Secondary Characteristic Classes and Cyclic Cohomology of Hopf Algebras

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Abstract

Let X be a manifold on which a discrete (pseudo)group of diffeomorphisms Γ acts, and let E be a Γ-equivariant vector bundle on X. We give a construction of cyclic cocycles on the cross product algebra $C_0^\infty(X) \rtimes \Gamma$ representing the equivariant characteristic classes of E. Our formulas can be viewed as a higher-dimensional analogue of Connes' Godbillon-Vey cyclic cocycle. An essential tool for our construction, which allows us to overcome difficulties arising in the higher-dimensional case, is Connes-Moscovici's theory of cyclic cohomology of Hopf algebras.

1 Introduction

In the paper [4] A. Connes provided an explicit construction of the Godbillon-Vey cocycle in cyclic cohomology. The goal of this paper is to give a similar construction for higher secondary classes.

First, let us recall Connes' construction. Let M be a smooth oriented manifold and let $\Gamma \subset \mathrm{Diff}^+(M)$ be a discrete group of orientation-preserving diffeomorphisms of M. Let ω be a volume form on M. Define the following function on $M \times \Gamma$: $\delta(g) = \frac{\omega^g}{\omega}$, where the superscript denotes the group action. Then one can define a one-parametric group of diffeomorphisms of the algebra $\mathcal{A} = C_0^\infty(M) \rtimes \Gamma$ by

$$\phi_t(aU_g) = a\delta(g)^{it}U_g \tag{1.1}$$

This is the Tomita-Takesaki group of automorphisms, associated to the weight on $\mathcal{A}$ given by ω.

Consider now the transverse fundamental class –the cyclic q-cocycle on $\mathcal{A}$

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given by

$$\tau(a_0U_{g_0}, a_1U_{g_1}, \ldots, a_qU_{g_q}) = \begin{cases} \frac{1}{q!}\int\limits_M a_0 da_1^{g_0} da_2^{g_0g_1} \ldots da_q^{g_0g_1\ldots g_{q-1}} & \text{if } g_0g_1\ldots g_q = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

To study the behavior of this cocycle under the 1-parametric group (1.1), consider the “Lie derivative” $\mathcal{L}$ acting on the cyclic complex by $\mathcal{L}\xi = \frac{d}{dt}|_{t=0}\phi_t^*\xi$, ξ being a cyclic cochain. It turns out that in general τ is not invariant under the group (1.1), and that $\mathcal{L}\tau \neq 0$.

However, it was noted by Connes that one always has

$$\mathcal{L}^{q+1}\tau = 0 \tag{1.3}$$

and that $\mathcal{L}^q\tau$ is invariant under the action of the group (1.1). One deduces from this that if ι_δ is the analogue of the interior derivative (see [4]), then $\iota_\delta\mathcal{L}^q\tau$ is a cyclic cocycle.

This is Connes' Godbillon-Vey cocycle. It can be related to the Godbillon-Vey class as follows. Let $[GV] \in H^*(M_\Gamma)$ be the Godbillon-Vey class, where $M_\Gamma = M \times_\Gamma \mathrm{E}\Gamma$ is the homotopy quotient. Connes defines a canonical map $\Phi : H^*(M_\Gamma) \to HP^*(\mathcal{A})$. Then one has

$$\Phi([GV]) = [\iota_\delta\mathcal{L}^q\tau] \tag{1.4}$$

The class of this cocycle is independent of the choice of the volume form. To prove this one can use Connes' noncommutative Radon-Nicodym theorem to conclude that if one changes the volume form, the one-parametric group ϕ_t remains the same modulo inner automorphisms.

A natural problem then is to extend this construction to cocycles corresponding to other secondary characteristic classes. It was noted by Connes [5] that if instead of a 1-dimensional bundle of q-forms on M one considers Γ equivariant trivial bundle of rank n, then in place of the 1-parametric group (1.1) one encounters a coaction of the group $GL_n(\mathbb{R})$ on the algebra $\mathcal{A}$. The difficulty is that for $n > 1$ this group is not commutative, and one cannot replace this coaction by the action of the dual group, similarly to (1.1).

In this paper we show that the Connes-Moscovici theory of cyclic cohomology for Hopf algebras (cf [6,7]) provides a natural framework for the higher-dimensional situation and allows one to give a construction of the secondary characteristic cocycles.

The situation we consider is the following. We have an orientation-preserving action of a discrete group (or pseudogroup) Γ on the oriented manifold M, and a trivial bundle E on M that is equivariant with respect to this action. Well-known examples in which such a situation arises are the following (cf. [5], [16],[6]). Let V be a manifold on which a discrete group (or pseudogroup) G acts, and let E_0 be a bundle (not necessarily trivial) on V, equivariant with respect to the action of G. Let U_i, $i \in I$, be an open covering of V such that the restriction of F on each U_i is trivial. Put $M = \sqcup U_i$, and let E be the pull-back of E_0 to M by the natural projection. Then we have an action of the following pseudogroup Γ on M: $\Gamma = \{g_{i,j} | \, g \in G \, i, j \in I\} \cup id$, where $\operatorname{Dom} g_{i,j} = g^{-1}(U_j) \cap U_i \subset U_i$, $\operatorname{Ran} g_{i,j} = g(U_i) \cap U_j \subset U_j$, and the natural composition rules hold. The bundle E is clearly equivariant with respect to this action. Our construction, described below, provides classes in the cyclic cohomology of the cross-product algebra $C_0^\infty(M) \rtimes \Gamma$, rather than in the cyclic cohomology of $C_0^\infty(V) \rtimes G$. However, the cross-product algebras $C_0^\infty(M) \rtimes \Gamma$ and $C_0^\infty(V) \rtimes G$ are Morita equivalent, and hence have the same cyclic cohomology.

Another natural example is provided by a manifold V with a foliation F, and a bundle E_0 which is holonomy equivariant. We can always choose a complete (possibly disconnected) transversal M, such that the restriction E of E_0 to M is trivial. Let Γ be the holonomy pseudogroup acting on M. E is clearly equivariant with respect to this action. In this case again the cross-product algebra $C_0^\infty(M) \rtimes \Gamma$ is Morita equivalent to the full algebra of the foliation $C_0^\infty(V, F)$.

We then construct a map from the cohomology of the truncated Weil algebra (cf. e.g. [14]) $W(\mathfrak{g}, O_n)_q$ to the periodic cyclic cohomology $HP^*(\mathcal{A})$ of the algebra $\mathcal{A} = C_0^\infty(M) \rtimes \Gamma$. The construction is the following. We consider the action of the *differential graded* Hopf algebra $\mathcal{H}(GL_n(\mathbb{R}))$ of differential forms on the group $GL_n(\mathbb{R})$ on the differential graded algebra $\Omega_0^*(M) \rtimes \Gamma$, where $\Omega_0^*(M)$ denotes the algebra of compactly supported differential forms on M. The use of differential graded algebras allows one to conveniently encode different identities, similar to (1.3). We then show that Connes-Moscovici theory (or rather a differential graded version of it) allows one to define a map from the cyclic complex of $\mathcal{H}(GL_n(\mathbb{R}))$ to the cyclic complex of $\Omega_0^*(M) \rtimes \Gamma$. After this we the relate cyclic complex of $\mathcal{H}(GL_n(\mathbb{R}))$ to the Weil algebra, and the cyclic cohomology of $\Omega_0^*(M) \rtimes \Gamma$ to the cyclic cohomology of $\mathcal{A}$.

It would be interesting to extend the methods of this paper to the situation of nontrivial bundles, and obtain the answer involving connections and curvatures, as done for the *primary* classes in [12,13]. It would also be interesting to relate our constructions with the work of E. Getzler [10], where Hopf algebra of differential forms on Lie group is used to give construction of equivariant characteristic classes for noncompact Lie groups.

The paper is organized as follows. In the next two sections we discuss cyclic complexes for differential graded algebras and differential graded Hopf algebras respectively. In section 4 we show that two different Hopf actions, which coincide "modulo inner automorphisms", induce the same Connes-Moscovici characteristic map in cyclic cohomology and discuss some other properties of the characteristic map. In section 5 we construct the action of $\mathcal{H}(GL_n(\mathbb{R}))$ on $\Omega_0^*(M) \rtimes \Gamma$. In section 6 we relate cyclic complex of the Hopf algebra $\mathcal{H}(GL_n(\mathbb{R}))$ with the Weil algebras. Finally, in section 7 we prove an analogue of formula (1.4) for the cocycles we construct.

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2 Cyclic complex for differential graded algebras

In this section we collect some preliminary standard facts about cyclic cohomology of differential graded algebras, and give cohomological version of some results of [11].

Recall that a cyclic module X^* is given by a cosimplicial module with the face maps $\delta_i : X^{n-1} \to X^n$ and degeneracy maps $\sigma_i : X^n \to X^{n-1}$ $0 \le i \le n$, satisfying the usual axioms. In addition, we have for each n an action of $\mathbb{Z}_{n+1}$ on X^n, with the generator τ_n, satisfying

$$\tau_n \delta_i = \delta_{i-1} \tau_{n-1} \text{ for } 1 \le i \le n \text{ and } \tau_n \delta_0 = \delta_n \tag{2.1}$$

$$\tau_n \sigma_i = \sigma_{i-1} \tau_{n+1} \text{ for } 1 \le i \le n \text{ and } \tau_n \sigma_0 = \sigma_n \tau_{n+1}^2 \tag{2.2}$$

$$\tau_n^{n+1} = id \tag{2.3}$$

For every cyclic object one can construct operators $b : X^n \to X^{n+1}$ and $B : X^n \to X^{n-1}$, defined by the formulas

$$b = \sum_{j=0}^{n} (-1)^j \delta_j \tag{2.4}$$

$$B = \left(\sum_{j=0}^{n-1} (-1)^{j(n-1)} \tau_{n-1}^j \right) \sigma_{n+1} (1 - (-1)^{n-1} \tau_n) \tag{2.5}$$

where

$$\sigma_{n+1} = \sigma_n \tau_{n+1} \tag{2.6}$$

These operators satisfy

$$b^2 = 0 \tag{2.7}$$
$$B^2 = 0 \tag{2.8}$$
$$bB + Bb = 0 \tag{2.9}$$

Hence for any cyclic object X^* we can construct a bicomplex $\mathcal{B}^{*,*}(X)$ as follows: $\mathcal{B}^{p,q}$, p, $q \geq 0$ is X^{p-q}, or 0 if $p < q$, and the differential $\mathcal{B}^{p,q} \to \mathcal{B}^{p+1,q}$ (resp. $\mathcal{B}^{p,q+1}$) is given by b (resp. B). By removing the restriction p, $q \geq 0$ we obtain a periodic bicomplex $\mathcal{B}_{\text{per}}$. Notice that it has periodicity induced by the tautological shift $S : \mathcal{B}^{p,q}_{\text{per}} \to \mathcal{B}^{p+1,q+1}_{\text{per}}$.

Now let Ω^* be a unital positively graded algebra. We can associate with it a cyclic object as follows (the differential d is not used in this definition).

Let $C^k(\Omega^*)$ be the space of continuous $(k+1)$-linear functionals on Ω^*. The face and degeneracy maps are given by

$$\begin{aligned}(\delta_j\phi)(a_0, a_1, \ldots, a_{k+1}) =& \phi(a_0, \ldots, a_j a_{j+1}, \ldots, a_{k+1}) \text{ for } 0 \leq j \leq n-1 \\ (\delta_n\phi)(a_0, a_1, \ldots, a_{k+1}) =& (-1)^{\deg a_{k+1}(\deg a_0 + \cdots + \deg a_k)}\phi(a_{k+1}a_0, a_1, \ldots a_k)\end{aligned} \tag{2.10}$$

$$(\sigma_j\phi)(a_0, \ldots, a_{k-1}) = \phi(a_0, \ldots, a_j, 1, a_{j+1}, \ldots a_{k-1}) \tag{2.11}$$

and the cyclic action is given by

$$(\tau_k\phi)(a_0, \ldots, a_k) = (-1)^{\deg a_k(\deg a_0 + \ldots \deg a_{k-1})}\phi(a_k, a_0, \ldots, a_{k-1}) \tag{2.12}$$

Cohomology of the total complex of the bicomplex $\mathcal{B}$ (resp. $\mathcal{B}_{\text{per}}$) where we consider only *finite* cochains, is the cyclic (resp. periodic cyclic) cohomology of Ω^*, which we denote $HC^*(\Omega^*)$ (resp. $HP^*(\Omega^*)$).

Suppose now that Ω^* is a differential graded (DG) algebra with the differential of degree 1. We say that $\phi \in C^k(\Omega^*)$ has weight m if $\phi(a_0, a_1, \ldots, a_k) = 0$ unless $\deg a_0 + \deg a_1 + \cdots + \deg a_k = m$. We denote by $C^{k,p}(\Omega^*) \subset C^k(\Omega^*)$ set of weight $(-p)$ functionals. Notice that in this case each $C^k(\Omega^*)$ is a complex in its own right, with grading defined above, and with differential $(-1)^k d$, where we extend d to $C^k(\Omega^*)$ by

$$d\phi(a_0, a_1, \ldots, a_k) = \sum_{j=0}^{k}(-1)^{\deg a_0 + \ldots \deg a_{j-1}}\phi(a_0, \ldots, da_j, \ldots, a_k) \tag{2.13}$$

Then $db - bd = 0$, $dB - Bd = 0$, and hence in this situation $\mathcal{B}$ and $\mathcal{B}_{\text{per}}$ become actually tricomplexes. Cyclic (resp. periodic cyclic) cohomology of the DG algebra (Ω^*, d) is then defined as the cohomology of the total complex of the tricomplex $\mathcal{B}$ (resp. $\mathcal{B}_{\text{per}}$), where we consider only *finite* cochains. Notations for the cyclic and periodic cyclic cohomologies are $HC^*\left((\Omega^*, d)\right)$ and $HP^*\left((\Omega^*, d)\right)$.

One can show that cyclic cohomology can be computed by the *normalized* complex, i.e. one where cochains satisfy

$$\phi(a_0, a_1, \ldots, a_k) = 0 \text{ if } a_i = 1,\ i \geq 1 \tag{2.14}$$

We will need the following result about the cyclic cohomology.

Theorem 1 *Let $\mathcal{A} = \Omega^0$ be the 0-degree part of Ω^* (which we consider as a trivially graded algebra with the zero differential). We then have a natural map of (total) complexes $I : \mathcal{B}_{per}(\mathcal{A}) \to \mathcal{B}_{per}\left((\Omega^*, d)\right)$ (extension of multilinear forms by 0 to Ω^*). Then the induced map in cohomology $HP^*(\mathcal{A}) \to HP^*\left((\Omega^*, d)\right)$ is an isomorphism.*

To prove the theorem and to write an explicit formula for the map $R : \mathcal{B}_{\text{per}}\left((\Omega^*, d)\right) \to \mathcal{B}_{\text{per}}(\mathcal{A})$, inducing the inverse isomorphism in the periodic cyclic cohomology, we need the following fact (Rinehart formula) which we use as stated in [11].

Let D be a derivation of the graded algebra Ω^* of degree $\deg D$, i.e. a linear map $D : \Omega^* \to \Omega^{*+\deg D}$ satisfying

$$D(ab) = (Da)b + (-1)^{\deg D \deg a} aD(b) \tag{2.15}$$

It defines an operator on the complex $\mathcal{B}(\Omega^*)$, by

$$\mathcal{L}_D \phi(a_0, a_1, \ldots, a_k) = \sum_{i=0}^{k} (-1)^{\deg D(a_0 + \cdots + a_{i-1})} \phi(a_0, \ldots, D(a_i), \ldots, a_k) \tag{2.16}$$

which commutes with both b and B. The action of this operator on the periodic cyclic bicomplex is homotopic to zero, with the homotopy constructed as follows. Define operators $e_D : C^{k-1}(\Omega^*) \to: C^k(\Omega^*)$, $E_D : C^{k+1}(\Omega^*) \to C^k(\Omega^*)$ by

$$e_D \phi(a_0, a_1, \ldots a_k) = \lambda \phi(D(a_k)a_0, a_1 \ldots, a_{k-1}) \tag{2.17}$$

$$E_D \phi(a_0, a_1, \ldots a_k) = \mu \sum_{1 \leq i \leq j \leq k} \phi(1, a_i, a_{i+1}, \ldots, a_{j-1}, Da_j, \ldots, a_k, a_0, \ldots) \tag{2.18}$$

where

$$\lambda = (-1)^{k+1+\deg a_k(\deg a_0 + \cdots + \deg a_{k-1})} \tag{2.19}$$

$$\mu = (-1)^{ik+1+(\deg a_i + \ldots \deg a_k)(\deg a_0 + \ldots \deg a_{i-1}) + \deg D(\deg a_i + \ldots \deg a_j)} \tag{2.20}$$

Then

$$[b+B, e_D+E_D] = \mathcal{L}_D \tag{2.21}$$

We now proceed with the proof of the Theorem 1

Proof of the Theorem 1 Consider the derivation D of degree 0 given by $Da = (\deg a)a$. On the multilinear form of weight m D acts by m. Define the homotopy h to be $\frac{1}{m}(e_D + E_D)$ on the forms of weight $m > 0$ and 0 on the forms of weight 0. We define map of complexes $R : \mathcal{B}_{\mathrm{per}}((\Omega^*, d)) \to \mathcal{B}_{\mathrm{per}}(\mathcal{A})$ by

$$R\phi = c_{k,m}(dh)^m\phi \text{ for } \phi \in C^{k,m} \tag{2.22}$$

where

$$c_{k,m} = (-1)^{km+\frac{m^2-m}{2}} \tag{2.23}$$

This is a map of (total) complexes. Indeed, using identities $(b+B)h+h(b+B) = id$ and $(b+B)d - d(b+B) = 0$, we have:

$$\begin{aligned}(b+B)R\phi - R(b+B+(-1)^k d)\phi = \\ c_{k,m}\left(((b+B)(dh)^m - (-1)^m(dh)^m(b+B)) + (-1)^m(dh)^{m-1}d\right)\phi = \\ c_{k,m}\left(-(-1)^m(dh)^{m-1}d + (-1)^m(dh)^{m-1}d\right)\phi = 0\end{aligned} \tag{2.24}$$

It is clear that

$$R \circ I = id \tag{2.25}$$

We denote by $\partial = b+B\pm d$ the total differential in the complex $\mathcal{B}_{\mathrm{per}}((\Omega^*, d))$, where we denote by $\pm d$ $(-1)^k d$acting on $C^{k,m}$, Then for $I \circ R$ we have

$$I \circ R = id - (\partial \circ H + H \circ \partial) \tag{2.26}$$

and the homotopy H is given by the formula

$$H\phi = \sum_{j=0}^{m-1} c_{k,j}h(dh)^j\phi \text{ for } \phi \in C^{k,m} \tag{2.27}$$

This equality can be verified by direct computation as follows.

For actions on $C^{k,-m}$

$$H \circ (b+B) = \sum_{j=0}^{m-1} c_{k,j}((dh)^j - (b+B)h(dh)^j) + \sum_{j=1}^{m-1} c_{k,j}(-1)^j (hd)^j \quad (2.28)$$

$$(b+B) \circ H = \sum_{j=0}^{m-1} c_{k,j}(b+B)h(dh)^j \quad (2.29)$$

$$H \circ (\pm d) = \sum_{j=0}^{m-2} (-1)^k c_{k,j}(hd)^{j+1} = \sum_{j=1}^{m-1} (-1)^k c_{k,j-1}(hd)^j \quad (2.30)$$

$$(\pm d) \circ H = \sum_{j=0}^{m-1} (-1)^{k-j-1} c_{k,j}(dh)^{j+1} = \sum_{j=1}^{m-1} (-1)^{k-j} c_{k,j-1}(dh)^j \quad (2.31)$$

and adding these equalities we get the desired result.

Remark 2 *We can give another formula for the map induced by R in cohomology. Define*

$$R'\phi = \frac{c_{k,m}}{m!}(e_d + E_d)^m \phi \text{ for } \phi \in C^{k,-m} \quad (2.32)$$

where as before

$$c_{k,m} = (-1)^{km + \frac{m^2-m}{2}} \quad (2.33)$$

Then it is easy to see that R' defines a map of complexes and $R' \circ I = id$, and hence R' and R induce the same map in cohomology.

We now consider an analogue of the notion of Connes' cycle over a differential graded algebra. Let (Ω^*, d) be a differential graded algebra. A cycle $\mathcal{C}$ of degree $p \in \mathbb{Z}$ over (Ω^*, d) is given by the following data:

(1) Bigraded bidifferential algebra $(\mathcal{C}^{*,*}, \delta, \bar{\delta})$, $\delta : \mathcal{C}^{i,j} \to \mathcal{C}^{i+1,j}$, $\bar{\delta} : \mathcal{C}^{i,j} \to \mathcal{C}^{i,j+1}$. Here δ, $\bar{\delta}$ must be differentials of $\mathcal{C}^{*,*}$ satisfying

$$\delta^2 = 0,\ \bar{\delta}^2 = 0,\ \delta\bar{\delta} + \bar{\delta}\delta = 0. \quad (2.34)$$

We use the following notation: for $\alpha \in \mathcal{C}^{i,j}$ $\deg \alpha = i + j$, $\deg' \alpha = i$, $\deg'' \alpha = j$, and $\partial = \delta + \bar{\delta}$. Notice that $\partial^2 = 0$, due to (2.34).

(2) Homomorphism of DG algebras

$$\rho : (\Omega^*, d) \to (\mathcal{C}^{0,*}, \bar{\delta}) \quad (2.35)$$

(3) A finite collection of linear functionals $\{\int_k\}$ on $\mathcal{C}^{k,-p+k}$, $k \geq \max\{0, p\}$

(extended by 0 to all $\mathcal{C}^{*,*}$) such that

$$\int_k \delta\alpha = 0 \text{ for all } k \tag{2.36}$$

$$\int_k [\alpha, \beta] = (-1)^{\deg \alpha} \int_{k+1} \bar{\delta}(\alpha\delta\beta) \text{ for } \alpha \in \mathcal{C}^{*,*}, \beta \in \mathcal{C}^{0,*} \tag{2.37}$$

With such a cycle one can associate its character, which is a p-cocycle in $\mathcal{B}((\Omega^*, d))$. Its component in $C^{k,p-k}$ is given by

$$\chi^k_{\mathcal{C}}(\omega_0, \omega_1, \dots, \omega_k) = (-1)^{\sum_{i=0}^{k}(k-i)\deg\omega_i} \int_k \rho(\omega_0)\delta\rho(\omega_1)\dots\delta\rho(\omega_k) \tag{2.38}$$

One verifies by a direct computation that

$$B\chi^k_{\mathcal{C}} = 0 \tag{2.39}$$

$$b\chi^k_{\mathcal{C}} + d\chi^{k+1}_{\mathcal{C}} = 0 \tag{2.40}$$

Hence $\{\chi^k_{\mathcal{C}}\}$ is indeed a cocycle in the complex $\mathcal{B}((\Omega^*, d))$. Note that this cocycle has a property of cyclicity, i.e. all of its component satisfy

$$\phi(\omega_0, \omega_1, \dots, \omega_k) = (-1)^{k+\deg\omega_k(\deg\omega_0+\dots\deg\omega_{k-1})}\phi(\omega_k, \omega_0, \dots, \omega_{k-1}) \tag{2.41}$$

One shows, as in Connes [3] (this material can also be found in [5], III.1.α) , that one can compute cyclic cohomology using the complex of cyclic cochains. Moreover, the proof from [3] shows that every cyclic cocycle is a character of a cycle in the above sense. Cyclic cocycles which are coboundaries can be interpreted as characters of vanishing cycles, defined as follows (as in [3]). We say that DG algebra (Ω^*, d) is *algebraically contractible* if there exist an automorphism p of (Ω^*, d) and an $X \in M_2(\Omega^*)$ with $dX = 0$ such that

$$X^{-1}\begin{pmatrix} a & 0 \\ 0 & p(a) \end{pmatrix} X = \begin{pmatrix} 0 & 0 \\ 0 & p(a) \end{pmatrix} \tag{2.42}$$

A cycle is called *vanishing* if $(\mathcal{C}^{0,*}, \bar{\delta})$ is algebraically contractible. Note that Connes' proof [3] (cf. also [5] p.191) applies without changes and shows that the cyclic cocycle is a coboundary if and only if it is a character of a vanishing cycle. Note also that if (Ω^*, d) is algebraically contractible, then Ω^0 is algebraically contractible as well (as an algebra, and not as DG algebra), and hence $HC^*(\Omega^0) = 0$.

Example 3 *We will now describe the map I introduced above in this setting. Let (Ω^*, d) be a DG algebra, and $\mathcal{A} = \Omega^0$. Let τ be a cyclic p-cocycle on $\mathcal{A}$. We will describe its image under the map $I : \mathcal{B}_{per}(\mathcal{A}) \to \mathcal{B}_{per}((\Omega^*, d))$ by an* *explicit cycle of degree p over the DG algebra (Ω^*, d). As the algebra $\mathcal{C}^{*,*}$ we take a universal DG algebra $\Omega^*(\Omega^*)$ of the algebra Ω^*, where we view Ω^* just as a graded algebra, and not DG algebra. (I.e. it is an algebra spanned by the expressions of the form $\alpha_0\delta\alpha_1\ldots\delta\alpha_r$, where $\alpha_q \in \Omega^*$. The product, as usual, is defined so that the relation $\delta(\alpha\alpha') = \delta(\alpha)\alpha' + (-1)^{\deg\alpha}\alpha\delta\alpha'$ holds, and the differential is defined by $\delta(\alpha_0\delta\alpha_1\ldots\delta\alpha_r) = \delta\alpha_0\delta\alpha_1\ldots\delta\alpha_r$). The bigrading is introduced by putting $\alpha_0\delta\alpha_1\ldots\delta\alpha_r \in \mathcal{C}^{i,j}$ if and only if $r = i$ and $\deg\alpha_0 + \deg\alpha_1 + \ldots \deg\alpha_r = j$. The differential δ is already defined. We define the graded differential $\bar{\delta}$ so that $\bar{\delta}\alpha = d\alpha$ for $\alpha \in \Omega^*$, and relations (2.34) hold, i.e.*

$$\bar{\delta}(\alpha_0\delta\alpha_1\ldots\delta\alpha_r) = d\alpha_0\delta\alpha_1\ldots\delta\alpha_r + \sum_{q=1}^{r}(-1)^{\deg\alpha_0+\ldots\deg\alpha_{q-1}+q}\alpha_0\delta\alpha_1\ldots\delta(d\alpha_q)\ldots\delta\alpha_r \quad (2.43)$$

Finally we define $\{\int_k\}$ as follows. $\int_k$ is nonzero only if $k = p$, and in this case it is a functional on $\mathcal{C}^{p,0}$. $\mathcal{C}^{p,0}$ is spanned by the elements of the form $a_0\delta a_1\ldots\delta a_p$, $a_i \in \mathcal{A}$, and we define

$$\int_p a_0\delta a_1\ldots\delta a_p = \tau(a_0, a_1, \ldots, a_p) \quad (2.44)$$

It is now straightforward to verify that the data above indeed defines a cycle over (Ω^*, d), whose character is $I\tau$.

Our goal now is to describe explicitly the image of the character of a cycle under the isomorphism $R : HP^*((\Omega^*, d)) \to HP^*(\mathcal{A})$, $\mathcal{A} = \Omega^0$.

Theorem 4 *Let the cyclic cocycle $\{\chi^k_{\mathcal{C}}\}$ be a character of a cycle $\mathcal{C}\left(\mathcal{C}^{*,*}, \delta, \bar{\delta}, \int\right)$. Then $[R\chi] \in HP^*(\mathcal{A})$ is the class of the cocycle in the (b, B) bicomplex given by*

$$\Phi^k_{\mathcal{C}}(a_0, a_1, \ldots a_{2k-p}) = \frac{k!}{(-p+2k+1)!}\sum_{i=0}^{2k-p}(-1)^{i(m-i)}\int_k \partial\rho(a_{i+1})\ldots\partial\rho(a_{2k-p})\rho(a_0)\ldots\partial\rho(a_i). \quad (2.45)$$

PROOF. First we note that $\Phi_{\mathcal{C}}$ is indeed a cocycle. This is verified by a direct computation, which can be found in [5], pp. 220-221. Next, we need to show that the cocycle $\Phi_{\mathcal{C}}$ constructed from the vanishing cycle $\mathcal{C}$ is cohomologous to 0. This do this notice that $\Phi_{\mathcal{C}}$ is a pull back of a cyclic cocycle from $\mathcal{C}^{0,0}$ via the map ρ. As $\mathcal{C}$ is a vanishing cycle, $HC^*(\mathcal{C}^{0,0}) = 0$, and the statement follows.

This shows that the correspondence $\{\chi^k_{\mathcal{C}}\} \to \Phi_{\mathcal{C}}$ is a well defined map $R'' : HC^*((\Omega^*, d)) \to HC^*(\mathcal{A})$. Notice that $R'' \circ I = id$, as follows from the description of the map I given in Example 3. This, together with the fact that $I : HC^*(\mathcal{A}) \to HC^*((\Omega^*, d))$ is an isomorphism, implies that R'' defines a map $HC^*((\Omega^*, d)) \to HC^*(\mathcal{A})$, which is inverse to I. As I is an isomorphism, R and R'' must coincide, and the statement of the theorem follows.

3 Cyclic complex for differential graded Hopf algebras

In this section we reproduce Connes-Moscovici's construction of the cyclic module of a Hopf algebra (cf. [6,7]) in the differential graded context.

Let us start with the graded Hopf algebra $\mathcal{H}^*$. We need to fix a modular pair, i.e. a homomorphism $\delta : \mathcal{H}^* \to \mathbb{C}$ and a group-like element $\sigma \in \mathcal{H}^0$. Using the standard notations for the coproduct and antipode, define the twisted antipode $\widetilde{S}_\delta$ by

$$\widetilde{S}_\delta(h) = \sum S(h_{(0)})\delta(h_{(1)}) \tag{3.1}$$

Suppose that the following condition holds:

$$\left(\sigma^{-1}\widetilde{S}_\delta\right)^2 = id \tag{3.2}$$

Then Connes and Moscovici show that one can define a cyclic object $(\mathcal{H}^*)^\sharp = \left\{(\mathcal{H}^*)^{\otimes n}\right\}_{n\geq 1}$ as follows. Face and degeneracy operators are given by

$$\begin{aligned}
\delta_0(h^1 \otimes \ldots \otimes h^{n-1}) &= 1 \otimes h^1 \otimes \ldots \otimes h^{n-1} \\
\delta_j(h^1 \otimes \ldots \otimes h^{n-1}) &= h^1 \otimes \ldots \otimes \Delta h^j \otimes \ldots \otimes h^n \text{ for } 1 \leq j \leq n-1, \\
\delta_n(h^1 \otimes \ldots \otimes h^{n-1}) &= h^1 \otimes \ldots \otimes h^{n-1} \otimes \sigma \\
\sigma_i(h^1 \otimes \ldots \otimes h^{n+1}) &= h^1 \otimes \ldots \otimes \varepsilon(h^{i+1}) \otimes \ldots \otimes h^{n+1}
\end{aligned} \tag{3.3}$$

The cyclic operators are given by

$$\begin{aligned}
&\tau_n(h^1 \otimes \ldots \otimes h^{n+1}) = \\
&\sum (-1)^{\sum\limits_{j>i\geq 0} \deg h_i^1 \deg h^j} \left(\widetilde{S}h^1\right)_{(0)} h^2 \otimes \cdots \otimes \left(\widetilde{S}h^1\right)_{(n-2)} h^n \otimes \left(\widetilde{S}h^1\right)_{(n-1)} \sigma
\end{aligned} \tag{3.4}$$

where

$$\left(\Delta^{n-1}\widetilde{S}h^1\right) = \sum \left(\widetilde{S}h^1\right)_{(0)} \otimes \ldots \left(\widetilde{S}h^1\right)_{(n-1)}$$

It is verified in [7] that the above operations indeed define a structure of a cyclic module.

Hence we can define cyclic and periodic cyclic complexes of this cyclic module. Suppose now that our Hopf algebra $\mathcal{H}^*$ is a DG Hopf algebra with the differential d of degree 1. Then complexes $\mathcal{B}$ and $\mathcal{B}_{\rm per}$ have an extra differential defined to be $(-1)^n d$ on $(\mathcal{H}^*)^{\otimes n}$, where we extend d by

$$d(h^1 \otimes h^2 \cdots \otimes h^n) = \sum_{i=1}^{n} (-1)^{\deg h^1 + \ldots \deg h^{i-1}} h^1 \otimes h^2 \ldots dh^i \cdots \otimes h^n \tag{3.5}$$

We consider the total complexes of the *finite* cochains in the resulting tricomplexes and define cyclic and periodic cyclic cohomology of DG Hopf algebra as cohomology of these complexes.

Suppose now that we are given an action π of a differential graded Hopf algebra $\mathcal{H}^*$ on Ω^*, which agrees with the differential graded structures on $\mathcal{H}^*$ and Ω^*, i.e. in addition to the general properties of Hopf algebra action we have

$$\deg \pi(h)(a) = \deg h + \deg a \tag{3.6}$$
$$d\left(\pi(h)(a)\right) = \pi(dh)(a) + (-1)^{\deg h} \pi(h)(da) \tag{3.7}$$

where $h \in \mathcal{H}^*$, $a \in \Omega^*$. We will often omit π from our notations and write just $h(a)$ if it is clear what action we are talking about.

Suppose that $⨍$ is a closed graded σ-trace on Ω^*, δ-invariant under the action of $\mathcal{H}^*$, i.e.

$$⨍ da = 0 \tag{3.8}$$
$$⨍ \pi(h)(a) b = ⨍ a \pi\left(\widetilde{Sh}\right) b \tag{3.9}$$
$$⨍ ab = ⨍ b \pi(\sigma)(a) \tag{3.10}$$

Then one has a map of cyclic modules $\chi_\pi : (\mathcal{H}^*)^\sharp \to (\Omega^*)^\sharp$, defined by

$$\chi_\pi(h^1 \otimes h^2 \cdots \otimes h^k)(a_0, a_1, \ldots a_k) = \lambda ⨍ a_0 \pi(h^1)(a_1) \ldots \pi(h^k)(a_k) \tag{3.11}$$

where

$$\lambda = (-1)^{\sum\limits_{j > i \geq 0} \deg h^j \deg a_i}$$

This map also commutes with the differential d, and hence induces a characteristic map $\chi_\pi : \mathcal{B}(\mathcal{H}^*, d) \to \mathcal{B}(\Omega^*, d)$, as well as corresponding maps in cohomology.

4 Properties of the characteristic map

In this section we show that the characteristic map in cohomology does not change if we twist the Hopf action by a cocycle (see (4.5) for the precise definition). We also discuss certain truncated version of cyclic complex for differential graded Hopf algebras.

We will consider now two actions of $\mathcal{H}^*$ on Ω^* which are conjugated by the inner automorphism (cf. [15]). We will work under the assumption that Ω^* is unital, indicating the changes which need to be made in the nonunital case in Remark 10. More precisely, let ρ^+ and ρ^- be two degree-preserving linear maps from $\mathcal{H}^*$ to Ω^*, which commute with the differentials. We suppose that they are inverse to each other with respect to convolution:

$$\sum \rho^+(h_{(0)})\rho^-(h_{(1)}) = \varepsilon(h)1 \tag{4.1}$$

and satisfy cocycle identities:

$$\rho^+(hg) = \sum \rho^+(h_{(0)})\pi(h_{(1)})(\rho^+(g)) \tag{4.2}$$
$$\rho^-(gh) = \sum \pi(h_{(0)})(\rho^-(g))\rho^-(h_{(1)}) \tag{4.3}$$
$$\rho^+(1) = \rho^-(1) = \rho^+(\sigma) = \rho^-(\sigma) = 1 \tag{4.4}$$

Then one can define a new action π' of $\mathcal{H}^*$ on Ω^* by

$$\pi'(h)(a) = \sum(-1)^{\deg h_{(2)} \deg a}\rho^+(h_{(0)})\pi(h_{(1)})(a)\rho^-(h_{(2)}) \tag{4.5}$$

Lemma 5 *Equation* (4.5) *defines an action of the DG Hopf algebra* $\mathcal{H}^*$ *on the DG algebra* Ω^*.

PROOF. We check that all the required properties are satisfied. First

$$\begin{aligned}
\pi'(h)(ab) = \sum(-1)^{\deg ab \deg h_{(2)}}\rho^+(h_{(0)})\pi(h_{(1)})(ab)\rho^-(h_{(2)}) = \\
\sum(-1)^{(\deg a+\deg b)\deg h_{(3)}}(-1)^{\deg a \deg h_{(2)}}\rho^+(h_{(0)})\pi(h_{(1)})(a)\pi(h_{(2)})(b)\rho^-(h_{(3)}) = \\
\sum(-1)^{(\deg a+\deg b)\deg h_{(5)}}(-1)^{\deg a(\deg h_{(2)}+\deg h_{(3)}+\deg h_{(4)})} \\
\rho^+(h_{(0)})\pi(h_{(1)})(a)\rho^-(h_{(2)})\rho^+h_{(3)}\pi(h_{(4)})(b)\rho^-(h_{(5)}) = \\
\sum(-1)^{\deg a \deg h_{(1)}}\pi'(h_{(0)})(a)\pi'(h_{(1)})(b)
\end{aligned} \tag{4.6}$$

Then

$$\pi'(hg)(a) = \\
\sum(-1)^{\deg a(\deg h_{(2)}+\deg g_{(3)})}\rho^{+}(h_{(0)}g_{(0)})\pi(h_{(1)}g_{(1)})(a)\rho^{-}(h_{(2)}g_{(2)}) = \\
\sum(-1)^{\deg a(\deg h_{(3)}+\deg g_{(2)}+\deg h_{(4)})} \\
\rho^{+}(h_{(0)})\pi(h_{(1)})(\rho^{+}(g_{(0)}))\pi(h_{(2)}g_{(1)})(a)\pi(h_{(3)})\rho^{-}(g_{(2)})\rho^{-}(h_{(4)}) = \\
\pi'(h)\left(\pi'(g)(a)\right) \tag{4.7}$$

Also

$$\pi'(h)(1) = \sum \rho^{+}(h_{(0)})h_{(1)}(1)\rho^{-}(h_{(3)}) = \varepsilon(h)1 \tag{4.8}$$
$$\pi'(1)(a) = \rho^{+}(1)a\rho^{-}(1) = a \tag{4.9}$$

and

$$d\left(\pi'(h)(a)\right) = d\sum(-1)^{\deg a \deg h_{(3)}}\rho^{+}(h_{(0)})\pi(h_{(1)})(a)\rho^{-}(h_{(2)}) = \\
\sum(-1)^{\deg a \deg h_{(3)}}\rho^{+}(dh_{(0)})\pi(h_{(1)})(a)\rho^{-}(h_{(2)}) + \\
\sum(-1)^{\deg a \deg h_{(3)}+\deg h_{(0)}} \\
\rho^{+}(h_{(0)})\left(\pi(dh_{(1)})(a) + (-1)^{\deg h_{(1)}}\pi(h_{(1)})(da)\right)\rho^{-}(h_{(2)}) + \\
\sum(-1)^{\deg a \deg h_{(3)}+\deg h_{(0)}+\deg h_{(1)}+\deg a}\rho^{+}(h_{(0)})\pi(h_{(1)})(a)\rho^{-}(dh_{(2)}) = \\
\pi'(dh)(a) + (-1)^{\deg h}\pi'(h)(da) \tag{4.10}$$

Suppose now that $⨍$ is the closed δ-invariant σ-trace for both actions π and π'. In this case we have two characteristic maps χ_π and $\chi_{\pi'}$ from $\mathcal{B}(\mathcal{H}^*, d)$ to $\mathcal{B}(\Omega^*, d)$. Then we have the following

Proposition 6 *Let π and π' be two actions of $\mathcal{H}^*$ on Ω^*, conjugated by inner automorphisms, and suppose that they both conditions* (3.9),(3.10) *are satisfied. Let χ_π, $\chi_{\pi'}$ be the corresponding characteristic maps. Then the induced maps in cohomology $HC^*(\mathcal{H}^*, d) \to HC^*(\Omega^*)$ coincide.*

PROOF. Let $M_2(\Omega^*) = \Omega^* \otimes M_2(\mathbb{C})$ be the differential graded algebra of 2×2 matrices over the algebra Ω^*.We can define an action π_2 of $\mathcal{H}^*$ on $\Omega^* \otimes M_2(\mathbb{C})$ by $\pi_2(h)(a \otimes m) = \pi(h)(a) \otimes m$, where $h \in \mathcal{H}^*$, $a \in \Omega^*$, $m \in M_2(\mathbb{C})$. Put now

$$\rho_2^{+}(h) = \begin{pmatrix} \rho^{+}(h) & 0 \\ 0 & \varepsilon(h) \end{pmatrix} \quad \rho_2^{-}(h) = \begin{pmatrix} \rho^{-}(h) & 0 \\ 0 & \varepsilon(h) \end{pmatrix} \tag{4.11}$$

It is easy to see that ρ_2^+, ρ_2^- satisfy equations (4.1)-(4.4), and hence we can twist the action π_2 by ρ_2^+, ρ_2^- to define a new action π_2', as in (4.5):

$$\pi_2'(h)(a\otimes m)=\sum(-1)^{\deg h_{(2)}\deg a}\rho_2^+(h_{(0)})\pi_2(h_{(1)})(a\otimes m)\rho_2^-(h_{(2)}) \tag{4.12}$$

Consider now the linear functional $⨍_2$ on $M_2(\Omega^*)$ defined by

$$⨍_2(a\otimes m)=\left(⨍a\right)(\operatorname{tr} m) \tag{4.13}$$

Then $⨍_2$ is a closed graded δ-invariant σ-trace on $M_2(\Omega^*)$ with respect to the action π_2'. Hence we can define the characteristic map $\chi_{\pi_2'}:\mathcal{B}(\mathcal{H}^*,d)\to\mathcal{B}\left(M_2(\Omega^*),d\right)$

Consider now two imbeddings i, $i':\Omega^*\hookrightarrow M_2(\Omega^*)$ defined by

$$i(a)=\begin{pmatrix}0&0\\0&a\end{pmatrix}\quad i'(a)=\begin{pmatrix}a&0\\0&0\end{pmatrix} \tag{4.14}$$

It is easy to see that $i^*\circ\chi_{\pi_2'}=\chi_\pi$ and $(i')^*\circ\chi_{\pi_2'}=\chi_{\pi'}$. Now to finish the proof it is enough to recall the well-known fact that i and i' induce the same map in cyclic cohomology. Since we will later need an explicit homotopy between χ_π and $\chi_{\pi'}$ we give the proof below.

Put $u_t=\exp\left(t\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right)=\begin{pmatrix}\cos t&-\sin t\\ \sin t&\cos t\end{pmatrix}$ We define now homotopy between i and i' by

$$i_t(a)=u_t i(a)u_t^{-1}=\begin{pmatrix}\cos t&-\sin t\\ \sin t&\cos t\end{pmatrix}\begin{pmatrix}0&0\\0&a\end{pmatrix}\begin{pmatrix}\cos t&\sin t\\ -\sin t&\cos t\end{pmatrix} \tag{4.15}$$

.

Notice that $i_0=i$, $i_{\pi/2}=i'$. Consider the family of maps $i_t^*:\mathcal{B}\left(M_2(\Omega^*)\right)\to\mathcal{B}\left(\Omega^*\right)$. Since we have $\frac{d}{dt}i_t(a)=[g,i_t(a)]$, where $g=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ these maps satisfy $\frac{d}{dt}i_t^*=i_t^*L_g$, where $L_g:C^k(M_2(\Omega^*))\to C^k(M_2(\Omega^*))$ is the operator defined by

$$L_g\phi(x_0,\ldots,x_k)=\sum_{j=0}^{k}\phi(x_0,\ldots,[g,x_j],\ldots,x_k)$$

Define also an operator $I_g : C^k(M_2(\Omega^*)) \to C^{k-1}(M_2(\Omega^*))$ by

$$I_g\phi(x_0,\ldots,x_{k-1}) = \sum_{j=0}^{k-1} \phi(x_0,\ldots,x_j,g,x_{j+1},\ldots,x_{k-1}) \tag{4.16}$$

Then it is easy to verify that $[b, I_g] = L_g$, $[B, I_g] = 0$ and $[d, I_g] = 0$. Hence $L_g = \partial I_g + I_g\partial$, $\partial = \pm d + b + B$. We conclude that $i_1^* - i_0^* = K\partial + \partial K$, where the homotopy K is given by $K\phi = \int_0^{\pi/2} i_t^* I_g dt$.

Hence

$$\chi_{\pi'} - \chi_\pi = \partial H + H\partial \tag{4.17}$$

where

$$\begin{aligned} H(h^1 \otimes h^2 \cdots \otimes h^k)(a_0, a_1, \ldots a_{k-1}) = \\ K\chi_{\pi_2'}(h^1 \otimes h^2 \cdots \otimes h^k)(a_0, a_1, \ldots a_{k-1}) = \\ = (-1)^{\sum\limits_{j>i\geq 0} \deg h^j \deg a_i} \sum_{j=0}^{k-1} \int_0^{\pi/2} ⨍ \operatorname{tr} i_t(a_0)\pi_2'(h^1)(i_t(a_1))\ldots \\ \pi_2'(h^j)(i_t(a_j))\pi_2'(h^{j+1})(g)\pi_2'(h^{j+2})(i_t(a_{j+1}))\ldots\pi_2'(h^k)(i_t(a_{k-1}))dt \end{aligned} \tag{4.18}$$

Now note that the complex $\mathcal{B}(\mathcal{H}^*)$ has a natural weight filtration by subcomplexes $F^l\mathcal{B}(\mathcal{H}^*, d)$, where

$$F^l\mathcal{B}(\mathcal{H}^*, d) = \{\alpha_1 \otimes \alpha_2 \cdots \otimes \alpha_j \mid \deg\alpha_1 + \deg\alpha_2 + \ldots \deg\alpha_j \geq l\} \tag{4.19}$$

Suppose now that $⨍$ has weight q, i.e.

$$⨍ a = 0 \text{ if } \deg a \neq q \tag{4.20}$$

Notice that in this case χ_π reduces the total degree by q Then following then proposition is clear:

Proposition 7 *The characteristic map is* 0 *on* $F^l\mathcal{B}(\mathcal{H}^*)$ *for* $l > q$.

Let $\mathcal{B}(\mathcal{H}^*, d)_l$ denote the truncated cyclic bicomplex:

$$\mathcal{B}(\mathcal{H}^*, d)_l = \mathcal{B}(\mathcal{H}^*, d) / F^{l+1}\mathcal{B}(\mathcal{H}^*, d) \tag{4.21}$$

Then we immediately have the following

Corollary 8 *The characteristic map* χ_π *defined in* (3.11) *induces the map from the complex* $\mathcal{B}(\mathcal{H}^*, d)_q$ *to the cyclic complex of the differential graded algebra* Ω^*.

This new map will also be denoted χ_π. We use the notation

$$HC^*(\mathcal{H}^*, d)_l = H^*\left(\mathcal{B}\left(\mathcal{H}^*, d\right)_l\right) \tag{4.22}$$

for the cohomology of the complex $\mathcal{B}\left(\mathcal{H}^*, d\right)_l$. The explicit form of the homotopy in the Proposition 6 now implies the following

Corollary 9 *Assume that in addition to the conditions of the Proposition 6 the property* (4.20) *holds. Then the two maps in cohomology induced by* $\chi_\pi, \chi_{\pi'} : \mathcal{B}\left(\mathcal{H}^*, d\right)_q \to \mathcal{B}\left(\Omega^*\right)$ *coincide.*

PROOF. We use the notations of the proof of the Proposition 6. There we established that $\chi_{\pi'} - \chi_\pi = \partial H + H\partial$. We need only to verify that H is well defined on the quotient complex $\mathcal{B}\left(\mathcal{H}^*, d\right)_q$. But since $H = K \circ \chi_{\pi_2'}$, and $\chi_{\pi_2'}$ is easily seen to be 0 on $F^{q+1}\mathcal{B}\left(\mathcal{H}^*, d\right)$, the result follows.

Remark 10 *We worked above in the assumption that the DG algebra* Ω^* *is unital. Here we will indicate which modifications should be made to treat the nonunital case. First of all,* $\rho^+(h)$, $\rho^-(h)$ *now don't have to be elements of the algebra, but rather multipliers, such that* π' *defined in* (4.5) *is a Hopf action. We need to require that if* m *is such a multiplier, then*

$$⨍ ma = ⨍ \pi(\sigma)(a)m \tag{4.23}$$

$\forall a \in \Omega^*$. *Then the the Proposition 6 remains true; characteristic maps in this case take values in the* $\mathcal{B}$ *complex of the algebra* Ω^* *with unit adjoined. To see this, we note that the action* π *of Hopf algebra* $\mathcal{H}^*$ *on the algebra* Ω^* *can be extended to the algebra of all multipliers satisfying* (4.23)*; if* ρ *is such a multiplier and* $h \in \mathcal{H}^*$ *we define* $\pi(h)(\rho)$ *by the formulas*

$$\pi(h)(\rho)a = \sum \pi(h_{(0)})\left(\rho\pi(S(h_{(1)})a\right) \tag{4.24}$$

$$a\pi(h)(\rho) = \sum \pi(h_{(1)})\left(\pi(S(h_{(0)})a\rho\right) \tag{4.25}$$

where $a \in \Omega^*$. *With this definition one can construct actions* π_2 *and* π_2' *exactly as before. Then homotopy between two characteristic maps is still given by explicit formula* (4.17)*, which continues to make sense in the nonunital situation. Actually one needs to use formulas* (4.24) *only to define* $\pi_2'(h)(g)$, *which is easily seen to be* $\begin{pmatrix} 0 & \rho^+(h) \\ -\rho^-(h) & 0 \end{pmatrix}$.

Finally, we collect all the information we will need to use in the next sections.

Theorem 11 *Let* (Ω^*, d) *be a differential graded algebra, and* $⨍$ *a linear functional on* Ω^* *of weight* q*, and let* $\mathcal{A} = \Omega^0$ *be the degree* 0 *part of* Ω^*. *Let* π

be an action of the DG Hopf algebra $\mathcal{H}^*$ act on the DGA Ω^*. Suppose that $⨍$ is a δ-invariant σ-trace with respect to π. Then characteristic map (3.11) *defines a map in cohomology $HC^i(\mathcal{H}^*)_q \to HP^{i-q}(\mathcal{A})$. Suppose now that π' is another action of $\mathcal{H}^*$ on Ω^*, obtained from π by twisting by a cocycle* (4.5). *Then if $⨍$ is a δ-invariant σ-trace with respect to π', the maps in cohomology $HC^i(\mathcal{H}^*)_q \to HP^{i-q}(\mathcal{A})$ induced by χ_π, $\chi_{\pi'}$ are the same.*

5 Secondary characteristic classes

Let M be a manifold, and let Γ be a discrete pseudogroup of diffeomorphisms of M, acting from the right.

By this we mean a set Γ such that every element g of Γ defines a local diffeomorphism of M, i.e. diffeomorphism $g : \operatorname{Dom} g \to \operatorname{Ran} g$, where $\operatorname{Dom} g$, $\operatorname{Ran} g \subset M$ – open subsets of M, and that we have partially defined operations of composition and inverse such that

(1) If $g \in \Gamma$ then $g^{-1} : \operatorname{Ran} g \to \operatorname{Dom} g$ is also in Γ.
(2) If $g_1, g_2 \in \Gamma$ then $g_1 g_2$ with domain $g_1^{-1}(\operatorname{Dom} g_2 \cap \operatorname{Ran} g_1)$ and range $g_2(\operatorname{Dom} g_2 \cap \operatorname{Ran} g_1)$ is in Γ.
(3) $id : M \to M$ is in Γ.

Note that we use a wide definition of pseudogroups, and do not include any saturation axioms.

Let E be a trivial vector bundle on M, equivariant with respect to the action of Γ. In other words, every $g \in \Gamma$ defines for every $x \in \operatorname{Dom} g$ a linear map $E_x \to E_{xg}$.

For the rest of the paper we suppose the following:

If g_1 and $g_2 \in \Gamma$ are such that they induce the same diffeomorphisms and the same action on the bundle, then $g_1 = g_2$.

With this data one can associate the following groupoid $\mathcal{G}$: the objects are the points of M and the morphisms $x \to y$, $x, y \in M$ are given by $g \in \Gamma$ such that $g(x) = y$, with the composition given by the product in Γ. Let $\mathcal{A}$ denote the convolution algebra of this groupoid, i.e. the cross-product $C_0^\infty(M) \rtimes \Gamma$. Let $\Omega^* = (\Omega^*(M) \rtimes \Gamma, d)$ denote the differential graded algebra of forms on $\mathcal{G}$ with the convolution product, where the differential d is the de Rham differential. We will use the usual cross-product notations ωU_g for the elements of this algebra, where $\omega \in \Omega^*(M)$, $g \in \Gamma$. Since Γ is, in general a pseudogroup, we suppose that

$$\operatorname{supp}\omega \subset \operatorname{Dom} g \tag{5.1}$$

Fix a trivialization of E. The action of Γ on the bundle defines then a homomorphism

$$h : \mathcal{G} \to GL_n(\mathbb{R}) \tag{5.2}$$

Let $\mathcal{H}\left(GL_n(\mathbb{R})\right)$ denote the differential graded Hopf algebra of the forms on $GL_n(\mathbb{R})$, with the product given exterior multiplication, coproduct, antipode and counit induced respectively by the product $GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, inverse $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ and the inclusion $1 \to GL_n(\mathbb{R})$. The differential is given by the de Rham differential on forms.

We now show that the map (5.2) allows one to define an action of $\mathcal{H}\left(GL_n(\mathbb{R})\right)$ on Ω^*.

Proposition 12 *The map $\mathcal{H}\left(GL_n(\mathbb{R})\right) \otimes \Omega^* \to \Omega^*$ given by*

$$\pi(\alpha)(\omega) = h^*(\alpha)\omega \tag{5.3}$$

where $\alpha \in \mathcal{H}\left(GL_n(\mathbb{R})\right)$, $\omega \in \Omega^*$ defines an action of the differential graded Hopf algebra $\mathcal{H}\left(GL_n(\mathbb{R})\right)$ on the differential graded algebra Ω^*.

PROOF. We have:

$$\pi(\alpha_1\alpha_2)(\omega) = h^*(\alpha_1\alpha_2)\omega = h^*(\alpha_1)h^*(\alpha_2)\omega = \pi(\alpha_1)\left(\pi(\alpha_2)(\omega)\right) \tag{5.4}$$

Next, if we write $\Delta\alpha = \sum_k \alpha_{(0)} \otimes \alpha_{(1)}$ we have:

$$\begin{aligned}
\pi(\alpha)(\omega_0\omega_1)(g) = h^*(\alpha)(g)\omega_0\omega_1(g) = h^*(\alpha)(g) \sum_{g_0 g_1 = g} \omega_0(g_0)\omega_1^{g_0}(g_1) = \\
\sum_{g_0 g_1 = g} \sum_k h^*(\alpha_{(0)})(g_0) h^*(\alpha_{(1)})^{g_0}(g_1)\omega_0(g_0)\omega_1^{g_0}(g_1) = \\
\sum_{g_0 g_1 = g} \sum_k (-1)^{\deg\omega_0 \deg\alpha_{(0)}} h^*(\alpha_{(0)})(g_0)\omega_0(g_0) h^*(\alpha_{(2)})(g_1)\omega_1^{g_0}(g_1) = \\
\sum_k (-1)^{\deg\omega_0 \deg\alpha_{(0)}} \pi(\alpha_{(0)})(\omega_0)\pi(\alpha_{(1)})(\omega_1)
\end{aligned} \tag{5.5}$$

Also, if M is compact the algebra Ω^* has a unit given by the function

$$e(g) = \begin{cases} 1 & \text{if } g \text{ is a unit} \\ 0 & \text{otherwise} \end{cases} \tag{5.6}$$

Then

$$\pi(\alpha)(e) = h^*(\alpha)e = \mathrm{e}(\alpha)e \tag{5.7}$$

Finally, we have

$$\begin{aligned} d(\pi(\alpha(\omega))) = d\left(h^*(\alpha)\omega\right) = \\ h^*(d\alpha)\omega + (-1)^{\deg\alpha} h^*(\alpha)d\omega = \pi((d\alpha))(\omega) + (-1)^{\deg\alpha}\pi(\alpha)(d\omega) \end{aligned} \tag{5.8}$$

We now have a natural inclusion $i : M \hookrightarrow \mathcal{G}$ as the space of units. Then we define a graded trace $⨍$ on Ω^* by

$$⨍\omega = \int\limits_M i^*\omega \tag{5.9}$$

Proposition 13 *The graded trace* $⨍$ *is closed under the de Rham differential and is invariant under the action of* $\mathcal{H}$, *i.e.*

$$⨍ d\omega = 0 \tag{5.10}$$

$$⨍\alpha(\omega) = \mathrm{e}(\alpha)⨍\omega \tag{5.11}$$

PROOF. The first identity is clear, the second follows from the fact that $h \circ i : M \to GL_n(\mathbb{R})$ is a constant map, taking the value 1.

Hence we have a characteristic map $\mathcal{B}_q\left(\mathcal{H}\left(GL_n(\mathbb{R})\right), d\right) \to \mathcal{B}\left(\Omega^*, d\right)$ where $q = \dim M$, which also gives us a map

$$\chi : HC_q^*\left(\mathcal{H}\left(GL_n(\mathbb{R})\right), d\right) \to HP^*\left(C_0^\infty(M) \rtimes \Gamma\right) \tag{5.12}$$

Definition of the action of $\mathcal{H}\left(GL_n(\mathbb{R})\right)$ on Ω^*, and hence the definition of the characteristic map given by (5.12) depends apriori on the choice of trivialization of E. However, as the following proposition in conjunction with corollary 9 shows, characteristic map is independent of the choice of trivialization.

Proposition 14 *Suppose we use another trivialization of* E *to define an action of* $\mathcal{H}\left(GL_n(\mathbb{R})\right)$ *on* Ω^*. *Then the two actions are conjugated by the inner automorphisms.*

PROOF. Let us chose another trivialization of the bundle E, and let $U(x)$ $x \in M$ be a transition matrix between the two bases of the fiber E_x. Then we have a new map $h' : \mathcal{G} \to GL_n(\mathbb{R})$, related to h by

$$h'(\gamma) = U(s(\gamma))h(\gamma)U^{-1}(r(\gamma)) \tag{5.13}$$

Let π' denote the action corresponding to the map h'.

Consider now the pull-back $U^* : \Omega^*(GL_n(\mathbb{R})) \to \Omega^*(M)$ as a map $\mathcal{H}\left(GL_n(\mathbb{R})\right) \to \Omega^*$, where we consider forms on M as the form on $\mathcal{G}$ which is 0 outside the space of units. When M is not compact, we obtain not an element in algebra, but rather a multiplier. Hence we see that if we define

$$\rho^+(\alpha) = U^*(\alpha) \tag{5.14}$$

and

$$\rho^-(\alpha) = (U^{-1})^*(\alpha) = \rho^+(S\alpha) \tag{5.15}$$

we will have

$$\pi'(\alpha)(\omega) = \sum (-1)^{\deg \alpha_{(2)} \deg \omega} \rho^+(\alpha_{(0)})\pi(\alpha_{(1)})(\omega)\rho^-(\alpha_{(2)}) \tag{5.16}$$

We can now summarize the results as follows.

Theorem 15 *Let Γ be a discrete pseudogroup acting on the manifold M of dimension q by orientation preserving diffeomorphisms. Let E be a Γ-equivariant trivial bundle of rank n on M. Let $\mathcal{H}\left(GL_n(\mathbb{R})\right)$ be the DG Hopf algebra of the differential forms on the Lie group $GL_n(\mathbb{R})$. Then we have a map*

$$\chi : HC^i_q(\mathcal{H}\left(GL_n(\mathbb{R})\right), d) \to HP^{i-q}\left(C_0^\infty(M) \rtimes \Gamma\right) \tag{5.17}$$

which is independent of the trivialization of E.

We will make this statement more explicit in the next section by identifying $HC^i_q(\mathcal{H}\left(GL_n(\mathbb{R})\right), d)$.

6 Relation with Weil algebras

In this section we use methods of [14], [17] and [8,9] to identify the cyclic cohomology $HC^*(\mathcal{H}\left(GL_n(\mathbb{R})\right), d)_q$. It turns out that

$$HC^i(\mathcal{H}\left(GL_n(\mathbb{R})\right), d)_q = \bigoplus_{m \geq 0} H^{i-2m}\left(W(\mathfrak{gl}_\mathfrak{n}, O_n)_q\right), \tag{6.1}$$

where $H^*(W(\mathfrak{gl}_\mathfrak{n}, O_n)_q)$ is the cohomology of the truncated Weil algebra (cf. [14]). As a matter of fact, one can work with the DG Hopf algebra $\mathcal{H}(G)$ of differential forms on any almost connected Lie group, and the result in this case is

$$HC^i(\mathcal{H}(G), d)_q = \bigoplus_{m \geq 0} H^{i-2m}\left(W(\mathfrak{g}, K)_q\right), \tag{6.2}$$

where K is the maximal compact subgroup of G. Computation of the Hochschild cohomology of this Hopf algebra is essentially contained in [14], and, with a little care using ideas from [8,9], one recovers the cyclic cohomology.

Let G be a Lie group with finitely many connected components. In the same manner as in the previous section we can define a differential graded Hopf algebra $\mathcal{H}(G)$ of differential forms on G. Let K be the maximal compact subgroup of G. We will now construct the map of complexes from the truncated relative Weil algebra $W(\mathfrak{g}, K)_q$ to the complex $\mathcal{B}_q(\mathcal{H}(G))$.

Let NG denote the simplicial manifold with $NG_p = \underbrace{G \times G \times \ldots G}_{p}$ The simplicial structure is given by the face maps

$$\partial_i(g_1, g_2, \ldots, g_k) = \begin{cases} (g_2, \ldots, g_k) & \text{if } i = 0 \\ (g_1, g_2, \ldots g_i g_{i+1}, \ldots, g_k) & \text{if } 1 \leq i \leq k-1 \\ (g_1, \ldots, g_{k-1}) & \text{if } i = k \end{cases} \tag{6.3}$$

and degeneracy maps

$$\sigma_i(g_1, g_2, \ldots, g_k) = (g_1, \ldots, g_{i-1}, 1, g_i, \ldots, g_k) \tag{6.4}$$

The geometric realization of this simplicial manifold is the classifying space BG. It is a union of manifolds $\Delta^p \times NG_p$, modulo the equivalence relation (cf. [9]). Note also that we have at our disposal an "integration over simplices" map $\int_\Delta$ from the de Rham complex of simplicial forms on BG to the simplicial-de Rham complex of NG.

We will also consider simplicial manifold $\bar{N}G$, with $\bar{N}G_p = \underbrace{G \times G \times \cdots \times G}_{p+1}$. The face and degeneracy maps are given by

$$\partial_i(g_0, g_1, g_2, \ldots, g_k) = (g_0, \ldots \hat{g}_i, \ldots, g_k) \tag{6.5}$$

$$\sigma_i(g_0, g_1, g_2, \ldots, g_k) = (g_0, \ldots, g_{i-1}, g_i, g_i, g_{i+1}, g_k) \tag{6.6}$$

The geometric realization of this simplicial manifold is EG. The map pr :

$\bar{N}G \to NG$ given by

$$pr(g_0, g_1, \dots, g_p) = (g_0 g_1^{-1}, g_1 g_2^{-1}, \dots, g_{p-1} g_p^{-1}) \tag{6.7}$$

defines a simplicial principal G-bundle $\mathrm{E}G \to \mathrm{B}G$. Simplicial manifolds $\bar{N}G$ and NG moreover have a cyclic structure, i.e. an action of the cyclic groups $\mathbb{Z}_{p+1}$ on the p-th component, which satisfy all the necessary relations with the face and degeneracy maps. The actions are given on $\bar{N}G$ by

$$\tau_p(g_0, g_1, \dots, g_p) = (g_1, g_2, \dots, g_p, g_0) \tag{6.8}$$

Since the maps τ_p are G-equivariant, they induce corresponding actions on NG:

$$\tau_p(g_1, g_2, \dots, g_p) = (g_2, g_3, \dots, g_p, (g_1 g_2 \cdots g_p)^{-1}) \tag{6.9}$$

We will identify the p-cochains for the Hopf algebra $\mathcal{H}(G)$ with the forms on NG_p. Under this identification the simplicial structure on the Hopf cochains corresponds to the one induced by the simplicial structure on NG, the de Rham differential on the Hopf cochains corresponds to the de Rham differential on NG, and the cyclic structure on the Hopf cochains is induced by the cyclic structure on NG. Filtration by the degree of the differential forms on the Hopf algebra cochains corresponds to the filtration by the degree of the differential forms on the manifold NG.

We will now construct the map μ from the complex $W(\mathfrak{g}, K)$ to the simplicial-de Rham complex of forms on NG, which preserves the filtration on these complexes, following [9]. We do it by constructing the map from $W(\mathfrak{g}, K)$ to the complex of simplicial forms on $\mathrm{B}G$, and then applying the integration map. The complex of simplicial forms on $\mathrm{B}G$ has a natural bigrading. Let θ be the Maurer-Cartan form on G. Let $p_i : \bar{N}G = G^{p+1} \to G$ be the projection on i-th component. Consider on $\mathrm{E}G_p$ the $\mathfrak{g}$-valued differential form $\omega = \sum t_i \theta_i$, where $\theta_i = p_i^* \theta$. It defines a simplicial connection in the bundle $\mathrm{E}G \to \mathrm{B}G$. The standard construction defines a differential graded algebra homomorphism ψ from $W(\mathfrak{g}, K)$ to the complex of K-basic simplicial forms on $\mathrm{E}G$, which we identify with forms on the space $\mathrm{E}G/K$. This space is a bundle over $\mathrm{B}G$ with fiber G/K. This bundle has a section, which can be explicitly described as follows. Since G/K has a natural structure of a manifold of nonpositive curvature, for any finite set of points x_0, x_1, ..., $x_k \in G/K$ one can construct a canonical simplex in G/K, i.e. a map $\sigma(x_0, x_1, \dots, x_k) : \Delta^k \to G/K$, with vertices x_0, x_1, ..., x_k, and this construction agrees with taking faces of a simplex, and is G-equivariant:

$$\sigma(gx_0, gx_1, \dots, gx_k)(t_0, t_1, \dots, t_k) = g\sigma(x_0, x_1, \dots, x_k)(t_0, t_1, \dots, t_k) \tag{6.10}$$

Denote by π the canonical projection $G \to G/K$. Then the section s is given by the simplicial map defined by the following formula, where we write just σ

for $\sigma(\pi(1), \pi(g_1), \pi(g_1 g_2), \ldots \pi(g_1 \ldots g_k)(t_0, t_1, \ldots, t_k)$:

$$s(g_1, g_2, \ldots, g_k; t_0, t_1, \ldots t_k) = \\ \left(\sigma, g_1^{-1}\sigma, (g_1 g_2)^{-1}\sigma, \ldots (g_1 \ldots g_k)^{-1}\sigma; t_0, t_1, \ldots t_k\right) \quad (6.11)$$

Consider now the map $s^* \circ \psi$. It is clearly a homomorphism from the differential graded algebra $W(\mathfrak{g}, K)$ to the differential graded algebra of simplicial forms on BG. We will show now that it preserves the filtrations on both algebras. First we need the following statement:

Lemma 16 *If ξ is a horizontal form on* EG *of the type (k, l),then $s^*\xi$ is also of the type (k, l).*

PROOF. Since ξ is horizontal, it can be written as a sum of the expressions of the form $fpr^*\zeta$, where f is a function, pr : E$G \to$ BG – projection, and ζ is a form on BG of the type (k, l). Then $s^*(fpr^*\zeta) = (s^*f)\zeta$ is also of the type (k, l).

Proposition 17 *The homomorphism $s^* \circ \psi$ agrees with the filtrations on the Weil algebra and on the forms on* BG.

PROOF. The curvature of the connection ω is a horizontal form Ω on BG, given by

$$\Omega = \sum dt_i\theta_i + \sum t_i d\theta_i + \sum_{i<j} t_i t_j [\theta_i, \theta_j]. \quad (6.12)$$

Hence Ω has only components of the type $(1, 1)$ and $(0, 2)$. The statement of the lemma will then follow from the fact that $s^*\Omega$ also has only components of the type $(1, 1)$ and $(0, 2)$. But this follows from the Lemma 16.

We can average the map s^* with respect to the action of symmetric groups on simplicial manifolds EG/K and BG to obtain a map $\widetilde{s^*}$. We can now apply the integration map and define the map μ from the Weil algebra to the simplicial-de Rham complex of NG as $\mu = \int_\Delta \circ \widetilde{s^*} \circ \psi$. Since the integration map respects filtrations, the resulting map μ also respects filtrations. We identify the Hochschild complex of $\mathcal{H}(G)$ with the simplicial-de Rham complex of NG. Results of [14], [17] imply that this is actually an isomorphism, i.e.

$$HH^i(\mathcal{H}(G), d)_q = H^i\left(W(\mathfrak{g}, K)_q\right) \quad (6.13)$$

Notice that the resulting Hochschild cochains are actually cylcic (they are graded antisymmetric). This implies that Connes' long exact sequence is equivalent to the collection of short exact sequnces

$$0 \to HC^{i-2}(\mathcal{H}(G), d)_q \xrightarrow{S} HC^i(\mathcal{H}(G), d)_q \xrightarrow{I} \\ \xrightarrow{I} HH^i(\mathcal{H}(G), d)_q \to 0, \quad (6.14)$$

and the map I splits. Hence

$$HC^i(\mathcal{H}(G), d)_q = \\ \bigoplus_{m \geq 0} HH^{i-2m}(\mathcal{H}(G), d)_q = \bigoplus_{m \geq 0} H^{i-2m}\left(W(\mathfrak{g}, K)_q\right). \quad (6.15)$$

Explicitly, maps $H^{i-2m}\left(W(\mathfrak{g}, K)_q\right) \to HC^i(\mathcal{H}(G), d)_q$ are given by $S^m \circ \mu$, where we consider μ as a map into the cyclic complex.

We can now formulate a more explicit version of the Theorem 15.

Theorem 18 *Let Γ be a discrete pseudogroup acting on the manifold M of dimension q by orientation preserving diffeomorphisms. Let E be a Γ-equivariant trivial bundle of rank n on M. Then our previous constructions define a map*

$$\chi : \bigoplus_{m \in \mathbb{Z}} H^{i-2m}\left(W(\mathfrak{gl}_\mathfrak{n}, O_n)_q\right) \to HP^{i-q}\left(C_0^\infty(M) \rtimes \Gamma\right) \quad (6.16)$$

7 Relation with other constructions

Suppose, as before, that we have an orientation-preserving action of a discrete group Γ on an oriented manifold M, and an equivariant trivial bundle E over M. Then results from previous sections provide us a map $H^*(W(\mathfrak{g}, O_n)) \to HP^*(\mathcal{A})$, where $\mathcal{A} = C_0^\infty(M) \rtimes \Gamma$. We also have a construction of the map $H^*(W(\mathfrak{g}, O_n)) \to H^*(M_\Gamma)$ (see e.g. [14,1,2]), where $M_\Gamma = M \times_\Gamma \mathrm{E}\Gamma$ is the homotopy quotient when Γ is a group, or more generally, M_Γ is the classifying space $\mathrm{B}\mathcal{G}$ of the groupoid constructed from the action of Γ on M. In this section we prove that these constructions are compatible, i.e. that the following diagram is commutative

$$\begin{tikzcd} H^*(W(\mathfrak{g}, O_n)) \arrow[r] \arrow[dr] & H^*(M_\Gamma) \arrow[d, "\Phi"'] \\ & HP^*(\mathcal{A}) \end{tikzcd} \quad (7.1)$$

where Φ is the canonical map given by Connes [4,5].

The proof goes as follows. We construct a map Ψ from the complex computing $H^*(M_\Gamma)$ to the cyclic complex $\mathcal{B}(\Omega^*, d)$, where $\Omega^* = \Omega_0^*(M) \rtimes \Gamma$, which has the following properties. First, it agrees with the map Φ, in the sense that the following diagram is commutative:

$$\begin{tikzcd} H^*(M_\Gamma) \arrow[r, "\Phi"] \arrow[dr, "\Psi"'] & HP^*(\mathcal{A}) \\ & HP^*(\Omega^*, d) \arrow[u, "R"'] \end{tikzcd} \tag{7.2}$$

where the map R is defined by (2.22). Then it is clear from the definitions that the diagram similar to (7.1) is valid with the map Ψ already on the level of cochains, not just cohomology.

The definition of the map Ψ is the following. Recall that the cohomology of M_Γ can be computed by the following bicomplex $\mathcal{C}^{*,*}$. $C^{k,l}$ denotes the set of totally antisymmetric functions τ on $\underbrace{\Gamma \times \Gamma \cdots \times \Gamma}_{k+1}$ with values in $-l$-currents on $\mathrm{Dom}\, g_0 \cap \mathrm{Dom}\, g_1 \cdots \cap \mathrm{Dom}\, g_k$, which satisfy the invariance condition

$$\tau(gg_0, gg_1, \ldots, gg_k) = \tau(g_0, g_1, \ldots, g_k)^{g^{-1}}. \tag{7.3}$$

The two differentials of this complex are given by the group cohomology complex differential given on $C^{k,l}$ by

$$(d_1\tau)(g_0, g_1, \ldots, g_k, g_{k+1}) = (-1)^l \sum_{j=0}^{k+1} (-1)^j \tau(g_0, g_1, \ldots, \hat{g}_j, \ldots, g_{k+1}) \tag{7.4}$$

and the de Rham differential d given by

$$(d_2\tau)(g_0, g_1 \ldots, g_k) = d\,(\tau(g_0, g_1, \ldots, g_k)) \tag{7.5}$$

We now define the map Ψ from the complex $C^{*,*}$ to the cyclic complex $\mathcal{B}(\Omega^*, d)$, where $\Omega^* = \Omega_0^*(M) \rtimes \Gamma$, by the following formula.

$$\Psi(\tau)(\omega_0 U_{g_0}, \omega_1 U_{g_1}, \ldots, \omega_k U_{g_k}) = \begin{cases} (-1)^{kl} \langle \tau(1, g_0, g_0 g_1, \ldots, g_0 \ldots g_{k-1}), \omega_0 \omega_1^{g_0} \ldots \omega_k^{g_0 \ldots g_{k-1}} \rangle & \text{if } g_0 \ldots g_k = 1 \\ 0 & \text{otherwise} \end{cases} \tag{7.6}$$

Map Ψ satisfies the following identities

$$b\Psi(\tau) = \Psi(d_1\tau) \tag{7.7}$$
$$d\Psi(\tau) = \Psi(d_2\tau) \tag{7.8}$$
$$B\Psi(\tau) = 0 \tag{7.9}$$

and hence it is indeed a map of complexes. It is clear from the definition of the map Ψ that the diagram obtained from the diagram (7.1) by replacing Φ by Ψ commutes, even on the level of complexes. It remains to prove that the map Ψ induces the same map in cohomology as the map Φ.

Theorem 19 *The maps Φ and $R \circ \Psi$ from $H^*(M_\Gamma)$ to $HP^*(C_0^\infty \rtimes \Gamma)$ coincide.*

PROOF. This statement follows easily from Theorem 4. Indeed, if $\{\tau_k\}$ is a cocycle in the d_1, d_2 bicompolex. The algebra $(\mathcal{C}^{*,*}, d', d'')$ that Connes defined in construction of the map Φ defines a cycle over the differential graded algebra $\Omega^*(M) \rtimes \Gamma$, with

$$\int \omega \otimes \delta_{g_1} \ldots \delta_{g_k} = \langle \tau_k(1, g_1, \ldots, g_k), \omega \rangle \tag{7.10}$$

It is clear that $\Psi(\tau)$ is the character of this cycle, and the result is now immediate from Theorem 4.

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