FORMALITY FOR ALGEBROID STACKS

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ABSTRACT. We extend the formality theorem of M. Kontsevich from deformations of the structure sheaf on a manifold to deformations of gerbes.

1. Introduction

In the fundamental paper [11] M. Kontsevich showed that the set of equivalence classes of formal deformations the algebra of functions on a manifold is in one-to-one correspondence with the set of equivalence classes of formal Poisson structures on the manifold. This result was obtained as a corollary of the formality of the Hochschild complex of the algebra of functions on the manifold conjectured by M. Kontsevich (cf. [10]) and proven in [11]. Later proofs by a different method were given in [14] and in [5].

In this paper we extend the formality theorem of M. Kontsevich to deformations of gerbes on smooth manifolds, using the method of [5]. Let X be a smooth manifold; we denote by $\mathcal{O}_X$ the sheaf of complex valued C^∞ functions on X. For a twisted form $\mathcal{S}$ of $\mathcal{O}_X$ regarded as an algebroid stack (see Section 2.5) we denote by $[\mathcal{S}]_{dR} \in H^3_{dR}(X)$ the de Rham class of $\mathcal{S}$. The main result of this paper establishes an equivalence of 2-groupoid valued functors of Artin $\mathbb{C}$-algebras between $\mathrm{Def}(\mathcal{S})$ (the formal deformation theory of $\mathcal{S}$, see [2]) and the Deligne 2-groupoid of Maurer-Cartan elements of L_∞-algebra of multivector fields on X twisted by a closed three-form representing $[\mathcal{S}]_{dR}$:

Theorem 6.1. *Suppose that $\mathcal{S}$ is a twisted form of $\mathcal{O}_X$. Let H be a closed 3-form on X which represents $[\mathcal{S}]_{dR} \in H^3_{dR}(X)$. For any Artin algebra R with maximal ideal $\mathfrak{m}_R$ there is an equivalence of 2-groupoids*

$$\mathrm{MC}^2(\mathfrak{s}(\mathcal{O}_X)_H \otimes \mathfrak{m}_R) \cong \mathrm{Def}(\mathcal{S})(R)$$

natural in R.

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Here, $\mathfrak{s}(\mathcal{O}_X)_H$ denotes the L_∞-algebra of multivector fields with the trivial differential, the binary operation given by Schouten bracket, the ternary operation given by H (see 5.3) and all other operations equal to zero. As a corollary of this result we obtain that the isomorphism classes of formal deformations of $\mathcal{S}$ are in a bijective correspondence with equivalence classes of the formal *twisted Poisson structures* defined by P. Severa and A. Weinstein in [13].

The proof of the Theorem proceeds along the following lines. As a starting point we use the construction of the Differential Graded Lie Algebra (DGLA) controlling the deformations of $\mathcal{S}$. This construction was obtained in [1, 2]. Next we construct a chain of L_∞-quasi-isomorphisms between this DGLA and $\mathfrak{s}(\mathcal{O}_X)_H$, using the techniques of [5]. Since L_∞-quasi-isomorphisms induce equivalences of respective Deligne groupoids, the result follows.

The paper is organized as follows. Section 2 contains the preliminary material on jets and deformations. Section 3 describes the results on the deformations of algebroid stacks. Section 4 is a short exposition of [5]. Section 5 contains the main technical result of the paper: the construction of the chain of quasi-isomorphisms mentioned above. Finally, in Section 6 the main theorem is deduced from the results of Section 5.

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2. Preliminaries

2.1. **Notations.** Throughout this paper, unless specified otherwise, X will denote a C^∞ manifold. By $\mathcal{O}_X$ we denote the sheaf of complex-valued C^∞ functions on X. $\mathcal{A}^\bullet_X$ denotes the sheaf of differential forms on X, and $\mathcal{T}_X$ the sheaf of vector fields on X. For a ring K we denote by $K^\times$ the group of invertible elements of K.

2.2. **Jets.** Let $\mathtt{pr}_i \colon X \times X \to X$, $i = 1, 2$, denote the projection on the i^{th} factor. Let $\Delta_X \colon X \to X \times X$ denote the diagonal embedding. Let $\mathcal{I}_X := \ker(\Delta_X^*)$.

For a locally-free $\mathcal{O}_X$-module of finite rank $\mathcal{E}$ let

$$\begin{aligned} \mathcal{J}^k_X(\mathcal{E}) &:= (\mathtt{pr}_1)_* \left(\mathcal{O}_{X\times X}/\mathcal{I}_X^{k+1} \otimes_{\mathtt{pr}_2^{-1}\mathcal{O}_X} \mathtt{pr}_2^{-1}\mathcal{E} \right) , \\ \mathcal{J}^k_X &:= \mathcal{J}^k_X(\mathcal{O}_X) . \end{aligned}$$

It is clear from the above definition that $\mathcal{J}^k_X$ is, in a natural way, a commutative algebra and $\mathcal{J}^k_X(\mathcal{E})$ is a $\mathcal{J}^k_X$-module.

Let

$$\mathbf{1}^{(k)}\colon \mathcal{O}_X \to \mathcal{J}_X^k$$

denote the composition

$$\mathcal{O}_X \xrightarrow{\mathtt{pr}_1^*} (\mathtt{pr}_1)_*\mathcal{O}_{X\times X} \to \mathcal{J}_X^k$$

In what follows, unless stated explicitly otherwise, we regard $\mathcal{J}_X^k(\mathcal{E})$ as a $\mathcal{O}_X$-module via the map $\mathbf{1}^{(k)}$.

Let

$$j^k\colon \mathcal{E} \to \mathcal{J}_X^k(\mathcal{E})$$

denote the composition

$$\mathcal{E} \xrightarrow{e\mapsto 1\otimes e} (\mathtt{pr}_1)_*\mathcal{O}_{X\times X} \otimes_{\mathbb{C}} \mathcal{E} \to \mathcal{J}_X^k(\mathcal{E})$$

The map j^k is not $\mathcal{O}_X$-linear unless $k = 0$.

For $0 \leq k \leq l$ the inclusion $\mathcal{I}_X^{l+1} \to \mathcal{I}_X^{k+1}$ induces the surjective map $\pi_{l,k}\colon \mathcal{J}_X^l(\mathcal{E}) \to \mathcal{J}_X^k(\mathcal{E})$. The sheaves $\mathcal{J}_X^k(\mathcal{E})$, $k = 0, 1, \ldots$ together with the maps $\pi_{l,k}$, $k \leq l$ form an inverse system. Let $\mathcal{J}_X(\mathcal{E}) = \mathcal{J}_X^\infty(\mathcal{E}) := \varprojlim \mathcal{J}_X^k(\mathcal{E})$. Thus, $\mathcal{J}_X(\mathcal{E})$ carries a natural topology.

The maps $\mathbf{1}^{(k)}$ (respectively, j^k), $k = 0, 1, 2, \ldots$ are compatible with the projections $\pi_{l,k}$, i.e. $\pi_{l,k} \circ \mathbf{1}^{(l)} = \mathbf{1}^{(k)}$ (respectively, $\pi_{l,k} \circ j^l = j^k$). Let $\mathbf{1} := \varprojlim \mathbf{1}^{(k)}$, $j^\infty := \varprojlim j^k$.

Let

$$d_1\colon \mathcal{O}_{X\times X} \otimes_{\mathtt{pr}_2^{-1}\mathcal{O}_X} \mathtt{pr}_2^{-1}\mathcal{E} \longrightarrow \mathtt{pr}_1^{-1}\mathcal{A}_X^1 \otimes_{\mathtt{pr}_1^{-1}\mathcal{O}_X} \mathcal{O}_{X\times X} \otimes_{\mathtt{pr}_2^{-1}\mathcal{O}_X} \mathtt{pr}_2^{-1}\mathcal{E}$$

denote the exterior derivative along the first factor. It satisfies

$$d_1(\mathcal{I}_X^{k+1} \otimes_{\mathtt{pr}_2^{-1}\mathcal{O}_X} \mathtt{pr}_2^{-1}\mathcal{E}) \subset \mathtt{pr}_1^{-1}\mathcal{A}_X^1 \otimes_{\mathtt{pr}_1^{-1}\mathcal{O}_X} \mathcal{I}_X^k \otimes_{\mathtt{pr}_2^{-1}\mathcal{O}_X} \mathtt{pr}_2^{-1}\mathcal{E}$$

for each k and, therefore, induces the map

$$d_1^{(k)}\colon \mathcal{J}^k(\mathcal{E}) \to \mathcal{A}_X^1 \otimes_{\mathcal{O}_X} \mathcal{J}^{k-1}(\mathcal{E})$$

The maps $d_1^{(k)}$ for different values of k are compatible with the maps $\pi_{l,k}$ giving rise to the *canonical flat connection*

$$\nabla^{can}\colon \mathcal{J}_X(\mathcal{E}) \to \mathcal{A}_X^1 \otimes_{\mathcal{O}_X} \mathcal{J}_X(\mathcal{E}) .$$

2.3. **Deligne groupoids.** In [4] P. Deligne and, independently, E. Getzler in [8] associated to a nilpotent DGLA $\mathfrak{g}$ concentrated in degrees grater than or equal to -1 the 2-groupoid, referred to as *the Deligne 2-groupoid* and denoted $\mathrm{MC}^2(\mathfrak{g})$ in [1], [2] and below. The objects of $\mathrm{MC}^2(\mathfrak{g})$ are the Maurer-Cartan elements of $\mathfrak{g}$. We refer the reader to [8] (as well as to [2]) for a detailed description. The above notion was extended and generalized by E. Getzler in [7] as follows.

To a nilpotent L_∞-algebra $\mathfrak{g}$ Getzler associates a (Kan) simplicial set $\gamma_\bullet(\mathfrak{g})$ which is functorial for L_∞ morphisms. If $\mathfrak{g}$ is concentrated in degrees greater than or equal to $1-l$, then the simplicial set $\gamma_\bullet(\mathfrak{g})$ is an l-dimensional hypergroupoid in the sense of J.W. Duskin (see [6]) by [7], Theorem 5.4.

Suppose that $\mathfrak{g}$ is a nilpotent L_∞-algebra concentrated in degrees grater than or equal to -1. Then, according to [6], Theorem 8.6 the simplicial set $\gamma_\bullet(\mathfrak{g})$ is the nerve of a bigroupoid, or, a 2-groupoid in our terminology. If $\mathfrak{g}$ is a DGLA concentrated in degrees grater than or equal to -1 this 2-groupoid coincides with $\mathrm{MC}^2(\mathfrak{g})$ of Deligne and Getzler alluded to earlier. We extend our notation to the more general setting of nilpotent L_∞-algebras as above and denote by $\mathrm{MC}^2(\mathfrak{g})$ the 2-groupoid furnished by [6], Theorem 8.6.

For an L_∞-algebra $\mathfrak{g}$ and a nilpotent commutative algebra $\mathfrak{m}$ the L_∞-algebra $\mathfrak{g}\otimes\mathfrak{m}$ is nilpotent, hence the simplicial set $\gamma_\bullet(\mathfrak{g}\otimes\mathfrak{m})$ is defined and enjoys the following homotopy invariance property ([7], Proposition 4.9, Corollary 5.11):

Theorem 2.1. *Suppose that $f\colon\mathfrak{g}\to\mathfrak{h}$ is a quasi-isomorphism of L_∞ algebras and let $\mathfrak{m}$ be a nilpotent commutative algebra. Then the induced map*

$$\gamma_\bullet(f\otimes\mathtt{Id})\colon\gamma_\bullet(\mathfrak{g}\otimes\mathfrak{m})\to\gamma_\bullet(\mathfrak{h}\otimes\mathfrak{m})$$

is a homotopy equivalence.

2.4. **Algebroid stacks.** Here we give a very brief overview, referring the reader to [3, 9] for the details. Let k be a field of characteristic zero, and let R be a commutative k-algebra.

Definition 2.2. A stack in R-linear categories $\mathcal{C}$ on X is an *R-algebroid stack* if it is locally nonempty and locally connected, i.e. satisfies

(1) any point $x\in X$ has a neighborhood U such that $\mathcal{C}(U)$ is nonempty;
(2) for any $U\subseteq X$, $x\in U$, $A,B\in\mathcal{C}(U)$ there exits a neighborhood $V\subseteq U$ of x and an isomorphism $A|_V\cong B|_V$.

For a prestack $\mathcal{C}$ we denote by $\widetilde{\mathcal{C}}$ the associated stack.

For a category C denote by iC the subcategory of isomorphisms in C; equivalently, iC is the maximal subgroupoid in C. If $\mathcal{C}$ is an algebroid stack then the stack associated to the substack of isomorphisms $\widetilde{i\mathcal{C}}$ is a gerbe.

For an algebra K we denote by K^+ the linear category with a single object whose endomorphism algebra is K. For a sheaf of algebras $\mathcal{K}$ on X we denote by $\mathcal{K}^+$ the prestack in linear categories given by $U \mapsto \mathcal{K}(U)^+$. Let $\widetilde{\mathcal{K}^+}$ denote the associated stack. Then, $\widetilde{\mathcal{K}^+}$ is an algebroid stack equivalent to the stack of locally free $\mathcal{K}^{\mathtt{op}}$-modules of rank one.

By a *twisted form of* $\mathcal{K}$ we mean an algebroid stack locally equivalent to $\widetilde{\mathcal{K}^+}$. It is easy to see that the equivalence classes of twisted forms of $\mathcal{K}$ are bijective correspondence with $H^2(X; \mathtt{Z}(\mathcal{K})^\times)$, where $\mathtt{Z}(\mathcal{K})$ denotes the center of $\mathcal{K}$.

2.5. **Twisted forms of $\mathcal{O}$.** Twisted forms of $\mathcal{O}_X$ are in bijective correspondence with $\mathcal{O}_X^\times$-gerbes: if $\mathcal{S}$ is a twisted form of $\mathcal{O}_X$, the corresponding gerbe is the substack $i\mathcal{S}$ of isomorphisms in $\mathcal{S}$. We shall not make a distinction between the two notions.

The equivalence classes of twisted forms of $\mathcal{O}_X$ are in bijective correspondence with $H^2(X; \mathcal{O}_X^\times)$. The composition

$$\mathcal{O}_X^\times \to \mathcal{O}_X^\times/\mathbb{C}^\times \xrightarrow{\log} \mathcal{O}_X/\mathbb{C} \xrightarrow{j^\infty} \mathtt{DR}(\mathcal{J}_X/\mathcal{O}_X)$$

induces the map $H^2(X; \mathcal{O}_X^\times) \to H^2(X; \mathtt{DR}(\mathcal{J}_X/\mathcal{O}_X)) \cong H^2(\Gamma(X; \mathcal{A}_X^\bullet \otimes \mathcal{J}_X/\mathcal{O}_X), \nabla^{can})$. We denote by $[\mathcal{S}]$ the image in the latter space of the class of $\mathcal{S}$.

The short exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{\mathbf{1}} \mathcal{J}_X \to \mathcal{J}_X/\mathcal{O}_X \to 0$$

gives rise to the short exact sequence of complexes

$$0 \to \Gamma(X; \mathcal{A}_X^\bullet) \to \Gamma(X; \mathtt{DR}(\mathcal{J}_X)) \to \Gamma(X; \mathtt{DR}(\mathcal{J}_X/\mathcal{O}_X)) \to 0,$$

hence to the map (connecting homomorphism) $H^2(X; \mathtt{DR}(\mathcal{J}_X/\mathcal{O}_X)) \to H^3_{dR}(X)$. Namely, if $B \in \Gamma(X; \mathcal{A}_X^2 \otimes \mathcal{J}_X)$ maps to $\overline{B} \in \Gamma(X; \mathcal{A}_X^2 \otimes \mathcal{J}_X/\mathcal{O}_X)$ which represents $[\mathcal{S}]$, then there exists a unique $H \in \Gamma(X; \mathcal{A}^3)$ such that $\nabla^{can} B = \mathtt{DR}(\mathbf{1})(H)$. The form H is closed and represents the image of the class of $\overline{B}$ under the connecting homomorphism.

Notation. We denote by $[\mathcal{S}]_{dR}$ the image of $[\mathcal{S}]$ under the map

$$H^2(X; \mathtt{DR}(\mathcal{J}_X/\mathcal{O}_X)) \to H^3_{dR}(X).$$

3. Deformations of algebroid stacks

3.1. **Deformations of linear stacks.** Here we describe the notion of 2-groupoid of deformations of an algebroid stack. We follow [2] and refer the reader to that paper for all the proofs and additional details.

For an R-linear category $\mathcal{C}$ and homomorphism of algebras $R \to S$ we denote by $\mathcal{C} \otimes_R S$ the category with the same objects as $\mathcal{C}$ and morphisms defined by $\operatorname{Hom}_{\mathcal{C}\otimes_R S}(A, B) = \operatorname{Hom}_{\mathcal{C}}(A, B) \otimes_R S$.

For a prestack $\mathcal{C}$ in R-linear categories we denote by $\mathcal{C} \otimes_R S$ the prestack associated to the fibered category $U \mapsto \mathcal{C}(U) \otimes_R S$.

Lemma 3.1 ([2], Lemma 4.13)**.** *Suppose that $\mathcal{A}$ is a sheaf of R-algebras and $\mathcal{C}$ is an R-algebroid stack. Then $\widetilde{\mathcal{C} \otimes_R S}$ is an algebroid stack.*

Suppose now that $\mathcal{C}$ is a stack in k-linear categories on X and R is a commutative Artin k-algebra. We denote by $\operatorname{Def}(\mathcal{C})(R)$ the 2-category with

- objects: pairs $(\mathcal{B}, \varpi)$, where $\mathcal{B}$ is a stack in R-linear categories flat over R and $\varpi : \widetilde{\mathcal{B} \otimes_R k} \to \mathcal{C}$ is an equivalence of stacks in k-linear categories
- 1-morphisms: a 1-morphism $(\mathcal{B}^{(1)}, \varpi^{(1)}) \to (\mathcal{B}^{(2)}, \varpi^{(2)})$ is a pair (F, θ) where $F : \mathcal{B}^{(1)} \to \mathcal{B}^{(2)}$ is a R-linear functor and $\theta : \varpi^{(2)} \circ (F \otimes_R k) \to \varpi^{(1)}$ is an isomorphism of functors
- 2-morphisms: a 2-morphism $(F', \theta') \to (F'', \theta'')$ is a morphism of R-linear functors $\kappa : F' \to F''$ such that $\theta'' \circ (\mathtt{Id}_{\varpi^{(2)}} \otimes (\kappa \otimes_R k)) = \theta'$

The 2-category $\operatorname{Def}(\mathcal{C})(R)$ is a 2-groupoid.

Let $\mathcal{B}$ be a prestack on X in R-linear categories. We say that $\mathcal{B}$ is *flat* if for any $U \subseteq X$, $A, B \in \mathcal{B}(U)$ the sheaf $\underline{\operatorname{Hom}}_{\mathcal{B}}(A, B)$ is flat (as a sheaf of R-modules).

Lemma 3.2 ([2], Lemma 6.2)**.** *Suppose that $\mathcal{B}$ is a flat R-linear stack on X such that $\widetilde{\mathcal{B} \otimes_R k}$ is an algebroid stack. Then $\mathcal{B}$ is an algebroid stack.*

3.2. **Deformations of twisted forms of $\mathcal{O}$.** Suppose that $\mathcal{S}$ is a twisted form of $\mathcal{O}_X$. We will now describe the DGLA controlling the deformations of $\mathcal{S}$.

The complex $\Gamma(X; \mathtt{DR}(C^\bullet(\mathcal{J}_X)) = (\Gamma(X; \mathcal{A}^\bullet_X \otimes C^\bullet(\mathcal{J}_X)), \nabla^{can} + \delta)$ is a differential graded brace algebra in a canonical way. The abelian Lie algebra $\mathcal{J}_X = C^0(\mathcal{J}_X)$ acts on the brace algebra $C^\bullet(\mathcal{J}_X)$ by derivations of degree -1 by Gerstenhaber bracket. The above action factors through an action of $\mathcal{J}_X/\mathcal{O}_X$. Therefore, the abelian Lie algebra

$\Gamma(X;\mathcal{A}^2_X\otimes\mathcal{J}_X/\mathcal{O}_X)$ acts on the brace algebra $\mathcal{A}^\bullet_X\otimes C^\bullet(\mathcal{J}_X)$ by derivations of degree $+1$. Following longstanding tradition, the action of an element a is denoted by i_a.

Due to commutativity of $\mathcal{J}_X$, for any $\omega\in\Gamma(X;\mathcal{A}^2_X\otimes\mathcal{J}_X/\mathcal{O}_X)$ the operation ι_ω commutes with the Hochschild differential δ. If, moreover, ω satisfies $\nabla^{can}\omega=0$, then $\nabla^{can}+\delta+i_\omega$ is a square-zero derivation of degree one of the brace structure. We refer to the complex

$$\Gamma(X;\mathtt{DR}(C^\bullet(\mathcal{J}_X))_\omega := (\Gamma(X;\mathcal{A}^\bullet_X\otimes C^\bullet(\mathcal{J}_X)),\nabla^{can}+\delta+i_\omega)$$

as the *ω-twist* of $\Gamma(X;\mathtt{DR}(C^\bullet(\mathcal{J}_X))$.

Let

$$\mathfrak{g}_{\mathtt{DR}}(\mathcal{J})_\omega := \Gamma(X;\mathtt{DR}(C^\bullet(\mathcal{J}_X))[1])_\omega$$

regarded as a DGLA. The following theorem is proved in [2] (Theorem 1 of loc. cit.):

Theorem 3.3. *For any Artin algebra R with maximal ideal $\mathfrak{m}_R$ there is an equivalence of 2-groupoids*

$$\mathrm{MC}^2(\mathfrak{g}_{\mathtt{DR}}(\mathcal{J}_X)_\omega\otimes\mathfrak{m}_R)\cong\mathrm{Def}(\mathcal{S})(R)$$

natural in R.

4. Formality

We give a synopsis of the results of [5] in the notations of loc. cit. Let k be a field of characteristic zero. For a k-cooperad $\mathcal{C}$ and a complex of k-vector spaces V we denote by $\mathbb{F}_{\mathcal{C}}(V)$ the cofree $\mathcal{C}$-coalgebra on V.

We denote by $\mathbf{e_2}$ the operad governing Gerstenhaber algebras. The latter is Koszul, and we denote by $\mathbf{e_2}^\vee$ the dual cooperad.

For an associative k-algebra A the Hochschild complex $C^\bullet(A)$ has a canonical structure of a brace algebra, hence a structure of homotopy $\mathbf{e_2}$-algebra. The latter structure is encoded in a differential (i.e. a coderivation of degree one and square zero) $M\colon\mathbb{F}_{\mathbf{e_2}^\vee}(C^\bullet(A))\to\mathbb{F}_{\mathbf{e_2}^\vee}(C^\bullet(A))[1]$.

Suppose from now on that A is regular commutative algebra over a field of characteristic zero (the regularity assumption is not needed for the constructions). Let $V^\bullet(A)=\mathrm{Sym}^\bullet_A(\mathtt{Der}(A)[-1])$ viewed as a complex with trivial differential. In this capacity $V^\bullet(A)$ has a canonical structure of an $\mathbf{e_2}$-algebra which gives rise to the differential $d_{V^\bullet(A)}$ on $\mathbb{F}_{\mathbf{e_2}^\vee}(V^\bullet(A))$; we have: $\mathtt{B}_{\mathbf{e_2}^\vee}(V^\bullet(A))=(\mathbb{F}_{\mathbf{e_2}^\vee}(V^\bullet(A)),d_{V^\bullet(A)})$ (see [5], Theorem 1 for notations).

In addition, the authors introduce a sub-$\mathbf{e_2}^\vee$-coalgebra $\Xi(A)$ of both $\mathbb{F}_{\mathbf{e_2}^\vee}(C^\bullet(A))$ and $\mathbb{F}_{\mathbf{e_2}^\vee}(V^\bullet(A))$. We denote by $\sigma\colon\Xi(A)\to\mathbb{F}_{\mathbf{e_2}^\vee}(C^\bullet(A))$ and $\iota\colon\Xi(A)\to\mathbb{F}_{\mathbf{e_2}^\vee}(V^\bullet(A))$ respective inclusions and identify $\Xi(A)$

with its image under the latter one. By [5], Proposition 7 the differential $d_{V^\bullet(A)}$ preserves $\Xi(A)$; we denote by $d_{V^\bullet(A)}$ its restriction to $\Xi(A)$. By Theorem 3, loc. cit. the inclusion σ is a morphism of complexes. Hence, we have the following diagram in the category of differential graded $\mathbf{e_2}^\vee$-coalgebras:

$$(\mathbb{F}_{\mathbf{e_2}^\vee}(C^\bullet(A)), M) \xleftarrow{\sigma} (\Xi(A), d_{V^\bullet(A)}) \xrightarrow{\iota} \mathtt{B}_{\mathbf{e_2}^\vee}(V^\bullet(A)) \tag{4.0.1}$$

Applying the functor $\Omega_{\mathbf{e_2}}$ (adjoint to the functor $\mathtt{B}_{\mathbf{e_2}^\vee}$, see [5], Theorem 1) to (4.0.1) we obtain the diagram

$$\begin{aligned} \text{(4.0.2)}\quad \Omega_{\mathbf{e_2}}(\mathbb{F}_{\mathbf{e_2}^\vee}(C^\bullet(A)), M) &\xleftarrow{\Omega_{\mathbf{e_2}}(\sigma)} \\ &\Omega_{\mathbf{e_2}}(\Xi(A), d_{V^\bullet(A)}) \xrightarrow{\Omega_{\mathbf{e_2}}(\iota)} \Omega_{\mathbf{e_2}}(\mathtt{B}_{\mathbf{e_2}^\vee}(V^\bullet(A))) \end{aligned}$$

of differential graded $\mathbf{e_2}$-algebras. Let $\nu = \eta_{\mathbf{e_2}} \circ \Omega_{\mathbf{e_2}}(\iota)$, where $\eta_{\mathbf{e_2}} : \Omega_{\mathbf{e_2}}(\mathtt{B}_{\mathbf{e_2}^\vee}(V^\bullet(A))) \to V^\bullet(A)$ is the counit of adjunction. Thus, we have the diagram

$$\Omega_{\mathbf{e_2}}(\mathbb{F}_{\mathbf{e_2}^\vee}(C^\bullet(A)), M) \xleftarrow{\Omega_{\mathbf{e_2}}(\sigma)} \Omega_{\mathbf{e_2}}(\Xi(A), d_{V^\bullet(A)}) \xrightarrow{\nu} V^\bullet(A) \tag{4.0.3}$$

of differential graded $\mathbf{e_2}$-algebras.

Theorem 4.1 ([5], Theorem 4)**.** *The maps $\Omega_{\mathbf{e_2}}(\sigma)$ and ν are quasi-isomorphisms.*

Additionally, concerning the DGLA structures relevant to applications to deformation theory, deduced from respective $\mathbf{e_2}$-algebra structures we have the following result.

Theorem 4.2 ([5], Theorem 2)**.** *The DGLA $\Omega_{\mathbf{e_2}}(\mathbb{F}_{\mathbf{e_2}^\vee}(C^\bullet(A)), M)[1]$ and $C^\bullet(A)[1]$ are canonically L_∞-quasi-isomorphic.*

Corollary 4.3 (Formality)**.** *The DGLA $C^\bullet(A)[1]$ and $V^\bullet(A)[1]$ are L_∞-quasi-isomorphic.*

4.1. **Some (super-)symmetries.** For applications to deformation theory of algebroid stacks we will need certain equivariance properties of the maps described in 4.

For $a \in A$ let $i_a \colon C^\bullet(A) \to C^\bullet(A)[-1]$ denote the adjoint action (in the sense of the Gerstenhaber bracket and the identification $A = C^0(A)$). It is given by the formula

$$i_a D(a_1, \ldots, a_n) = \sum_{i=0}^{n} (-1)^k D(a_1, \ldots, a_i, a, a_{k+1}, \ldots, a_n).$$

The operation i_a extends uniquely to a coderivation of $\mathbb{F}_{\mathbf{e_2}^\vee}(C^\bullet(A))$; we denote this extension by i_a as well. Furthermore, the subcoalgebra $\Xi(A)$ is preserved by i_a.

Since the operation i_a is a derivation of the cup product as well as of all of the brace operations on $C^\bullet(A)$ and the homotopy-$\mathbf{e_2}$-algebra structure on $C^\bullet(A)$ given in terms of the cup product and the brace operations it follows that i_a anti-commutes with the differential M. Hence, the coderivation i_a induces a derivation of the differential graded $\mathbf{e_2}$-algebra $\Omega_{\mathbf{e_2}}(\mathbb{F}_{\mathbf{e_2}^\vee}(C^\bullet(A)), M)$ which will be denoted by i_a as well. For the same reason the DGLA $\Omega_{\mathbf{e_2}}(\mathbb{F}_{\mathbf{e_2}^\vee}(C^\bullet(A)), M)[1]$ and $C^\bullet(A)[1]$ are quasi-isomorphic in a way which commutes with the respective operations i_a.

On the other hand, let $i_a \colon V^\bullet(A) \to V^\bullet(A)[-1]$ denote the adjoint action in the sense of the Schouten bracket and the identification $A = V^0(A)$. The operation i_a extends uniquely to a coderivation of $\mathbb{F}_{\mathbf{e_2}^\vee}(V^\bullet(A))$ which anticommutes with the differential $d_{V^\bullet(A)}$ because i_a is a derivation of the $\mathbf{e_2}$-algebra structure on $V^\bullet(A)$. We denote this coderivation as well as its unique extension to a derivation of the differential graded $\mathbf{e_2}$-algebra $\Omega_{\mathbf{e_2}}(\mathtt{B}_{\mathbf{e_2}^\vee}(V^\bullet(A)))$ by i_a. The counit map $\eta_{\mathbf{e_2}} \colon \Omega_{\mathbf{e_2}}(\mathtt{B}_{\mathbf{e_2}^\vee}(V^\bullet(A))) \to V^\bullet(A)$ commutes with respective operations i_a.

The subcoalgebra $\Xi(A)$ of $\mathbb{F}_{\mathbf{e_2}^\vee}(C^\bullet(A))$ and $\mathbb{F}_{\mathbf{e_2}^\vee}(V^\bullet(A))$ is preserved by the respective operations i_a. Moreover, the restrictions of the two operations to $\Xi(A)$ coincide, i.e. the maps in (4.0.1) commute with i_a and, therefore, so do the maps in (4.0.2) and (4.0.3).

4.2. **Deformations of $\mathcal{O}$ and Kontsevich formality.** Suppose that X is a manifold. Let $\mathcal{O}_X$ (respectively, $\mathcal{T}_X$) denote the structure sheaf (respectively, the sheaf of vector fields). The construction of the diagram localizes on X yielding the diagram of sheaves of differential graded $\mathbf{e_2}$-algebras

(4.2.1)
$$\Omega_{\mathbf{e_2}}(\mathbb{F}_{\mathbf{e_2}^\vee}(C^\bullet(\mathcal{O}_X)), M) \xleftarrow{\Omega_{\mathbf{e_2}}(\sigma)} \Omega_{\mathbf{e_2}}(\Xi(\mathcal{O}_X), d_{V^\bullet(\mathcal{O}_X)}) \xrightarrow{\nu} V^\bullet(\mathcal{O}_X),$$

where $C^\bullet(\mathcal{O}_X)$ denotes the sheaf of multidifferential operators and $V^\bullet(\mathcal{O}_X) := \mathrm{Sym}^\bullet_{\mathcal{O}_X}(\mathcal{T}_X[-1])$ denotes the sheaf of multivector fields. Theorem 4.1 extends easily to this case stating that the morphisms $\Omega_{\mathbf{e_2}}(\sigma)$ and ν in (4.2.1) are quasi-isomorphisms of sheaves of differential graded $\mathbf{e_2}$-algebras.

5. Formality for the algebroid Hochschild complex

5.1. **A version of [5] for jets.** Let $C^\bullet(\mathcal{J}_X)$ denote sheaf of continuous (with respect to the adic topology) $\mathcal{O}_X$-multilinear Hochschild cochains on $\mathcal{J}_X$. Let $V^\bullet(\mathcal{J}_X) = \mathrm{Sym}^\bullet_{\mathcal{J}_X}(\mathtt{Der}^{cont}_{\mathcal{O}_X}(\mathcal{J}_X)[-1])$.

Working now in the category of graded $\mathcal{O}_X$-modules we have the diagram
(5.1.1)

$$\Omega_{\mathbf{e_2}}(\mathbb{F}_{\mathbf{e_2}^\vee}(C^\bullet(\mathcal{J}_X)), M) \xleftarrow{\Omega_{\mathbf{e_2}}(\sigma)} \Omega_{\mathbf{e_2}}(\Xi(\mathcal{J}_X), d_{V^\bullet(\mathcal{J}_X)}) \xrightarrow{\nu} V^\bullet(\mathcal{J}_X)$$

of sheaves of differential graded $\mathcal{O}_X$-$\mathbf{e_2}$-algebras. Theorem 4.1 extends easily to this situation: the morphisms $\Omega_{\mathbf{e_2}}(\sigma)$ and ν in (5.1.1) are quasi-isomorphisms. The sheaves of DGLA $\Omega_{\mathbf{e_2}}(\mathbb{F}_{\mathbf{e_2}^\vee}(C^\bullet(\mathcal{J}_X)), M)[1]$ and $C^\bullet(\mathcal{J}_X)[1]$ are canonically L_∞-quasi-isomorphic.

The canonical flat connection ∇^{can} on $\mathcal{J}_X$ induces a flat connection which we denote ∇^{can} as well on each of the objects in the diagram (5.1.1). Moreover, the maps $\Omega_{\mathbf{e_2}}(\sigma)$ and ν are flat with respect to ∇^{can} hence induce the maps of respective de Rham complexes

$$(5.1.2)\quad \mathtt{DR}(\Omega_{\mathbf{e_2}}(\mathbb{F}_{\mathbf{e_2}^\vee}(C^\bullet(\mathcal{J}_X)), M)) \xleftarrow{\mathtt{DR}(\Omega_{\mathbf{e_2}}(\sigma))} \mathtt{DR}(\Omega_{\mathbf{e_2}}(\Xi(\mathcal{J}_X), d_{V^\bullet(\mathcal{J}_X)})) \xrightarrow{\mathtt{DR}(\nu)} \mathtt{DR}(V^\bullet(\mathcal{J}_X))$$

where, for $(K^\bullet, d)$ one of the objects in (5.1.1) we denote by $\mathtt{DR}(K^\bullet, d)$ the total complex of the double complex $(\mathcal{A}^\bullet_X \otimes K^\bullet, d, \nabla^{can})$. All objects in the diagram (5.1.2) have canonical structures of differential graded $\mathbf{e_2}$-algebras and the maps are morphisms thereof.

The DGLA $\Omega_{\mathbf{e_2}}(\mathbb{F}_{\mathbf{e_2}^\vee}(C^\bullet(\mathcal{J}_X)), M)[1]$ and $C^\bullet(\mathcal{J}_X)[1]$ are canonically L_∞-quasi-isomorphic in a way compatible with ∇^{can}. Hence, the DGLA $\mathtt{DR}(\Omega_{\mathbf{e_2}}(\mathbb{F}_{\mathbf{e_2}^\vee}(C^\bullet(\mathcal{J}_X)), M)[1])$ and $\mathtt{DR}(C^\bullet(\mathcal{J}_X)[1])$ are canonically L_∞-quasi-isomorphic.

5.2. **A version of** [5] **for jets with a twist.** Suppose that $\omega \in \Gamma(X; \mathcal{A}^2_X \otimes \mathcal{J}_X/\mathcal{O}_X)$ satisfies $\nabla^{can}\omega = 0$.

For each of the objects in (5.1.2) we denote by i_ω the operation which is induced by the one described in 4.1 and the wedge product on $\mathcal{A}^\bullet_X$. Thus, for each differential graded $\mathbf{e_2}$-algebra $(N^\bullet, d)$ in (5.1.2) we have a derivation of degree one and square zero i_ω which anticommutes with d and we denote by $(N^\bullet, d)_\omega$ the *ω-twist* of $(N^\bullet, d)$, i.e. the differential graded $\mathbf{e_2}$-algebra $(N^\bullet, d + i_\omega)$. Since the morphisms in (5.1.2) commute with the respective operations i_ω, they give rise to morphisms of respective ω-twists

$$(5.2.1)\quad \mathtt{DR}(\Omega_{\mathbf{e_2}}(\mathbb{F}_{\mathbf{e_2}^\vee}(C^\bullet(\mathcal{J}_X)), M))_\omega \xleftarrow{\mathtt{DR}(\Omega_{\mathbf{e_2}}(\sigma))} \mathtt{DR}(\Omega_{\mathbf{e_2}}(\Xi(\mathcal{J}_X), d_{V^\bullet(\mathcal{J}_X)}))_\omega \xrightarrow{\mathtt{DR}(\nu)} \mathtt{DR}(V^\bullet(\mathcal{J}_X))_\omega.$$

Let $F_\bullet\mathcal{A}^\bullet_X$ denote the stupid filtration: $F_i\mathcal{A}^\bullet_X = \mathcal{A}^{\geq -i}_X$. The filtration $F_\bullet\mathcal{A}^\bullet_X$ induces a filtration denoted $F_\bullet\mathtt{DR}(K^\bullet, d)_\omega$ for each object $(K^\bullet, d)$

of (5.1.1) defined by $F_i\mathtt{DR}(K^\bullet, d)_\omega = F_i\mathcal{A}^\bullet_X \otimes K^\bullet$. As is easy to see, the associated graded complex is given by

$$Gr_{-p}\mathtt{DR}(K^\bullet, d)_\omega = (\mathcal{A}^p_X \otimes K^\bullet, \mathtt{Id} \otimes d). \tag{5.2.2}$$

It is clear that the morphisms $\mathtt{DR}(\Omega_{\mathbf{e_2}}(\sigma))$ and $\mathtt{DR}(\nu)$ are filtered with respect to $F_\bullet$.

Theorem 5.1. *The morphisms in* (5.2.1) *are filtered quasi-isomorphisms, i.e. the maps $Gr_i\mathtt{DR}(\Omega_{\mathbf{e_2}}(\sigma))$ and $Gr_i\mathtt{DR}(\nu)$ are quasi-isomorphisms for all $i \in \mathbb{Z}$.*

Proof. We consider the case of $\mathtt{DR}(\Omega_{\mathbf{e_2}}(\sigma))$ leaving $Gr_i\mathtt{DR}(\nu)$ to the reader.

The map $Gr_{-p}\mathtt{DR}(\Omega_{\mathbf{e_2}}(\sigma))$ induced by $\mathtt{DR}(\Omega_{\mathbf{e_2}}(\sigma))$ on the respective associated graded objects in degree $-p$ is equal to the map of complexes
(5.2.3)
$$\mathtt{Id}\otimes\Omega_{\mathbf{e_2}}(\sigma)\colon \mathcal{A}^p_X\otimes\Omega_{\mathbf{e_2}}(\Xi(\mathcal{J}_X), d_{V^\bullet(\mathcal{J}_X)}) \to \mathcal{A}^p_X\otimes\Omega_{\mathbf{e_2}}(\mathbb{F}_{\mathbf{e_2}^\vee}(C^\bullet(\mathcal{J}_X)), M).$$

The map σ is a quasi-isomorphism by Theorem 4.1, therefore so is $\Omega_{\mathbf{e_2}}(\sigma)$. Since $\mathcal{A}^p_X$ is flat over $\mathcal{O}_X$, the map (5.2.3) is a quasi-isomorphism. □

Corollary 5.2. *The maps $\mathtt{DR}(\Omega_{\mathbf{e_2}}(\sigma))$ and $\mathtt{DR}(\nu)$ in* (5.2.1) *are quasi-isomorphisms of sheaves of differential graded $\mathbf{e_2}$-algebras.*

Additionally, the DGLA $\mathtt{DR}(\Omega_{\mathbf{e_2}}(\mathbb{F}_{\mathbf{e_2}^\vee}(C^\bullet(\mathcal{J}_X)), M)[1])$ and $\mathtt{DR}(C^\bullet(\mathcal{J}_X)[1])$ are canonically L_∞-quasi-isomorphic in a way which commutes with the respective operations i_ω which implies that the respective ω-twists $\mathtt{DR}(\Omega_{\mathbf{e_2}}(\mathbb{F}_{\mathbf{e_2}^\vee}(C^\bullet(\mathcal{J}_X)), M)[1])_\omega$ and $\mathtt{DR}(C^\bullet(\mathcal{J}_X)[1])_\omega$ are canonically L_∞-quasi-isomorphic.

5.3. **L_∞-structures on multivectors.** The canonical pairing $\langle\ ,\ \rangle\colon \mathcal{A}^1_X\otimes \mathcal{T}_X \to \mathcal{O}_X$ extends to the pairing

$$\langle\ ,\ \rangle\colon \mathcal{A}^1_X \otimes V^\bullet(\mathcal{O}_X) \to V^\bullet(\mathcal{O}_X)[-1]$$

For $k \geq 1$, $\omega = \alpha_1 \wedge \ldots \wedge \alpha_k$, $\alpha_i \in \mathcal{A}^1_X$, $i = 1, \ldots, k$, let

$$\Phi(\omega)\colon \operatorname{Sym}^k V^\bullet(\mathcal{O}_X)[2] \to V^\bullet(\mathcal{O}_X)[k]$$

denote the map given by the formula

$$\Phi(\omega)(\pi_1, \ldots, \pi_k) = (-1)^{(k-1)(|\pi_1|-1)+\ldots+2|(\pi_{k-3}|-1)+(|\pi_{k-2}|-1)}\times \sum_\sigma \operatorname{sgn}(\sigma)\langle\alpha_1, \pi_{\sigma(1)}\rangle \wedge \cdots \wedge \langle\alpha_k, \pi_{\sigma(k)}\rangle,$$

where $|\pi| = l$ for $\pi \in V^l(\mathcal{O}_X)$. For $\alpha \in \mathcal{O}_X$ let $\Phi(\alpha) = \alpha \in V^0(\mathcal{O}_X)$.

Recall that a graded vector space W gives rise to the graded Lie algebra $\mathtt{Der}(\mathrm{coComm}(W[1]))$. An element $\gamma \in \mathtt{Der}(\mathrm{coComm}(W[1]))$ of degree one which satisfies $[\gamma, \gamma] = 0$ defines a structure of an L_∞-algebra on W. Such a γ determines a differential $\partial_\gamma := [\gamma, .]$ on $\mathtt{Der}(\mathrm{coComm}(W[1]))$, such that $(\mathtt{Der}(\mathrm{coComm}(W[1])), \partial_\gamma)$ is a differential graded Lie algebra. If $\mathfrak{g}$ is a graded Lie algebra and γ is the element of $\mathtt{Der}(\mathrm{coComm}(\mathfrak{g}[1]))$ corresponding to the bracket on $\mathfrak{g}$, then $(\mathtt{Der}(\mathrm{coComm}(\mathfrak{g}[1])), \partial_\gamma)$ is equal to the shifted Chevalley cochain complex $C^\bullet(\mathfrak{g}; \mathfrak{g})[1]$.

In what follows we consider the (shifted) de Rham complex $\mathcal{A}^\bullet_X[2]$ as a differential graded Lie algebra with the trivial bracket.

Lemma 5.3. *The map $\omega \mapsto \Phi(\omega)$ defines a morphism of sheaves of differential graded Lie algebras*

$$\Phi\colon \mathcal{A}^\bullet_X[2] \to C^\bullet(V^\bullet(\mathcal{O}_X)[1]; V^\bullet(\mathcal{O}_X)[1])[1]. \tag{5.3.1}$$

Proof. Recall the explicit formulas for the Schouten bracket. If f and g are functions and X_i, Y_j are vector fields, then

$$[fX_1 \ldots X_k, gY_1 \ldots Y_l] = \sum_i (-1)^{k-i} f X_k(g) X_1 \ldots \widehat{X_i} \ldots X_k Y_1 \ldots Y_l +$$
$$\sum_j (-1)^j Y_j(f) g X_1 \ldots X_k Y_1 \ldots \widehat{Y_j} \ldots Y_l +$$
$$\sum_{i,j} (-1)^{i+j} f g X_1 \ldots \widehat{X_i} \ldots X_k Y_1 \ldots \widehat{Y_j} \ldots Y_l$$

Note that for a one-form ω and for vector fields X and Y

$$\langle \omega, [X, Y]\rangle - \langle [\omega, X], Y\rangle - \langle X, [\omega, Y]\rangle = \Phi(d\omega)(X, Y) \tag{5.3.2}$$

From the two formulas above we deduce by an explicit computation that

$$\langle \omega, [\pi, \rho]\rangle - \langle [\omega, \pi], \rho\rangle - (-1)^{|\pi|-1}\langle \pi, [\omega, \rho]\rangle = (-1)^{|\pi|-1}\Phi(d\omega)(\pi, \rho)$$

Note that Lie algebra cochains are invariant under the symmetric group acting by permutations multiplied by signs that are computed by the following rule: a permutation of π_i and π_j contributes a factor $(-1)^{|\pi_i||\pi_j|}$. We use the explicit formula for the bracket on the Lie algebra complex.

$$[\Phi, \Psi] = \Phi \circ \Psi - (-1)^{|\Phi||\Psi|}\Psi \circ \Phi$$

$$(\Phi \circ \Psi)(\pi_1, \ldots, \pi_{k+l-1}) = \sum_{I,J} \epsilon(I, J)\Phi(\Psi(\pi_{i_1}, \ldots, \pi_{i_k}), \pi_{j_1}, \ldots, \pi_{j_{l-1}})$$

Here $I = \{i_1, \ldots, i_k\}$; $J = \{j_1, \ldots, j_{l-1}\}$; $i_1 < \ldots < i_k$; $j_1 < \ldots < j_{l-1}$; $I \coprod J = \{1, \ldots, k+l-1\}$; the sign $\epsilon(I, J)$ is computed by the same sign rule as above. The differential is given by the formula

$$\partial\Phi = [m, \Phi]$$

where $m(\pi, \rho) = (-1)^{|\pi|-1}[\pi, \rho]$. Let $\alpha = \alpha_1 \ldots \alpha_k$ and $\beta = \beta_1 \ldots \beta_l$. We see from the above that both cochains $\Phi(\alpha) \circ \Phi(\beta)$ and $\Phi(\beta) \circ \Phi(\alpha)$ are antisymmetrizations with respect to α_i and β_j of the sums

$$\sum_{I,J,p} \pm \langle \alpha_1\beta_1, \pi_p\rangle\langle\alpha_2, \pi_{i_1}\rangle \ldots \langle\alpha_k, \pi_{i_{k-1}}\rangle\langle\beta_2, \pi_{j_1}\rangle \ldots \langle\beta_l, \pi_{j_{l-1}}\rangle$$

over all partitions $\{1, \ldots, k+l-1\} = I \coprod J \coprod \{p\}$ where $i_1 < \ldots < i_{k-1}$ and $j_1 < \ldots < j_{l-1}$; here $\langle\alpha\beta, \pi\rangle = \langle\alpha, \langle\beta, \pi\rangle\rangle$. After checking the signs, we conclude that $[\Phi(\alpha), \Phi(\beta)] = 0$. Also, from the definition of the differential, we see that $\partial\Phi(\alpha)(\pi_1, \ldots, \pi_{k+1})$ is the antisymmetrizations with respect to α_i and β_j of the sum

$$\sum_{i<j} \pm(\langle\alpha_1, [\pi_i, \pi_j]\rangle - \langle[\alpha_1, \pi_i], \pi_j\rangle - (-1)^{|\pi_i|-1}[\pi_i, \langle\alpha_1, \pi_j\rangle])\cdot$$
$$\langle\alpha_2, \pi_1\rangle \ldots \langle\alpha_i, \pi_{i-1}\rangle\langle\alpha_{i+1}, \pi_{i+1}\rangle \ldots \langle\alpha_{j-1}, \pi_{j-1}\rangle\langle\alpha_j, \pi_{j+1}\rangle\langle\alpha_k, \pi_{k+1}\rangle$$

We conclude from this and (5.3.2) that $\partial\Phi(\alpha) = \Phi(d\alpha)$. $\square$

Thus, according to Lemma 5.3, a closed 3-form H on X gives rise to a Maurer-Cartan element $\Phi(H)$ in $\Gamma(X; C^\bullet(V^\bullet(\mathcal{O}_X)[1]; V^\bullet(\mathcal{O}_X)[1])[1])$, hence a structure of an L_∞-algebra on $V^\bullet(\mathcal{O}_X)[1]$ which has the trivial differential (the unary operation), the binary operation equal to the Schouten-Nijenhuis bracket, the ternary operation given by $\Phi(H)$, and all higher operations equal to zero. Moreover, cohomologous closed 3-forms give rise to gauge equivalent Maurer-Cartan elements, hence to L_∞-isomorphic L_∞-structures.

Notation. For a closed 3-form H on X we denote the corresponding L_∞-algebra structure on $V^\bullet(\mathcal{O}_X)[1]$ by $V^\bullet(\mathcal{O}_X)[1]_H$. Let

$$\mathfrak{s}(\mathcal{O}_X)_H := \Gamma(X; V^\bullet(\mathcal{O}_X)[1])_H.$$

5.4. **L_∞-structures on multivectors via formal geometry.** In order to relate the results of 5.2 with those of 5.3 we consider the analog of the latter for jets.

Let $\widehat{\Omega}^k_{\mathcal{J}/\mathcal{O}} := \mathcal{J}_X(\mathcal{A}^k_X)$, the sheaf of jets of differential k-forms on X. Let $\widehat{\mathrm{d}}_{\mathrm{dR}}$ denote the ($\mathcal{O}_X$-linear) differential in $\widehat{\Omega}^\bullet_{\mathcal{J}/\mathcal{O}}$ induced by the de Rham differential in $\mathcal{A}^\bullet_X$. The differential $\widehat{\mathrm{d}}_{\mathrm{dR}}$ is horizontal with respect to the canonical flat connection ∇^{can} on $\widehat{\Omega}^\bullet_{\mathcal{J}/\mathcal{O}}$, hence we have the double complex $(\mathcal{A}^\bullet_X\otimes\widehat{\Omega}^\bullet_{\mathcal{J}/\mathcal{O}},\nabla^{can},\mathtt{Id}\otimes\widehat{\mathrm{d}}_{\mathrm{dR}})$ whose total complex is denoted $\mathtt{DR}(\widehat{\Omega}^\bullet_{\mathcal{J}/\mathcal{O}})$.

Let $\mathbf{1}\colon\mathcal{O}_X\to\mathcal{J}_X$ denote the unit map (not to be confused with the map j^∞); it is an isomorphism onto the kernel of $\widehat{\mathrm{d}}_{\mathrm{dR}}:\mathcal{J}_X\to\widehat{\Omega}^1_{\mathcal{J}/\mathcal{O}}$ and therefore defines the morphism of complexes $\mathbf{1}\colon\mathcal{O}_X\to\widehat{\Omega}^\bullet_{\mathcal{J}/\mathcal{O}}$ which is a quasi-isomorphism. The map $\mathbf{1}$ is horizontal with respect to the canonical flat connections on $\mathcal{O}_X$ and $\mathcal{J}_X$ (respectively, $\widehat{\Omega}^\bullet_{\mathcal{J}/\mathcal{O}}$), therefore we have the induced map of respective de Rham complexes $\mathtt{DR}(\mathbf{1})\colon\mathcal{A}^\bullet_X\to\mathtt{DR}(\mathcal{J}_X)$ (respectively, $\mathtt{DR}(\mathbf{1})\colon\mathcal{A}^\bullet_X\to\mathtt{DR}(\widehat{\Omega}^\bullet_{\mathcal{J}/\mathcal{O}})$, a quasi-isomorphism).

Let $C^\bullet(\mathfrak{g}(\mathcal{J}_X);\mathfrak{g}(\mathcal{J}_X))$ denote the complex of continuous $\mathcal{O}_X$-multilinear cochains. The map of differential graded Lie algebras

$$\widehat{\Phi}\colon\widehat{\Omega}^\bullet_{\mathcal{J}/\mathcal{O}}[2]\to C^\bullet(V^\bullet(\mathcal{J}_X)[1];V^\bullet(\mathcal{J}_X)[1])[1]\tag{5.4.1}$$

defined in the same way as (5.3.1) is horizontal with respect to the canonical flat connection ∇^{can} and induces the map

$$\mathtt{DR}(\widehat{\Phi})\colon\mathtt{DR}(\widehat{\Omega}^\bullet_{\mathcal{J}/\mathcal{O}})[2]\to\mathtt{DR}((C^\bullet(V^\bullet(\mathcal{J}_X)[1];V^\bullet(\mathcal{J}_X)[1])[1])\tag{5.4.2}$$

There is a canonical morphism of sheaves of differential graded Lie algebras

(5.4.3)
$$\mathtt{DR}(C^\bullet(V^\bullet(\mathcal{J}_X)[1];V^\bullet(\mathcal{J}_X)[1])[1])\to C^\bullet(\mathtt{DR}(V^\bullet(\mathcal{J}_X)[1]);\mathtt{DR}(V^\bullet(\mathcal{J}_X)[1]))[1]$$

Therefore, a degree three cocycle in $\Gamma(X;\mathtt{DR}(\widehat{\Omega}^\bullet_{\mathcal{J}/\mathcal{O}}))$ determines an L_∞-structure on $\mathtt{DR}(V^\bullet(\mathcal{J}_X)[1])$ and cohomologous cocycles determine L_∞-isomorphic structures.

Notation. For a section $B\in\Gamma(X;\mathcal{A}^2_X\otimes\mathcal{J}_X)$ we denote by $\overline{B}$ it's image in $\Gamma(X;\mathcal{A}^2_X\otimes\mathcal{J}_X/\mathcal{O}_X)$.

Lemma 5.4. *If $B\in\Gamma(X;\mathcal{A}^2_X\otimes\mathcal{J}_X)$ satisfies $\nabla^{can}\overline{B}=0$, then*

1. *$\widehat{\mathrm{d}}_{\mathrm{dR}}B$ is a (degree three) cocycle in $\Gamma(X;\mathtt{DR}(\widehat{\Omega}^\bullet_{\mathcal{J}/\mathcal{O}}))$;*
2. *there exist a unique $H\in\Gamma(X;\mathcal{A}^3_X)$ such that $dH=0$ and $\mathtt{DR}(\mathbf{1})(H)=\nabla^{can}B$.*

Proof. For the first claim it suffices to show that $\nabla^{can}B=0$. This follows from the assumption that $\nabla^{can}\overline{B}=0$ and the fact that $\widehat{\mathrm{d}}_{\mathrm{dR}}:\mathcal{A}^\bullet_X\otimes\mathcal{J}_X\to\mathcal{A}^\bullet_X\otimes\widehat{\Omega}^1_{\mathcal{J}/\mathcal{O}}$ factors through $\mathcal{A}^\bullet_X\otimes\mathcal{J}_X/\mathcal{O}_X$.

We have: $\widehat{\mathrm{d}}_{\mathrm{dR}}\nabla^{can}B=\nabla^{can}\widehat{\mathrm{d}}_{\mathrm{dR}}B=0$. Therefore, $\nabla^{can}B$ is in the image of $\mathtt{DR}(\mathbf{1}):\Gamma(X;\mathcal{A}^3_X)\to\Gamma(X;\mathcal{A}^3_X\otimes\mathcal{J}_X)$ which is injective, whence the existence and uniqueness of H. Since $\mathtt{DR}(\mathbf{1})$ is a morphism of complexes it follows that H is closed. $\square$

Suppose that $B \in \Gamma(X; \mathcal{A}_X^2 \otimes \mathcal{J}_X)$ satisfies $\nabla^{can}\overline{B} = 0$. Then, the differential graded Lie algebra $\mathtt{DR}(\mathfrak{g}(\mathcal{J}_X))_{\overline{B}}$ (the $\overline{B}$-twist of $\mathtt{DR}(\mathfrak{g}(\mathcal{J}_X))$) is defined. On the other hand, due to Lemma 5.4, (5.4.2) and (5.4.3), $\widehat{\mathrm{d}}_{\mathrm{dR}}B$ gives rise to an L_∞-structure on $\mathtt{DR}(V^\bullet(\mathcal{J}_X)[1])$.

Lemma 5.5. *The L_∞-structure induced by $\widehat{\mathrm{d}}_{\mathrm{dR}}B$ is that of a differential graded Lie algebra equal to* $\mathtt{DR}(V^\bullet(\mathcal{J}_X)[1])_{\overline{B}}$.

Proof. Left to the reader. □

Notation. For a 3-cocycle $\omega \in \Gamma(X; \mathtt{DR}(\widehat{\Omega}^\bullet_{\mathcal{J}/\mathcal{O}}))$ we will denote by $\mathtt{DR}(V^\bullet(\mathcal{J}_X)[1])_\omega$ the L_∞-algebra obtained from ω via (5.4.2) and (5.4.3). Let

$$\mathfrak{s}_{\mathtt{DR}}(\mathcal{J}_X)_\omega := \Gamma(X; \mathtt{DR}(V^\bullet(\mathcal{J}_X)[1]))_\omega.$$

Remark 5.6. Lemma 5.5 shows that this notation is unambiguous with reference to the previously introduced notation for the twist. In the notations introduced above, $\widehat{\mathrm{d}}_{\mathrm{dR}}B$ is the image of $\overline{B}$ under the *injective* map $\Gamma(X; \mathcal{A}_X^2 \otimes \mathcal{J}_X/\mathcal{O}_X) \to \Gamma(X; \mathcal{A}_X^2 \otimes \widehat{\Omega}^1_{\mathcal{J}/\mathcal{O}})$ which factors $\widehat{\mathrm{d}}_{\mathrm{dR}}$ and "allows" us to "identify" $\overline{B}$ with $\widehat{\mathrm{d}}_{\mathrm{dR}}B$.

Theorem 5.7. *Suppose that $B \in \Gamma(X; \mathcal{A}_X^2 \otimes \mathcal{J}_X)$ satisfies $\nabla^{can}\overline{B} = 0$. Let $H \in \Gamma(X; \mathcal{A}_X^3)$ denote the unique 3-form such that $\mathtt{DR}(\mathbf{1})(H) = \nabla^{can}B$ (cf. Lemma 5.4). Then, the L_∞-algebras $\mathfrak{g}_{\mathtt{DR}}(\mathcal{J}_X)_{\overline{B}}$ and $\mathfrak{s}(\mathcal{O}_X)_H$ are L_∞-quasi-isomorphic.*

Proof. The map $j^\infty \colon V^\bullet(\mathcal{O}_X) \to V^\bullet(\mathcal{J}_X)$ induces a quasi-isomorphism of sheaves of DGLA

$$j^\infty \colon V^\bullet(\mathcal{O}_X)[1] \to \mathtt{DR}(V^\bullet(\mathcal{J}_X)[1]). \tag{5.4.4}$$

Suppose that H is a closed 3-form on X. Then, the map (5.4.4) is a quasi-isomorphism of sheaves of L_∞-algebras

$$j^\infty \colon V^\bullet(\mathcal{O}_X)[1]_H \to \mathtt{DR}(V^\bullet(\mathcal{J}_X)[1])_{\mathtt{DR}(\mathbf{1})(H)}.$$

Passing to global section we obtain the quasi-isomorphism of L_∞-algebras

$$j^\infty \colon \mathfrak{s}(\mathcal{O}_X)_H \to \mathfrak{s}_{\mathtt{DR}}(\mathcal{J}_X)_{\mathtt{DR}(\mathbf{1})(H)}. \tag{5.4.5}$$

By assumption, B provides a homology between $\widehat{\mathrm{d}}_{\mathrm{dR}}B$ and $\nabla^{can}B = \mathtt{DR}(\mathbf{1})(H)$. Therefore, we have the corresponding L_∞-quasi-isomorphism (5.4.6)

$$\mathtt{DR}(V^\bullet(\mathcal{J}_X)[1])_{\mathtt{DR}(\mathbf{1})(H)} \overset{L_\infty}{\cong} \mathtt{DR}(V^\bullet(\mathcal{J}_X)[1])_{\widehat{\mathrm{d}}_{\mathrm{dR}}B} = \mathtt{DR}(V^\bullet(\mathcal{J}_X)[1])_{\overline{B}}$$

(the second equality is due to Lemma 5.5).

According to Corollary 5.2 the sheaf of DGLA $\mathtt{DR}(V^\bullet(\mathcal{J}_X)[1])_{\overline{B}}$ is L_∞-quasi-isomorphic to the DGLA deduced form the differential graded $\mathbf{e_2}$-algebra $\mathtt{DR}(\Omega_{\mathbf{e_2}}(\mathbb{F}_{\mathbf{e_2}^\vee}(C^\bullet(\mathcal{J}_X)), M))_{\overline{B}}$. The latter DGLA is L_∞-quasi-isomorphic to $\mathtt{DR}(C^\bullet(\mathcal{J}_X)[1])_{\overline{B}}$.

Passing to global sections we conclude that $\mathfrak{s}_{\mathtt{DR}}(\mathcal{J}_X)_{\mathtt{DR}(\mathbf{1})(H)}$ and $\mathfrak{g}_{\mathtt{DR}}(\mathcal{J}_X)_{\overline{B}}$ are L_∞-quasi-isomorphic. Together with (5.4.5) this implies the claim. □

6. Application to deformation theory

Theorem 6.1. *Suppose that $\mathcal{S}$ is a twisted form of $\mathcal{O}_X$ (2.5). Let H be a closed 3-form on X which represents $[\mathcal{S}]_{dR} \in H^3_{dR}(X)$. For any Artin algebra R with maximal ideal $\mathfrak{m}_R$ there is an equivalence of 2-groupoids*

$$\mathrm{MC}^2(\mathfrak{s}(\mathcal{O}_X)_H \otimes \mathfrak{m}_R) \cong \mathrm{Def}(\mathcal{S})(R)$$

natural in R.

Proof. Since cohomologous 3-forms give rise to L_∞-quasi-isomorphic L_∞-algebras we may assume, possibly replacing H by another representative of $[\mathcal{S}]_{dR}$, that there exists $B \in \Gamma(X; \mathcal{A}^2_X \otimes \mathcal{J}_X)$ such that $\overline{B}$ represents $[\mathcal{S}]$ and $\nabla^{can} B = \mathtt{DR}(\mathbf{1})(H)$. By Theorem 5.7 $\mathfrak{s}(\mathcal{O}_X)_H$ is L_∞-quasi-isomorphic to $\mathfrak{g}_{\mathtt{DR}}(\mathcal{J}_X)_{\overline{B}}$. By the Theorem 2.1 we have a homotopy equivalence of nerves of 2-groupoids $\gamma_\bullet(\mathfrak{s}(\mathcal{O}_X)_H \otimes \mathfrak{m}_R) \cong \gamma_\bullet(\mathfrak{g}_{\mathtt{DR}}(\mathcal{J}_X)_{\overline{B}} \otimes \mathfrak{m}_R)$. Therefore, there are equivalences

$$\mathrm{MC}^2(\mathfrak{s}(\mathcal{O}_X)_H \otimes \mathfrak{m}_R) \cong \mathrm{MC}^2(\mathfrak{g}_{\mathtt{DR}}(\mathcal{J}_X)_{\overline{B}} \otimes \mathfrak{m}_R) \cong \mathrm{Def}(\mathcal{S})(R),$$

the second one being that of Theorem 3.3. □

Remark 6.2. In particular, the isomorphism classes of formal deformations of $\mathcal{S}$ are in a bijective correspondence with equivalence classes of Maurer-Cartan elements of the L_∞-algebra $\mathfrak{s}_{\mathtt{DR}}(\mathcal{O}_X)_H \widehat{\otimes} t \cdot \mathbb{C}[[t]]$. These are the formal *twisted Poisson structures* in the terminology of [13], i.e. the formal series $\pi = \sum_{k=1}^\infty t^k \pi_k$, $\pi_k \in \Gamma(X; \bigwedge^2 \mathcal{T}_X)$, satisfying the equation

$$[\pi, \pi] = \Phi(H)(\pi, \pi, \pi).$$

A construction of an algebroid stack associated to a twisted Poisson structure was proposed by P. Ševera in [12].

References

[1] P. Bressler, A. Gorokhovsky, R. Nest, and B. Tsygan. Deformation quantization of gerbes. *Adv. Math.*, 214(1):230–266, 2007.

[2] P. Bressler, A. Gorokhovsky, R. Nest, and B. Tsygan. Deformations of gerbes on smooth manifolds. In *K-theory and noncommutative geometry*. European Mathematical Society, 2008.

[3] A. D'Agnolo and P. Polesello. Stacks of twisted modules and integral transforms. In *Geometric aspects of Dwork theory. Vol. I, II*, pages 463–507. Walter de Gruyter GmbH & Co. KG, Berlin, 2004.
[4] P. Deligne. Letter to L. Breen, 1994.
[5] V. Dolgushev, D. Tamarkin, and B. Tsygan. The homotopy Gerstenhaber algebra of Hochschild cochains of a regular algebra is formal. *J. Noncommut. Geom.*, 1(1):1–25, 2007.
[6] J. W. Duskin. Simplicial matrices and the nerves of weak n-categories. I. Nerves of bicategories. *Theory Appl. Categ.*, 9:198–308 (electronic), 2001/02. CT2000 Conference (Como).
[7] E. Getzler. Lie theory for nilpotent L-infinity algebras. *Ann. of Math.*, to appear, arXiv:math.AT/0404003.
[8] E. Getzler. A Darboux theorem for Hamiltonian operators in the formal calculus of variations. *Duke Math. J.*, 111(3):535–560, 2002.
[9] M. Kashiwara and P. Schapira. *Categories and sheaves*, volume 332 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.
[10] M. Kontsevich. Formality conjecture. In *Deformation theory and symplectic geometry (Ascona, 1996)*, volume 20 of *Math. Phys. Stud.*, pages 139–156. Kluwer Acad. Publ., Dordrecht, 1997.
[11] M. Kontsevich. Deformation quantization of Poisson manifolds. *Lett. Math. Phys.*, 66(3):157–216, 2003.
[12] P. Ševera. Quantization of Poisson families and of twisted Poisson structures. *Lett. Math. Phys.*, 63(2):105–113, 2003.
[13] P. Ševera and A. Weinstein. Poisson geometry with a 3-form background. *Progr. Theoret. Phys. Suppl.*, (144):145–154, 2001. Noncommutative geometry and string theory (Yokohama, 2001).
[14] D. Tamarkin. Another proof of M. Kontsevich formality theorem. preprint, arXiv:math/9803025v4 [math.QA].

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