenhaber bracket and commutes with the Hochschild differential  $\delta$ . Therefore we have the sheaf of DGLA  $\text{DR}(C^\bullet(\text{Mat}(\mathcal{J}(\underline{\mathcal{A}}))^p)^{loc}[1])$  with the underlying sheaf of graded Lie algebras  $\Omega_{N_p \mathcal{U}}^\bullet \otimes C^\bullet(\text{Mat}(\mathcal{J}(\underline{\mathcal{A}}))^p)^{loc}[1]$  and the differential  $\nabla^{can} + \delta$ .

For  $\lambda : [n] \rightarrow \Delta$  let

$$\mathfrak{G}_{\text{DR}}(\mathcal{J}(\underline{\mathcal{A}}))^\lambda = \Gamma(N_{\lambda(n)} \mathcal{U}; \lambda(0n)_* \text{DR}(C^\bullet(\text{Mat}(\mathcal{J}(\underline{\mathcal{A}}))^{\lambda(0)})^{loc}[1]))$$

be the DGLA of global sections. The ‘‘inclusion of horizontal sections’’ map induces the morphism of DGLA

$$j^\infty : \mathfrak{G}(\underline{\mathcal{A}})^\lambda \rightarrow \mathfrak{G}_{\text{DR}}(\mathcal{J}(\underline{\mathcal{A}}))^\lambda$$

For  $\phi : [m] \rightarrow [n]$  in  $\Delta$ ,  $\mu = \lambda \circ \phi$  there is a morphism of DGLA  $\phi_* : \mathfrak{G}_{\text{DR}}(\mathcal{J}(\underline{\mathcal{A}}))^\mu \rightarrow \mathfrak{G}_{\text{DR}}(\mathcal{J}(\underline{\mathcal{A}}))^\lambda$  making the diagram

$$\begin{array}{ccc} \mathfrak{G}(\underline{\mathcal{A}})^\lambda & \xrightarrow{j^\infty} & \mathfrak{G}_{\text{DR}}(\mathcal{J}(\underline{\mathcal{A}}))^\lambda \\ \phi_* \downarrow & & \downarrow \phi_* \\ \mathfrak{G}(\underline{\mathcal{A}})^\mu & \xrightarrow{j^\infty} & \mathfrak{G}_{\text{DR}}(\mathcal{J}(\underline{\mathcal{A}}))^\mu \end{array}$$

commutative.

Let  $\mathfrak{G}_{\text{DR}}(\mathcal{J}(\underline{\mathcal{A}}))^n = \prod_{[n] \xrightarrow{\lambda} \Delta} \mathfrak{G}_{\text{DR}}(\mathcal{J}(\underline{\mathcal{A}}))^\lambda$ . The assignment  $\Delta \ni [n] \mapsto \mathfrak{G}_{\text{DR}}(\mathcal{J}(\underline{\mathcal{A}}))^n$ ,  $\phi \mapsto \phi_*$  defines the cosimplicial DGLA  $\mathfrak{G}_{\text{DR}}(\mathcal{J}(\underline{\mathcal{A}}))$ .

**Proposition 7.7.** *The map  $j^\infty : \mathfrak{G}(\underline{\mathcal{A}}) \rightarrow \mathfrak{G}(\mathcal{J}(\underline{\mathcal{A}}))$  extends to a quasiisomorphism of cosimplicial DGLA.*

$$j^\infty : \mathfrak{G}(\underline{\mathcal{A}}) \rightarrow \mathfrak{G}_{\text{DR}}(\mathcal{J}(\underline{\mathcal{A}}))$$

The goal of this section is to construct a quasiisomorphism of the latter DGLA with the simpler DGLA.

The canonical flat connection  $\nabla^{can}$  on  $\mathcal{J}_X$  induces a flat connection on  $\overline{\mathcal{C}}^\bullet(\mathcal{J}_X)[1]$ , the complex of  $\mathcal{O}$ -linear continuous normalized Hochschild cochains, still denoted  $\nabla^{can}$  which acts by derivations of the Gerstenhaber bracket and commutes with the Hochschild differential  $\delta$ . Therefore we have the sheaf of DGLA  $\mathrm{DR}(\overline{\mathcal{C}}^\bullet(\mathcal{J}_X)[1])$  with the underlying graded Lie algebra  $\Omega_X^\bullet \otimes \overline{\mathcal{C}}^\bullet(\mathcal{J}_X)[1]$  and the differential  $\delta + \nabla^{can}$ .

Recall that the Hochschild differential  $\delta$  is zero on  $C^0(\mathcal{J}_X)$  due to commutativity of  $\mathcal{J}_X$ . It follows that the action of the sheaf of abelian Lie algebras  $\mathcal{J}_X = C^0(\mathcal{J}_X)$  on  $\overline{\mathcal{C}}^\bullet(\mathcal{J}_X)[1]$  via the restriction of the adjoint action (by derivations of degree  $-1$ ) commutes with the Hochschild differential  $\delta$ . Since the cochains we consider are  $\mathcal{O}_X$ -linear, the subsheaf  $\mathcal{O}_X = \mathcal{O}_X \cdot j^\infty(1) \subset \mathcal{J}_X$  acts trivially. Hence the action of  $\mathcal{J}_X$  descends to an action of the quotient  $\mathcal{J}_X/\mathcal{O}_X$ . This action induces the action of the abelian graded Lie algebra  $\Omega_X^\bullet \otimes \mathcal{J}_X/\mathcal{O}_X$  by derivations on the graded Lie algebra  $\Omega_X^\bullet \otimes \overline{\mathcal{C}}^\bullet(\mathcal{J}_X)[1]$ . Moreover, the subsheaf  $(\Omega_X^\bullet \otimes \mathcal{J}_X/\mathcal{O}_X)^{cl} := \ker(\nabla^{can})$  acts by derivation which commute with the differential  $\delta + \nabla^{can}$ . For  $\omega \in \Gamma(X; (\Omega_X^2 \otimes \mathcal{J}_X/\mathcal{O}_X)^{cl})$  we denote by  $\mathrm{DR}(\overline{\mathcal{C}}^\bullet(\mathcal{J}_X)[1])_\omega$  the sheaf of DGLA with the underlying graded Lie algebra  $\Omega_X^\bullet \otimes \overline{\mathcal{C}}^\bullet(\mathcal{J}_X)[1]$  and the differential  $\delta + \nabla^{can} + \iota_\omega$ . Let

$$\mathfrak{g}_{\mathrm{DR}}(\mathcal{J}_X)_\omega = \Gamma(X; \mathrm{DR}(\overline{\mathcal{C}}^\bullet(\mathcal{J}_X)[1])_\omega),$$

be the corresponding DGLA of global sections.

Suppose now that  $\mathcal{U}$  is a cover of  $X$ ; let  $\epsilon : N\mathcal{U} \rightarrow X$  denote the canonical map. For  $\lambda : [n] \rightarrow \Delta$  let

$$\mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega^\lambda = \Gamma(N_{\lambda(n)}\mathcal{U}; \epsilon^* \mathrm{DR}(\overline{\mathcal{C}}^\bullet(\mathcal{J}_X)[1])_\omega)$$

For  $\mu : [m] \rightarrow \Delta$  and a morphism  $\phi : [m] \rightarrow [n]$  in  $\Delta$  such that  $\mu = \lambda \circ \phi$  the map  $\mu(m) \rightarrow \lambda(n)$  induces the map

$$\phi_* : \mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega^\mu \rightarrow \mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega^\lambda$$

For  $n = 0, 1, \dots$  let  $\mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega^n = \prod_{[m] \xrightarrow{\lambda} \Delta} \mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega^\lambda$ . The assignment  $[n] \mapsto \mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega^n$  extends to a cosimplicial DGLA  $\mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega$ . We will also denote this DGLA by  $\mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega(\mathcal{U})$  if we need to explicitly indicate the cover.

**Lemma 7.8.** *The cosimplicial DGLA  $\mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega$  is acyclic, i.e. satisfies the condition (3.8).*

*Proof.* Consider the cosimplicial vector space  $V^\bullet$  with  $V^n = \Gamma(N_n\mathcal{U}; \epsilon^* \mathrm{DR}(\overline{\mathcal{C}}^\bullet(\mathcal{J}_X)[1])_\omega)$  and the cosimplicial structure induced by the simplicial structure of  $N\mathcal{U}$ . The cohomology of the complex  $(V^\bullet, \partial)$  is the Čech cohomology of  $\mathcal{U}$  with the coefficients in the soft sheaf of vector spaces  $\Omega^\bullet \otimes \overline{\mathcal{C}}^\bullet(\mathcal{J}_X)[1]$  and, therefore, vanishes in the positive degrees.  $\mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega$  as a cosimplicial vector space can be identified with  $\widehat{V}^\bullet$  in the notations of Lemma 2.1. Hence the result follows from the Lemma 2.1.  $\square$

We leave the proof of the following lemma to the reader.

**Lemma 7.9.** *The map  $\epsilon^* : \mathfrak{g}_{\text{DR}}(\mathcal{J}_X)_\omega \rightarrow \mathfrak{G}_{\text{DR}}(\mathcal{J})_\omega^0$  induces an isomorphism of DGLA*

$$\mathfrak{g}_{\text{DR}}(\mathcal{J}_X)_\omega \cong \ker(\mathfrak{G}_{\text{DR}}(\mathcal{J})_\omega^0 \rightrightarrows \mathfrak{G}_{\text{DR}}(\mathcal{J})_\omega^1)$$

where the two maps on the right are  $(\partial_0^0)_*$  and  $(\partial_1^0)_*$ .

Two previous lemmas together with the Corollary 3.13 imply the following:

**Proposition 7.10.** *Let  $\omega \in \Gamma(X; (\Omega_X^2 \otimes \mathcal{J}_X/\mathcal{O}_X)^{cl})$  and let  $\mathfrak{m}$  be a commutative nilpotent ring. Then the map  $\epsilon^*$  induces equivalence of groupoids:*

$$\text{MC}^2(\mathfrak{g}_{\text{DR}}(\mathcal{J}_X)_\omega \otimes \mathfrak{m}) \cong \text{Stack}(\mathfrak{G}_{\text{DR}}(\mathcal{J})_\omega \otimes \mathfrak{m})$$

For  $\beta \in \Gamma(X; \Omega^1 \otimes \mathcal{J}_X/\mathcal{O}_X)$  there is a canonical isomorphism of cosimplicial DGLA  $\exp(\iota_\beta) : \mathfrak{G}_{\text{DR}}(\mathcal{J})_{\omega+\nabla^{can}\beta} \rightarrow \mathfrak{G}_{\text{DR}}(\mathcal{J})_\omega$ . Therefore,  $\mathfrak{G}_{\text{DR}}(\mathcal{J})_\omega$  depends only on the class of  $\omega$  in  $H_{\text{DR}}^2(\mathcal{J}_X/\mathcal{O}_X)$ .

The rest of this section is devoted to the proof of the following theorem.

**Theorem 7.11.** *Suppose that  $(\mathcal{U}, \underline{\mathcal{A}})$  is a descent datum representing a twisted form  $\mathcal{S}$  of  $\mathcal{O}_X$ . There exists a quasi-isomorphism of cosimplicial DGLA  $\mathfrak{G}_{\text{DR}}(\mathcal{J})_{[\mathcal{S}]} \rightarrow \mathfrak{G}_{\text{DR}}(\mathcal{J}(\underline{\mathcal{A}}))$ .*

**7.4. Quasiisomorphism.** Suppose that  $(\mathcal{U}, \underline{\mathcal{A}})$  is a descent datum for a twisted form of  $\mathcal{O}_X$ . Thus,  $\mathcal{A}$  is identified with  $\mathcal{O}_{N_0\mathcal{U}}$  and  $\mathcal{A}_{01}$  is a line bundle on  $N_1\mathcal{U}$ .

**7.4.1. Multiplicative connections.** Let  $\mathcal{C}^\mu(\mathcal{A}_{01})$  denote the set of connections  $\nabla$  on  $\mathcal{A}_{01}$  which satisfy

1.  $\text{Ad}_{\mathcal{A}_{012}}((\text{pr}_{02}^2)^*\nabla) = (\text{pr}_{01}^2)^*\nabla \otimes \text{Id} + \text{Id} \otimes (\text{pr}_{12}^2)^*\nabla$
2.  $(\text{pr}_{00}^0)^*\nabla$  is the canonical flat connection on  $\mathcal{O}_{N_0\mathcal{U}}$ .

Let  $\text{Isom}_0^\mu(\mathcal{A}_{01} \otimes \mathcal{J}_{N_1\mathcal{U}}, \mathcal{J}(\mathcal{A}_{01}))$  denote the subset of  $\text{Isom}_0(\mathcal{A}_{01} \otimes \mathcal{J}_{N_1\mathcal{U}}, \mathcal{J}(\mathcal{A}_{01}))$  which consists of  $\sigma$  which satisfy

1.  $\text{Ad}_{\mathcal{A}_{012}}((\text{pr}_{02}^2)^*\sigma) = (\text{pr}_{01}^2)^*\sigma \otimes (\text{pr}_{12}^2)^*\sigma$
2.  $(\text{pr}_{00}^0)^*\sigma = \text{Id}$

Note that the vector space  $\overline{\mathcal{Z}}^1(\mathcal{U}; \Omega^1)$  of cocycles in the *normalized* Čech complex of the cover  $\mathcal{U}$  with coefficients in the sheaf of 1-forms  $\Omega^1$  acts on the set  $\mathcal{C}^\mu(\mathcal{A}_{01})$ , with the action given by

$$\alpha \cdot \nabla = \nabla + \alpha \tag{7.3}$$

Here  $\nabla \in \mathcal{C}^\mu(\mathcal{A}_{01})$ ,  $\alpha \in \overline{\mathcal{Z}}^1(\mathcal{U}; \Omega^1) \subset \Omega^1(N_1\mathcal{U})$ .

Similarly, the vector space  $\overline{\mathcal{Z}}^1(\mathcal{U}; \mathcal{J}_0)$  acts on the set  $\text{Isom}_0^\mu(\mathcal{A}_{01} \otimes \mathcal{J}_{N_1\mathcal{U}}, \mathcal{J}(\mathcal{A}_{01}))$ , with the action given as in Corollary 7.4.

Note that since the sheaves involved are soft, cocycles coincide with coboundaries:  $\overline{\mathcal{Z}}^1(\mathcal{U}; \Omega^1) = \overline{\mathcal{B}}^1(\mathcal{U}; \Omega^1)$ ,  $\overline{\mathcal{Z}}^1(\mathcal{U}; \mathcal{J}_0) = \overline{\mathcal{B}}^1(\mathcal{U}; \mathcal{J}_0)$ .

**Proposition 7.12.** *The set  $\mathcal{C}^\mu(\mathcal{A}_{01})$  (respectively,  $\text{Isom}_0^\mu(\mathcal{A}_{01} \otimes \mathcal{J}_{N_1\mathcal{U}}, \mathcal{J}(\mathcal{A}_{01}))$ ) is an affine space with the underlying vector space being  $\overline{\mathcal{Z}}^1(\mathcal{U}; \Omega^1)$  (respectively,  $\overline{\mathcal{Z}}^1(\mathcal{U}; \mathcal{J}_0)$ ).*

*Proof.* Proofs of the both statements are completely analogous. Therefore we explain the proof of the statement concerning  $\text{Isom}_0^\mu(\mathcal{A}_{01} \otimes \mathcal{J}_{N_1\mathcal{U}}, \mathcal{J}(\mathcal{A}_{01}))$  only.

We show first that  $\text{Isom}_0^\mu(\mathcal{A}_{01} \otimes \mathcal{J}_{N_1\mathcal{U}}, \mathcal{J}(\mathcal{A}_{01}))$  is nonempty. Choose an arbitrary  $\sigma \in \text{Isom}_0(\mathcal{A}_{01} \otimes \mathcal{J}_{N_1\mathcal{U}}, \mathcal{J}(\mathcal{A}_{ij}))$  such that  $(\text{pr}_{00}^0)^*\sigma = \text{Id}$ . Then, by Corollary 7.4, there exists  $c \in \Gamma(N_2\mathcal{U}; \mathcal{J}_0)$  such that  $c \cdot (\text{Ad } \mathcal{A}_{012}((d^1)^*\sigma_{02})) = (d^0)^*\sigma_{01} \otimes (d^2)^*\sigma_{12}$ . It is easy to see that  $c \in \overline{\mathcal{Z}}^2(\mathcal{U}; \exp \mathcal{J}_0)$ . Since the sheaf  $\exp \mathcal{J}_0$  is soft, corresponding Čech cohomology is trivial. Therefore, there exists  $\phi \in \overline{\mathcal{C}}^1(\mathcal{U}; \mathcal{J}_0)$  such that  $c = \check{\partial}\phi$ . Then,  $\phi \cdot \sigma \in \text{Isom}_0^\mu(\mathcal{A}_{01} \otimes \mathcal{J}_{N_1\mathcal{U}}, \mathcal{J}(\mathcal{A}_{01}))$ .

Suppose that  $\sigma, \sigma' \in \text{Isom}_0^\mu(\mathcal{A}_{01} \otimes \mathcal{J}_{N_1\mathcal{U}}, \mathcal{J}(\mathcal{A}_{01}))$ . By the Corollary 7.4  $\sigma = \phi \cdot \sigma'$  for some uniquely defined  $\phi \in \Gamma(N_2\mathcal{U}; \mathcal{J}_0)$ . It is easy to see that  $\phi \in \overline{\mathcal{Z}}^1(\mathcal{U}; \mathcal{J}_0)$ .  $\square$

We assume from now on that we have chosen  $\sigma \in \text{Isom}_0^\mu(\mathcal{A}_{01} \otimes \mathcal{J}_{N_1\mathcal{U}}, \mathcal{J}(\mathcal{A}_{01}))$ ,  $\nabla \in \mathcal{C}^\mu(\mathcal{A}_{01})$ . Such a choice defines  $\sigma_{ij}^p \in \text{Isom}_0(\mathcal{A}_{ij} \otimes \mathcal{J}_{N_p\mathcal{U}}, \mathcal{J}(\mathcal{A}_{ij}))$  for every  $p$  and  $0 \leq i, j \leq p$  by  $\sigma_{ij}^p = (\text{pr}_{ij}^p)^*\sigma$ . This collection of  $\sigma_{ij}^p$  induces for every  $p$  algebra isomorphism  $\sigma^p : \text{Mat}(\underline{\mathcal{A}})^p \otimes \mathcal{J}_{N_p\mathcal{U}} \rightarrow \text{Mat}(\mathcal{J}(\underline{\mathcal{A}}))^p$ . The following compatibility holds for these isomorphisms. Let  $f : [p] \rightarrow [q]$  be a morphism in  $\Delta$ . Then the following diagram commutes:

$$\begin{array}{ccc} f_*(\text{Mat}(\underline{\mathcal{A}})^p \otimes \mathcal{J}_{N_p\mathcal{U}}) & \xrightarrow{f_*} & f^\# \text{Mat}(\underline{\mathcal{A}} \otimes \mathcal{J}_{N_q\mathcal{U}})^q \\ f_*(\sigma^p) \downarrow & & \downarrow f^\#(\sigma^p) \\ f_* \text{Mat}(\mathcal{J}(\underline{\mathcal{A}}))^p & \xrightarrow{f_*} & f^\# \text{Mat}(\mathcal{J}(\underline{\mathcal{A}}))^q \end{array} \quad (7.4)$$

Similarly define the connections  $\nabla_{ij}^p = (\text{pr}_{ij}^p)^*\nabla$ . For  $p = 0, 1, \dots$  set  $\nabla^p = \bigoplus_{i,j=0}^p \nabla_{ij}$ ; the connections  $\nabla^p$  on  $\text{Mat}(\underline{\mathcal{A}})^p$  satisfy

$$f_* \nabla^p = (\text{Ad } f_*)(f^\# \nabla^q). \quad (7.5)$$

Note that  $F(\sigma, \nabla) \in \Gamma(N_1\mathcal{U}; \Omega_{N_1\mathcal{U}}^1 \otimes \mathcal{J}_{N_1\mathcal{U}})$  is a cocycle of degree one in  $\check{C}^\bullet(\mathcal{U}; \Omega_X^1 \otimes \mathcal{J}_X)$ . Vanishing of the corresponding Čech cohomology implies that there exists  $F^0 \in \Gamma(N_0\mathcal{U}; \Omega_{N_0\mathcal{U}}^1 \otimes \mathcal{J}_{N_0\mathcal{U}})$  such that

$$(d^1)^*F^0 - (d^0)^*F^0 = F(\sigma, \nabla). \quad (7.6)$$

For  $p = 0, 1, \dots$ ,  $0 \leq i \leq p$ , let  $F_{ii}^p = (\text{pr}_i^p)^*F^0$ ; put  $F_{ij}^p = 0$  for  $i \neq j$ . Let  $F^p \in \Gamma(N_p\mathcal{U}; \Omega_{N_p\mathcal{U}}^1 \otimes \text{Mat}(\underline{\mathcal{A}})^p) \otimes \mathcal{J}_{N_p\mathcal{U}}$  denote the diagonal matrix with components  $F_{ij}^p$ . For  $f : [p] \rightarrow [q]$  we have

$$f_* F^p = f^\# F^q. \quad (7.7)$$

Then, we obtain the following equality of connections on  $\text{Mat}(\underline{\mathcal{A}})^p \otimes \mathcal{J}_{N_p\mathcal{U}}$ :

$$(\sigma^p)^{-1} \circ \nabla^{can} \circ \sigma^p = \nabla^p \otimes \text{Id} + \text{Id} \otimes \nabla^{can} + \text{ad } F^p \quad (7.8)$$

The matrices  $F^p$  also have the following property. Let  $\nabla^{can} F^p$  be the diagonal matrix with the entries  $(\nabla^{can} F^p)_{ii} = \nabla^{can} F_{ii}^p$ . Denote by  $\overline{\nabla^{can} F^p}$  the image of  $\nabla^{can} F^p$  under the natural map  $\Gamma(N_p \mathcal{U}; \Omega_{N_p \mathcal{U}}^2 \otimes \text{Mat}(\underline{\mathcal{A}})^p \otimes \mathcal{J}_{N_p \mathcal{U}}) \rightarrow \Gamma(N_p \mathcal{U}; \Omega_{N_p \mathcal{U}}^2 \otimes \text{Mat}(\underline{\mathcal{A}})^p \otimes (\mathcal{J}_{N_p \mathcal{U}} / \mathcal{O}_{N_p \mathcal{U}}))$ . Recall the canonical map  $\epsilon_p : N_p \mathcal{U} \rightarrow X$ . Then, we have the following:

**Lemma 7.13.** *There exists a unique  $\omega \in \Gamma(X; (\Omega_X^2 \otimes \mathcal{J}_X / \mathcal{O}_X)^{cl})$  such that*

$$\overline{\nabla^{can} F^p} = -\epsilon_p^* \omega \otimes \text{Id}_p,$$

where  $\text{Id}_p$  denotes the  $(p+1) \times (p+1)$  identity matrix.

*Proof.* Using the definition of  $F^0$  and formula (7.2) we obtain:  $(d^1)^* \nabla^{can} F^0 - (d^0)^* \nabla^{can} F^0 = \nabla^{can} F(\sigma, \nabla) \in \Omega^2(N_1 \mathcal{U})$ . Therefore  $(d^1)^* \overline{\nabla^{can} F^0} - (d^0)^* \overline{\nabla^{can} F^0} = 0$ , and there exists a unique  $\omega \in \Gamma(X; \Omega_X^2 \otimes (\mathcal{J}_X / \mathcal{O}_X))$  such that  $\epsilon_0^* \omega = \overline{\nabla^{can} F^0}$ . Since  $(\nabla^{can})^2 = 0$  it follows that  $\nabla^{can} \omega = 0$ . For any  $p$  we have:  $(\overline{\nabla^{can} F^p})_{ii} = \text{pr}_i^* \overline{\nabla^{can} F^0} = \epsilon_p^* \omega$ , and the assertion of the Lemma follows.  $\square$

**Lemma 7.14.** *The class of  $\omega$  in  $H^2(\Gamma(X; \Omega_X^2 \otimes \mathcal{J}_X / \mathcal{O}_X), \nabla^{can})$  does not depend on the choices made in its construction.*

*Proof.* The construction of  $\omega$  depends on the choice of  $\sigma \in \text{Isom}_0^\mu(\mathcal{A}_{01} \otimes \mathcal{J}_{N_1 \mathcal{U}}, \mathcal{J}(\mathcal{A}_{01}))$ ,  $\nabla \in \mathcal{C}^\mu(\mathcal{A}_{01})$  and  $F^0$  satisfying the equation (7.6). Assume that we make different choices:  $\sigma' = (\partial \phi) \cdot \sigma$ ,  $\nabla' = (\partial \alpha) \cdot \nabla$  and  $(F^0)'$  satisfying  $\partial(F^0)' = F(\sigma', \nabla')$ . Here,  $\phi \in \overline{\mathcal{C}}^0(\mathcal{U}; \mathcal{J}_0)$  and  $\alpha \in \overline{\mathcal{C}}^0(\mathcal{U}; \Omega^1)$ . We have:  $F(\sigma', \nabla') = F(\sigma, \nabla) - \partial \alpha + \partial \nabla^{can} \phi$ . It follows that  $\partial((F^0)' - F^0 - \nabla^{can} \phi + \alpha) = 0$ . Therefore  $(F^0)' - F^0 - \nabla^{can} \phi + \alpha = -\epsilon_0^* \beta$  for some  $\beta \in \Gamma(X; \Omega_X^1 \otimes \mathcal{J}_X)$ . Hence if  $\omega'$  is constructed using  $\sigma'$ ,  $\nabla'$ ,  $(F^0)'$  then  $\omega' - \omega = \nabla^{can} \beta$  where  $\beta$  is the image of  $\beta$  under the natural projection  $\Gamma(X; \Omega_X^1 \otimes \mathcal{J}_X) \rightarrow \Gamma(X; \Omega_X^1 \otimes (\mathcal{J}_X / \mathcal{O}_X))$ .  $\square$

Let  $\rho : \mathcal{V} \rightarrow \mathcal{U}$  be a refinement of the cover  $\mathcal{U}$ , and  $(\mathcal{V}, \underline{\mathcal{A}}^\rho)$  the corresponding descent datum. Choice of  $\sigma, \nabla, F^0$  on  $\mathcal{U}$  induces the corresponding choice  $(N\rho)^* \sigma, (N\rho)^* \nabla, (N\rho)^*(F^0)$  on  $\mathcal{V}$ . Let  $\omega^\rho$  denotes the form constructed as in Lemma 7.13 using  $(N\rho)^* \sigma, (N\rho)^* \nabla, (N\rho)^* F^0$ . Then,

$$\omega^\rho = (N\rho)^* \omega.$$

The following result now follows easily and we leave the details to the reader.

**Proposition 7.15.** *The class of  $\omega$  in  $H^2(\Gamma(X; \Omega_X^2 \otimes \mathcal{J}_X / \mathcal{O}_X), \nabla^{can})$  coincides with the image  $[\mathcal{S}]$  of the class of the gerbe.*

**7.5. Construction of the quasiisomorphism.** For  $\lambda : [n] \rightarrow \Delta$  let

$$\mathfrak{H}^\lambda := \Gamma(N_{\lambda(n)} \mathcal{U}; \Omega_{N_{\lambda(n)} \mathcal{U}}^2 \otimes \lambda(0n)_* \mathcal{C}^\bullet(\text{Mat}(\underline{\mathcal{A}})^{\lambda(0)} \otimes \mathcal{J}_{N_{\lambda(0)} \mathcal{U}})^{loc}[1])$$

considered as a graded Lie algebra. For  $\phi : [m] \rightarrow [n]$ ,  $\mu = \lambda \circ \phi$  there is a morphism of graded Lie algebras  $\phi_* : \mathfrak{H}^\mu \rightarrow \mathfrak{H}^\lambda$ . For  $n = 0, 1, \dots$  let  $\mathfrak{H}^n := \prod_{[n] \xrightarrow{\lambda} \Delta} \mathfrak{H}^\lambda$ . The assignment  $\Delta \ni [n] \mapsto \mathfrak{H}^n$ ,  $\phi \mapsto \phi_*$  defines a cosimplicial graded Lie algebra  $\mathfrak{H}$ .

For each  $\lambda : [n] \rightarrow \Delta$  the map

$$\sigma_*^\lambda := \text{Id} \otimes \lambda(0n)_*(\sigma^{\lambda(0)}) : \mathfrak{H}^\lambda \rightarrow \mathfrak{G}_{\text{DR}}(\mathcal{J}(\underline{\mathcal{A}}))^\lambda$$

is an isomorphism of graded Lie algebras. It follows from (7.4) that the maps  $\sigma_*^\lambda$  yield an isomorphism of cosimplicial graded Lie algebras

$$\sigma_* : \mathfrak{H} \rightarrow \mathfrak{G}_{\text{DR}}(\mathcal{J}(\underline{\mathcal{A}})).$$

Moreover, the equation (7.8) shows that if we equip  $\mathfrak{H}$  with the differential given on  $\Gamma(N_{\lambda(n)}\mathcal{U}; \Omega_{N_{\lambda(n)}}^\bullet \mathcal{U} \otimes \lambda(0n)_* C^\bullet(\text{Mat}(\underline{\mathcal{A}})^{\lambda(0)} \otimes \mathcal{J}_{N_{\lambda(0)}\mathcal{U}})^{\text{loc}}[1])$  by

$$\begin{aligned} & \delta + \lambda(0n)_*(\nabla^{\lambda(0)}) \otimes \text{Id} + \text{Id} \otimes \nabla^{\text{can}} + \text{ad } \lambda(0n)_*(F^{\lambda(0)}) \\ &= \delta + \lambda(0n)^\sharp(\nabla^{\lambda(n)}) \otimes \text{Id} + \text{Id} \otimes \nabla^{\text{can}} + \text{ad } \lambda(0n)^\sharp(F^{\lambda(n)}) \end{aligned} \quad (7.9)$$

then  $\sigma_*$  becomes an isomorphism of DGLA. Consider now an automorphism  $\exp \iota_F$  of the cosimplicial graded Lie algebra  $\mathfrak{H}$  given on  $\mathfrak{H}^\lambda$  by  $\exp \iota_{\lambda(0n)_* F^p}$ . Note the fact that this morphism preserves the cosimplicial structure follows from the relation (7.7).

The following result is proved by the direct calculation; see [4], Lemma 16.

**Lemma 7.16.**

$$\begin{aligned} \exp(\iota_{F^p}) \circ (\delta + \nabla^p \otimes \text{Id} + \text{Id} \otimes \nabla^{\text{can}} + \text{ad } F^p) \circ \exp(-\iota_{F^p}) = \\ \delta + \nabla^p \otimes \text{Id} + \text{Id} \otimes \nabla^{\text{can}} - \iota_{\nabla F^p}. \end{aligned} \quad (7.10)$$

Therefore the morphism

$$\exp \iota_F : \mathfrak{H} \rightarrow \mathfrak{H} \quad (7.11)$$

conjugates the differential given by the formula (7.9) into the differential which on  $\mathfrak{H}^\lambda$  is given by

$$\delta + \lambda(0n)_*(\nabla^{\lambda(0)}) \otimes \text{Id} + \text{Id} \otimes \nabla^{\text{can}} - \iota_{\lambda(0n)_*(\nabla^{\text{can}} F^{\lambda(0)})} \quad (7.12)$$

Consider the map

$$\text{cotr} : \overline{C}^\bullet(\mathcal{J}_{N_p\mathcal{U}})[1] \rightarrow C^\bullet(\text{Mat}(\underline{\mathcal{A}})^p \otimes \mathcal{J}_{N_p\mathcal{U}})[1] \quad (7.13)$$

defined as follows:

$$\text{cotr}(D)(a_1 \otimes j_1, \dots, a_n \otimes j_n) = a_0 \dots a_n D(j_1, \dots, j_n). \quad (7.14)$$

The map  $\text{cotr}$  is a quasiisomorphism of DGLAs (cf. [15], section 1.5.6; see also [4] Proposition 14).

**Lemma 7.17.** *For every  $p$  the map*

$$\text{Id} \otimes \text{cotr} : \Omega_{N_p\mathcal{U}}^\bullet \otimes \overline{C}^\bullet(\mathcal{J}_{N_p\mathcal{U}})[1] \rightarrow \Omega_{N_p\mathcal{U}}^\bullet \otimes C^\bullet(\text{Mat}(\underline{\mathcal{A}})^p \otimes \mathcal{J}_{N_p\mathcal{U}})[1]. \quad (7.15)$$

*is a quasiisomorphism of DGLA, where the source and the target are equipped with the differentials  $\delta + \nabla^{\text{can}} + \iota_{\epsilon^*\omega}$  and  $\delta + \nabla^p \otimes \text{Id} + \text{Id} \otimes \nabla^{\text{can}} - \iota_{\nabla F^p}$  respectively.*

*Proof.* It is easy to see that  $\mathrm{Id} \otimes \mathrm{cotr}$  is a morphism of graded Lie algebras, which satisfies  $(\nabla^p \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{can}) \circ (\mathrm{Id} \otimes \mathrm{cotr}) = (\mathrm{Id} \otimes \mathrm{cotr}) \circ \nabla^{can}$  and  $\delta \circ (\mathrm{Id} \otimes \mathrm{cotr}) = (\mathrm{Id} \otimes \mathrm{cotr}) \circ \delta$ . Since the domain of  $(\mathrm{Id} \otimes \mathrm{cotr})$  is the normalized complex, in view of the Lemma 7.13 we also have  $\iota_{\nabla F^p} \circ (\mathrm{Id} \otimes \mathrm{cotr}) = -(\mathrm{Id} \otimes \mathrm{cotr}) \circ \iota_{\epsilon^* \omega}$ . This implies that  $(\mathrm{Id} \otimes \mathrm{cotr})$  is a morphism of DGLA.

To see that this map is a quasiisomorphism, introduce filtration on  $\Omega_{N_p \mathcal{U}}^\bullet$  by  $F_i \Omega_{N_p \mathcal{U}}^\bullet = \Omega_{N_p \mathcal{U}}^{\geq -i}$  and consider the complexes  $\overline{C}^\bullet(\mathcal{J}_{N_p \mathcal{U}})[1]$  and  $C^\bullet(\mathrm{Mat}(\underline{\mathcal{A}}))^p \otimes \mathcal{J}_{N_p \mathcal{U}}[1]$  equipped with the trivial filtration. The map (7.15) is a morphism of filtered complexes with respect to the induced filtrations on the source and the target. The differentials induced on the associated graded complexes are  $\delta$  (or, more precisely,  $\mathrm{Id} \otimes \delta$ ) and the induced map of the associated graded objects is  $\mathrm{Id} \otimes \mathrm{cotr}$  which is a quasi-isomorphism. Therefore, the map (7.15) is a quasiisomorphism as claimed.  $\square$

The map (7.15) therefore induces for every  $\lambda : [n] \rightarrow \Delta$  a morphism  $\mathrm{Id} \otimes \mathrm{cotr} : \mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega^\lambda \rightarrow \mathfrak{H}^\lambda$ . These morphisms are clearly compatible with the cosimplicial structure and hence induce a quasiisomorphism of cosimplicial DGLAs

$$\mathrm{Id} \otimes \mathrm{cotr} : \mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega \rightarrow \mathfrak{H}$$

where the differential in the right hand side is given by (7.12).

We summarize our consideration in the following:

**Theorem 7.18.** *For a any choice of  $\sigma \in \mathrm{Isom}_0^\mu(\mathcal{A}_{01} \otimes \mathcal{J}_{N_1 \mathcal{U}}, \mathcal{J}(\mathcal{A}_{01}))$ ,  $\nabla \in C^\mu(\mathcal{A}_{01})$  and  $F^0$  as in (7.6) the composition  $\Phi_{\sigma, \nabla, F} := \sigma_* \circ \exp(\iota_F) \circ (\mathrm{Id} \otimes \mathrm{cotr})$*

$$\Phi_{\sigma, \nabla, F} : \mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega \rightarrow \mathfrak{G}_{\mathrm{DR}}(\mathcal{J}(\underline{\mathcal{A}})) \quad (7.16)$$

*is a quasiisomorphism of cosimplicial DGLAs.*

Let  $\rho : \mathcal{V} \rightarrow \mathcal{U}$  be a refinement of the cover  $\mathcal{U}$  and let  $(\mathcal{V}, \underline{\mathcal{A}}^\rho)$  be the induced descent datum. We will denote the corresponding cosimplicial DGLAs by  $\mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega(\mathcal{U})$  and  $\mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega(\mathcal{V})$  respectively. Then the map  $N\rho : N\mathcal{V} \rightarrow N\mathcal{U}$  induces a morphism of cosimplicial DGLAs

$$(N\rho)^* : \mathfrak{G}_{\mathrm{DR}}(\mathcal{J}(\underline{\mathcal{A}})) \rightarrow \mathfrak{G}_{\mathrm{DR}}(\mathcal{J}(\underline{\mathcal{A}}^\rho)) \quad (7.17)$$

and

$$(N\rho)^* : \mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega(\mathcal{U}) \rightarrow \mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega(\mathcal{V}). \quad (7.18)$$

Notice also that the choice that the choice of data  $\sigma, \nabla, F^0$  on  $N\mathcal{U}$  induces the corresponding data  $(N\rho)^* \sigma, (N\rho)^* \nabla, (N\rho)^* F^0$  on  $N\mathcal{V}$ . This data allows one to construct using the equation (7.16) the map

$$\Phi_{(N\rho)^* \sigma, (N\rho)^* \nabla, (N\rho)^* F} : \mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega \rightarrow \mathfrak{G}_{\mathrm{DR}}(\mathcal{J}(\underline{\mathcal{A}}^\rho)) \quad (7.19)$$

The following Proposition is an easy consequence of the description of the map  $\Phi$ , and we leave the proof to the reader.

**Proposition 7.19.** *The following diagram commutes:*

$$\begin{array}{ccc}
\mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_{\omega}(\mathcal{U}) & \xrightarrow{(N\rho)^*} & \mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_{\omega}(\mathcal{V}) \\
\Phi_{\sigma, \nabla, F} \downarrow & & \downarrow \Phi_{(N\rho)^* \sigma, (N\rho)^* \nabla, (N\rho)^* F} \\
\mathfrak{G}_{\mathrm{DR}}(\mathcal{J}(\underline{\mathcal{A}})) & \xrightarrow{(N\rho)^*} & \mathfrak{G}_{\mathrm{DR}}(\mathcal{J}(\underline{\mathcal{A}}^{\rho}))
\end{array} \tag{7.20}$$

## 8. Proof of the main theorem

In this section we prove the main result of this paper. Recall the statement of the theorem from the introduction:

**Theorem 1.** *Suppose that  $X$  is a  $C^{\infty}$  manifold and  $\mathcal{S}$  is an algebroid stack on  $X$  which is a twisted form of  $\mathcal{O}_X$ . Then, there is an equivalence of 2-groupoid valued functors of commutative Artin  $\mathbb{C}$ -algebras*

$$\mathrm{Def}_X(\mathcal{S}) \cong \mathrm{MC}^2(\mathfrak{g}_{\mathrm{DR}}(\mathcal{J}_X)_{[\mathcal{S}]}) .$$

*Proof.* Suppose  $\mathcal{U}$  is a cover of  $X$  such that  $\epsilon_0^* \mathcal{S}(N_0 \mathcal{U})$  is nonempty. There is a descent datum  $(\mathcal{U}, \underline{\mathcal{A}}) \in \mathrm{Desc}_{\mathbb{C}}(\mathcal{U})$  whose image under the functor  $\mathrm{Desc}_{\mathbb{C}}(\mathcal{U}) \rightarrow \mathrm{AlgStack}_{\mathbb{C}}(X)$  is equivalent to  $\mathcal{S}$ .

The proof proceeds as follows. Recall the 2-groupoids  $\mathrm{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R)$  and  $\mathrm{Def}(\mathcal{U}, \underline{\mathcal{A}})(R)$  of deformations of and star-products on the descent datum  $(\mathcal{U}, \underline{\mathcal{A}})$  defined in the Section 6.2. Note that the composition  $\mathrm{Desc}_R(\mathcal{U}) \rightarrow \mathrm{Triv}_R(X) \rightarrow \mathrm{AlgStack}_R(X)$  induces functors  $\mathrm{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R) \rightarrow \mathrm{Def}(\mathcal{S})(R)$  and  $\mathrm{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) \rightarrow \mathrm{Def}(\mathcal{S})(R)$ , the second one being the composition of the first one with the equivalence  $\mathrm{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) \rightarrow \mathrm{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R)$ . We are going to show that for a commutative Artin  $\mathbb{C}$ -algebra  $R$  there are equivalences

1.  $\mathrm{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) \cong \mathrm{MC}^2(\mathfrak{g}_{\mathrm{DR}}(\mathcal{J}_X)_{[\mathcal{S}]})(R)$  and
2. the functor  $\mathrm{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) \cong \mathrm{Def}(\mathcal{S})(R)$  induced by the functor  $\mathrm{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) \rightarrow \mathrm{Def}(\mathcal{S})(R)$  above.

Let  $\mathcal{J}(\underline{\mathcal{A}}) = (\mathcal{J}(\mathcal{A}_{01}), j^{\infty}(\mathcal{A}_{012}))$ . Then,  $(\mathcal{U}, \mathcal{J}(\underline{\mathcal{A}}))$  is a descent datum for a twisted form of  $\mathcal{J}_X$ . Let  $R$  be a commutative Artin  $\mathbb{C}$ -algebra with maximal ideal  $\mathfrak{m}_R$ .

Then the first statement follows from the equivalences

$$\mathrm{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) \cong \mathrm{Stack}(\mathfrak{G}(\underline{\mathcal{A}}) \otimes \mathfrak{m}_R) \tag{8.1}$$

$$\cong \mathrm{Stack}(\mathfrak{G}_{\mathrm{DR}}(\mathcal{J}(\underline{\mathcal{A}})) \otimes \mathfrak{m}_R) \tag{8.2}$$

$$\cong \mathrm{Stack}(\mathfrak{G}_{\mathrm{DR}}(\mathcal{J}_X)_{[\mathcal{S}]} \otimes \mathfrak{m}_R) \tag{8.3}$$

$$\cong \mathrm{MC}^2(\mathfrak{g}_{\mathrm{DR}}(\mathcal{J}_X)_{[\mathcal{S}]} \otimes \mathfrak{m}_R) \tag{8.4}$$

Here the equivalence (8.1) is the subject of the Proposition 6.4. The inclusion of horizontal sections is a quasi-isomorphism and the induced map in (8.2) is an

equivalence by Theorem 3.6. In the Theorem 7.18 we have constructed a quasi-isomorphism  $\mathfrak{G}(\mathcal{J}_X)_{[\mathcal{S}]} \rightarrow \mathfrak{G}(\mathcal{J}(\underline{\mathcal{A}}))$ ; the induced map (8.3) is an equivalence by another application of Theorem 3.6. Finally, the equivalence (8.4) is shown in the Proposition 7.10

We now proceed with the proof of the second statement. We begin by considering the behavior of  $\text{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R)$  under the refinement. Consider a refinement  $\rho : \mathcal{V} \rightarrow \mathcal{U}$ . Recall that by the Proposition 7.10 the map  $\epsilon^*$  induces equivalences  $\text{MC}^2(\mathfrak{g}_{\text{DR}}(\mathcal{J}_X)_\omega \otimes \mathfrak{m}) \rightarrow \text{Stack}(\mathfrak{G}_{\text{DR}}(\mathcal{J})_\omega(\mathcal{U}) \otimes \mathfrak{m})$  and  $\text{MC}^2(\mathfrak{g}_{\text{DR}}(\mathcal{J}_X)_\omega \otimes \mathfrak{m}) \rightarrow \text{Stack}(\mathfrak{G}_{\text{DR}}(\mathcal{J})_\omega(\mathcal{V}) \otimes \mathfrak{m})$ . It is clear that the diagram

$$\begin{array}{ccc} & \text{MC}^2(\mathfrak{g}_{\text{DR}}(\mathcal{J}_X)_\omega \otimes \mathfrak{m}) & \\ & \swarrow \quad \searrow & \\ \text{Stack}(\mathfrak{G}_{\text{DR}}(\mathcal{J})_\omega(\mathcal{U}) \otimes \mathfrak{m}) & \xrightarrow{(N\rho)^*} & \text{Stack}(\mathfrak{G}_{\text{DR}}(\mathcal{J})_\omega(\mathcal{V}) \otimes \mathfrak{m}) \end{array}$$

commutes, and therefore  $(N\rho)^* : \text{Stack}(\mathfrak{G}_{\text{DR}}(\mathcal{J})_\omega(\mathcal{U}) \otimes \mathfrak{m}) \rightarrow \text{Stack}(\mathfrak{G}_{\text{DR}}(\mathcal{J})_\omega(\mathcal{V}) \otimes \mathfrak{m})$  is an equivalence. Then the Proposition 7.19 together with the Theorem 3.6 implies that  $(N\rho)^* : \text{Stack}(\mathfrak{G}_{\text{DR}}(\mathcal{J}(\underline{\mathcal{A}}))) \otimes \mathfrak{m} \rightarrow \text{Stack}(\mathfrak{G}_{\text{DR}}(\mathcal{J}(\underline{\mathcal{A}}^\rho))) \otimes \mathfrak{m}$  is an equivalence. It follows that the functor  $\rho^* : \text{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) \rightarrow \text{Def}(\mathcal{V}, \underline{\mathcal{A}}^\rho)(R)$  is an equivalence. Note also that the diagram

$$\begin{array}{ccc} \text{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) & \xrightarrow{\rho^*} & \text{Def}(\mathcal{V}, \underline{\mathcal{A}}^\rho)(R) \\ \downarrow & & \downarrow \\ \text{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R) & \xrightarrow{\rho^*} & \text{Def}'(\mathcal{V}, \underline{\mathcal{A}}^\rho)(R) \end{array}$$

is commutative with the top horizontal and both vertical maps being equivalences. Hence it follows that the bottom horizontal map is an equivalence.

Recall now that by the Proposition 6.3 the embedding  $\text{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) \rightarrow \text{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R)$  is an equivalence. Therefore it is sufficient to show that the functor  $\text{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R) \rightarrow \text{Def}(\mathcal{S})(R)$  is an equivalence. Suppose that  $\mathcal{C}$  is an  $R$ -deformation of  $\mathcal{S}$ . It follows from Lemma 6.2 that  $\mathcal{C}$  is an algebroid stack. Therefore, there exists a cover  $\mathcal{V}$  and an  $R$ -descent datum  $(\mathcal{V}, \underline{\mathcal{B}})$  whose image under the functor  $\text{Desc}_R(\mathcal{V}) \rightarrow \text{AlgStack}_R(X)$  is equivalent to  $\mathcal{C}$ . Replacing  $\mathcal{V}$  by a common refinement of  $\mathcal{U}$  and  $\mathcal{V}$  if necessary we may assume that there is a morphism of covers  $\rho : \mathcal{V} \rightarrow \mathcal{U}$ . Clearly,  $(\mathcal{V}, \underline{\mathcal{B}})$  is a deformation of  $(\mathcal{V}, \underline{\mathcal{A}}^\rho)$ . Since the functor  $\rho^* : \text{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R) \rightarrow \text{Def}'(\mathcal{V}, \underline{\mathcal{A}}^\rho)$  is an equivalence there exists a deformation  $(\mathcal{U}, \underline{\mathcal{B}}')$  such that  $\rho^*(\mathcal{U}, \underline{\mathcal{B}}')$  is isomorphic to  $(\mathcal{V}, \underline{\mathcal{B}})$ . Let  $\mathcal{C}'$  denote the image of  $(\mathcal{U}, \underline{\mathcal{B}}')$  under the functor  $\text{Desc}_R(\mathcal{V}) \rightarrow \text{AlgStack}_R(X)$ . Since the images of  $(\mathcal{U}, \underline{\mathcal{B}}')$  and  $\rho^*(\mathcal{U}, \underline{\mathcal{B}}')$  in  $\text{AlgStack}_R(X)$  are equivalent it follows that  $\mathcal{C}'$  is equivalent to  $\mathcal{C}$ . This shows that the functor  $\text{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R) \rightarrow \text{Def}(\mathcal{C})(R)$  is essentially surjective.

Suppose now that  $(\mathcal{U}, \underline{\mathcal{B}}^{(i)})$ ,  $i = 1, 2$ , are  $R$ -deformations of  $(\mathcal{U}, \underline{\mathcal{A}})$ . Let  $\mathcal{C}^{(i)}$  denote the image of  $(\mathcal{U}, \underline{\mathcal{B}}^{(i)})$  in  $\text{Def}(\mathcal{S})(R)$ . Suppose that  $F : \mathcal{C}^{(1)} \rightarrow \mathcal{C}^{(2)}$  is a 1-morphism.

Let  $L$  denote the image of the canonical trivialization under the composition

$$\epsilon_0^*(F) : \widetilde{\mathcal{B}^{(1)+}} \cong \epsilon_0^* \mathcal{C}^{(1)} \xrightarrow{F} \epsilon_0^* \mathcal{C}^{(1)} \cong \widetilde{\mathcal{B}^{(2)+}} .$$

Thus,  $L$  is a  $\mathcal{B}^{(1)} \otimes_R \mathcal{B}^{(2)op}$ -module such that the line bundle  $L \otimes_R \mathbb{C}$  is trivial. Therefore,  $L$  admits a non-vanishing global section. Moreover, there is an isomorphism  $f : \mathcal{B}_{01}^{(1)} \otimes_{(\mathcal{B}^{(1)})_1^1} (\mathbf{pr}_1^1)^* L \rightarrow (\mathbf{pr}_0^1)^* L \otimes_{(\mathcal{B}^{(2)})_0^1} \mathcal{B}_{01}^{(2)}$  of  $(\mathcal{B}^{(1)})_0^1 \otimes_R ((\mathcal{B}^{(2)})_1^1)^{op}$ -modules.

A choice of a non-vanishing global section of  $L$  gives rise to isomorphisms  $\mathcal{B}_{01}^{(1)} \cong \mathcal{B}_{01}^{(1)} \otimes_{(\mathcal{B}^{(1)})_1^1} (\mathbf{pr}_1^1)^* L$  and  $\mathcal{B}_{01}^{(2)} \cong (\mathbf{pr}_0^1)^* L \otimes_{(\mathcal{B}^{(2)})_0^1} \mathcal{B}_{01}^{(2)}$ .

The composition

$$\mathcal{B}_{01}^{(1)} \cong \mathcal{B}_{01}^{(1)} \otimes_{(\mathcal{B}^{(1)})_1^1} (\mathbf{pr}_1^1)^* L \xrightarrow{f} (\mathbf{pr}_0^1)^* L \otimes_{(\mathcal{B}^{(2)})_0^1} \mathcal{B}_{01}^{(2)} \cong \mathcal{B}_{01}^{(2)}$$

defines a 1-morphism of deformations of  $(\mathcal{U}, \underline{\mathcal{A}})$  such that the induced 1-morphism  $\mathcal{C}^{(1)} \rightarrow \mathcal{C}^{(2)}$  is isomorphic to  $F$ . This shows that the functor  $\text{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R) \rightarrow \text{Def}(\mathcal{C})$  induces essentially surjective functors on groupoids of morphisms. By similar arguments left to the reader one shows that these are fully faithful.

This completes the proof of Theorem 1.  $\square$

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Paul Bressler, Institute for Advanced Study, Princeton, NJ 08540, USA

E-mail: bressler@math.ias.edu

Alexander Gorokhovsky, Department of Mathematics, University of Colorado, UCB 395, Boulder, CO 80309-0395, USA

E-mail: Alexander.Gorokhovsky@colorado.edu

Ryszard Nest, Department of Mathematics, Copenhagen University, Universitetsparken 5, 2100 Copenhagen, Denmark

E-mail: rnest@math.ku.dk

Boris Tsygan, Department of Mathematics, Northwestern University, Evanston, IL 60208-2730, USA

E-mail: tsygan@math.northwestern.edu