# DEFORMATIONS OF AZUMAYA ALGEBRAS

PAUL BRESSLER, ALEXANDER GOROKHOVSKY, RYSZARD NEST, AND BORIS TSYGAN

## 1. Introduction.

In this paper we compute the deformation theory of a special class of algebras, namely of Azumaya algebras on a manifold ($C^\infty$ or complex analytic).

Deformation theory of associative algebras was initiated by Gerstenhaber in [G]. A deformation of an associative algebra $A$ over an Artinian ring $\mathfrak{a}$ is an $\mathfrak{a}$-linear associative algebra structure on $A \otimes \mathfrak{a}$ such that, for the maximal ideal $\mathfrak{m}$ of $\mathfrak{a}$, $A \otimes \mathfrak{m}$ is an ideal, and the quotient algebra on $A$ is the original one. Gerstenhaber showed that the Hochschild cochain complex of an associative algebra $A$ has a structure of a differential graded Lie algebra (DGLA), and that deformations of $A$ over an Artinian ring $\mathfrak{a}$ are classified by Maurer-Cartan elements of the DGLA $C^\bullet(A, A)[1] \otimes \mathfrak{m}$. A Maurer-Catan element of a DGLA $\mathcal{L}^\bullet$ with the differential $\delta$ is by definition an element $\lambda$ of $\mathcal{L}^1$ satisfying

$$\delta\lambda + \frac{1}{2}[\lambda, \lambda] = 0 \tag{1.0.1}$$

Isomorphic deformations correspond to equivalent Maurer-Cartan elements, and vice versa.

In subsequent works [GM], [D], [SS], [Dr] it was shown that deformation theories of many other objects are governed by appropriate DGLAs in the same sense as above. Moreover, if two DGLAs are quasi-isomorphic, then there is a bijection between the corresponding sets of equivalence classes of Maurer-Cartan elements. Therefore, to prove that deformation theories of two associative algebras are isomorphic, it is enough to construct a chain of quasi-isomorphisms of DGLAs whose endpoints are the Hochschild complexes of respective algebras.

This is exactly what is done in this paper. We construct a canonical isomorphism of deformation theories of two algebras: one is an Azumaya algebra on a $C^\infty$ manifold $X$, the other the algebra of $C^\infty$ functions on $X$. The systematic study of the deformation theory of the

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latter algebra was initiated in [BFFLS]. In addition to the definition above, it is required that the multiplication on $A \otimes \mathfrak{a}$ be given by bidifferential expressions. The corresponding Hochschild cochain complex consists of multi-differential maps $C^\infty(X)^{\otimes n} \to C^\infty(X)$. The complete classification of deformations of $C^\infty(X)$ is known from the formality theorem of Kontsevich [K1]. This theorem asserts that there is a chain of quasi-isomorphisms connecting the two DGLAs $C^\bullet(C^\infty(X), C^\infty(X))[1]$ and $\Gamma(X, \wedge^\bullet(TX))[1]$, the latter being the DGLA of multivector fields with the Schouten-Nijenhuis bracket. For the proofs in the case of a general manifold, cf. also [Do], [Ha], [DTT].

An Azumaya algebra on a manifold $X$ is a sheaf of algebras locally isomorphic to the algebra of $n \times n$ matrices over the algebra of functions. Such an algebra determines a second cohomology class with values in $\mathcal{O}_X^\times$ (here, as everywhere in this paper, in the $C^\infty$ case $\mathcal{O}_X$ denotes the sheaf of smooth functions). This cohomology class $c$ is necessarily $n$-torsion, i.e. $nc = 0$ in $H^2(X, \mathcal{O}_X^\times)$. The main result of this paper is that there is a chain of quasi-isomorphisms between the Hochschild complexes of multidifferential cochains of the algebra of functions and of an arbitrary Azumaya algebra. This chain of quasi-isomorphisms does depend on some choices but is essentially canonical. More precisely, what we construct is a canonical isomorphism in the derived category [H] of the closed model category [Q] of DGLAs.

Note that in [BGNT] we considered a related problem of deformation theory. Namely, any cohomology class in $H^2(X, \mathcal{O}_X^\times)$, whether torsion or not, determines an isomorphism class of a gerbe on $X$, cf. [Br]. A gerbe is a partial case of an algebroid stack; it is a sheaf of categories on $X$ satisfying certain properties. In [BGNT] we showed that deformation theory of algebroid stacks is governed by a certain DGLA. If an algebroid stack is a gerbe, we constructed a chain of quasi-isomorphisms between this DGLA and another one, much more closely related to the Hochschild complex. When the gerbe is an Azumaya algebra, this latter DGLA is just the Hochschild complex itself. In ther words, we solved a different deformation problem for an Azumaya algebra, and got the same answer.

Now let us turn to the case of a complex analytic manifold $X$. In this case, let $\mathcal{O}_X$ stand for the sheaf of algebras of holomorphic functions. It is natural to talk about deformations of this sheaf of algebras. Multidifferential multiholomorphic Hochschild cochains form a sheaf, and it can be shown that the corresponding deformation theory is governed by the DGLA of Dolbeault forms $\Omega^{0,\bullet}(X, C^{\bullet>0}(\mathcal{O}_X, \mathcal{O}_X)[1])$. The full DGLA $\Omega^{0,\bullet}(X, C^\bullet(\mathcal{O}_X, \mathcal{O}_X)[1])$ governs deformations of $\mathcal{O}_X$ as an algebroid stack. We show that our construction of a chain of quasi-isomorphisms can be carried out for this full DGLA in the holomorphic case, or in a more general case of a real manifold with a complex integrable distribution. This construction does not seem to work for the DGLA governing deformations of $\mathcal{O}_X$ as a sheaf of algebras. As shown in [NT] and [BGNT], such deformation theory can be in general more complicated.

Our motivations for studying deformations of gerbes and Azumaya algebras are the following:

1) A fractional index theorem from [MMS]. The algebra of pseudo-differential operators which is used there is closely related to formal deformations of Azumaya algebras.

2) Index theory of Fourier integral operators (FIOs). Guillemin and Sternberg [GS] have studied FIOs associated to a coisotropic submanifold of a cotangent bundle. It appears that higher index theorems for such operators are related to algebraic index theorems [BNT] for deformations of the trivial gerbe on a symplectic manifold with an étale groupoid. A similar algebraic index theorem in the holomorphic case should help establish a Riemann-Roch theorem in the setting of [KS], [PS].

3) Dualities between gerbes and noncommutative spaces ([MR1], [MR2], [MR3], [Bl], [BBP], [Ka]).

Hochschild and cyclic homology of Azumaya algebras were computed in [CW, S] and in the more general case of continuous trace algebras (in the cohomological setting) in [MS]. Here we require, however, much more precise statement which involves the whole Hochschild complex as a differential graded Lie algebra, rather then just its cohomology groups. It is conjectured in [S] that algebras which are similar in the sense of [S] have the same deformation theories. Some of our results can be considered as a verification of this conjecture in the particular case of Azumaya algebras.

A very recent preprint [Do1] contains results which have a significant overlap with ours. Namely, a chain of quasi-isomorphisms similar to ours is established in a partial case when the Azumaya algebra is an algebra of endomorphisms of a vector bundle. On the other hand, a broader statement is proven, namely that the above chain of quasi-isomorphisms extends to Hochschild chain complexes viewed as DGL modules over DGLAs of Hochschild cochains. This work, like ours, is motivated by problems of index theory.

## 2. Azumaya algebras.

Let $X$ be a smooth manifold. In what follows we will denote by $\mathcal{O}_X$ the sheaf of complex-valued $C^\infty$ functions on $X$.

**Definition 1.** An *Azumaya algebra* on $X$ is a sheaf of central $\mathcal{O}_X$-algebras locally isomorphic to $\mathtt{Mat}_n(\mathcal{O}_X)$.

Thus, by definition, the unit map $\mathcal{O}_X \hookrightarrow \mathcal{A}$ takes values in the center of $\mathcal{A}$.

Let $\mathcal{A}_0 := [\mathcal{A}, \mathcal{A}]$ denote the $\mathcal{O}_X$-submodule generated by the image of the commutator map.

We will now consider $\mathcal{A}$, $\mathcal{A}_0$ and $\mathcal{O}_X$ as Lie algebras under the commutator bracket. Note that the bracket on $\mathcal{A}$ takes values in $\mathcal{A}_0$ and the latter is a Lie ideal in $\mathcal{A}$.

**Lemma 2.** *The composition $\mathcal{O}_X \hookrightarrow \mathcal{A} \to \mathcal{A}/\mathcal{A}_0$ is an isomorphism.*

*Proof.* The issue is local so we may assume that $\mathcal{A} = \mathtt{Mat}_n(\mathcal{O}_X)$ in which case it is well known to be true. □

**Corollary 3.** *The map $\mathcal{O}_X \oplus \mathcal{A}_0 \to \mathcal{A}$ induced by the unit map and the inclusion is an isomorphism of Lie algebras.*

**Lemma 4.** *The sequence*

$$0 \to \mathcal{O}_X \to \mathcal{A} \stackrel{\mathrm{ad}}{\to} \mathtt{Der}_{\mathcal{O}_X}(\mathcal{A}) \to 0$$

*is exact. Moreover, the composition $\mathcal{A}_0 \hookrightarrow \mathcal{A} \stackrel{\mathrm{ad}}{\to} \mathtt{Der}_{\mathcal{O}_X}(\mathcal{A})$ is an isomorphism of Lie algebras.*

Let $\mathcal{C}(\mathcal{A})$ denote the sheaf of (locally defined) connections on $\mathcal{A}$ with respect to which the multiplication on $\mathcal{A}$ is horizontal; equivalently, such a connection $\nabla$ satisfies the Leibniz rule $\nabla(ab) = \nabla(a)b + a\nabla(b)$ in $\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{A}$ for all (locally defined) $a, b \in \mathcal{A}$. The sheaf $\mathcal{C}(\mathcal{A})$ is a torsor under $\Omega^1_X \otimes_{\mathcal{O}_X} \mathtt{Der}_{\mathcal{O}_X}(\mathcal{A})$.

Any connection $\nabla \in \mathcal{C}(\mathcal{A})$ satisfies $\nabla(\mathcal{A}_0) \subset \Omega^1_X \otimes \mathcal{A}_0$.

For $\nabla \in \mathcal{C}(\mathcal{A})$ there exists a unique $\theta = \theta(\nabla) \in \Omega^2_X \otimes \mathcal{A}_0$ such that $\mathrm{ad}(\theta) = \nabla^2 \in \Omega^2_X \otimes \mathtt{Der}_{\mathcal{O}_X}(\mathcal{A})$.

## 3. Jets

Let $\mathcal{J}_X$ be the sheaf of infinite jets of smooth functions on $X$. Let $p : \mathcal{J}_X \to \mathcal{O}_X$ denote the canonical projection. Suppose now that $\mathcal{A}$ is an Azumaya algebra. Let $\mathcal{J}(\mathcal{A})$ denote the sheaf of infinite jets of $\mathcal{A}$. Let $p_{\mathcal{A}} : \mathcal{J}(\mathcal{A}) \to \mathcal{A}$ denote the canonical projection. The sheaves $\mathcal{J}(\mathcal{A})$ and $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{J}_X$ have canonical structures of sheaves of central $\mathcal{J}_X$-algebras locally isomorphic to $\mathtt{Mat}_n(\mathcal{J}_X)$.

Let $\underline{\mathrm{Isom}}_0(\mathcal{A}\otimes\mathcal{J}_X, \mathcal{J}(\mathcal{A}))$ denote the sheaf of (locally defined) $\mathcal{J}_X$-algebra isomorphisms $\mathcal{A}\otimes\mathcal{J}_X \to \mathcal{J}(\mathcal{A})$ such that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{A}\otimes\mathcal{J}_X \arrow[r] \arrow[d, "\mathtt{Id}\otimes p"'] & \mathcal{J}(\mathcal{A}) \arrow[d, "p_{\mathcal{A}}"] \\ \mathcal{A} \arrow[r, "\mathtt{Id}"] & \mathcal{A} \end{tikzcd}$$

Similarly denote by $\underline{\mathrm{Aut}}_0(\mathcal{A}\otimes\mathcal{J}_X)$ the sheaf of (locally defined) $\mathcal{J}_X$-algebra automorphisms of $\mathcal{A}\otimes\mathcal{J}_X$ such that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{A}\otimes\mathcal{J}_X \arrow[r] \arrow[d, "\mathtt{Id}\otimes p"'] & \mathcal{A}\otimes\mathcal{J}_X \arrow[d, "\mathtt{Id}\otimes p"] \\ \mathcal{A} \arrow[r, "\mathtt{Id}"] & \mathcal{A} \end{tikzcd}$$

**Lemma 5.** *The sheaf* $\underline{\mathrm{Isom}}_0(\mathcal{A}\otimes\mathcal{J}_X, \mathcal{J}(\mathcal{A}))$ *is a torsor under the sheaf of groups* $\underline{\mathrm{Aut}}_0(\mathcal{A}\otimes\mathcal{J}_X)$.

*Proof.* Since both $\mathcal{J}(\mathcal{A})$ and $\mathcal{A}\otimes_{\mathcal{O}_X}\mathcal{J}_X$ are locally isomorphic to $\mathtt{Mat}_n(\mathcal{J}_X)$, the sheaf $\underline{\mathrm{Isom}}_0(\mathcal{A}\otimes\mathcal{J}_X, \mathcal{J}(\mathcal{A}))$ is locally non-empty, hence a torsor. □

**Lemma 6.** *The sheaf of groups* $\underline{\mathrm{Aut}}_0(\mathcal{A}\otimes\mathcal{J}_X)$ *is soft.*

*Proof.* Let $\mathtt{Der}_{\mathcal{J}_X,0}(\mathcal{A}\otimes\mathcal{J}_X)$ denote the sheaf of $\mathcal{J}_X$-linear derivations of the algebra $\mathcal{A}\otimes\mathcal{J}_X$ which reduce to the zero map modulo $\mathcal{J}_0$. The exponential map

$$\exp : \mathtt{Der}_{\mathcal{J}_X,0}(\mathcal{A}\otimes\mathcal{J}_X) \to \underline{\mathrm{Aut}}_0(\mathcal{A}\otimes\mathcal{J}_X)\ , \tag{3.0.2}$$

$\delta \mapsto \exp(\delta)$, is an isomorphism of sheaves (the inverse map is given by $a \mapsto \log a = \sum_{n=1}^{\infty} \frac{(a-1)^n}{n}$). Therefore, it suffices to show that the sheaf $\mathtt{Der}_{\mathcal{J}_X,0}(\mathcal{A}\otimes\mathcal{J}_X)$ is soft, but this is clear since it is a module over the sheaf $\mathcal{O}_X$ of $C^\infty$-functions. □

**Corollary 7.** *The torsor* $\underline{\mathrm{Isom}}_0(\mathcal{A}\otimes\mathcal{J}_X, \mathcal{J}(\mathcal{A}))$ *is trivial, i.e.* $\mathrm{Isom}_0(\mathcal{A}\otimes\mathcal{J}_X), \mathcal{J}(\mathcal{A})) := \Gamma(X; \underline{\mathrm{Isom}}_0(\mathcal{A}\otimes\mathcal{J}_X, \mathcal{J}(\mathcal{A}))) \neq \emptyset$.

*Proof.* Since the sheaf of groups $\underline{\mathrm{Aut}}_0(\mathcal{A}\otimes\mathcal{J}_X)$ is soft we have $H^1(X, \underline{\mathrm{Aut}}_0(\mathcal{A}\otimes\mathcal{J}_X) = 1$ ([DD], Lemme 22, cf. also [Br], Proposition 4.1.7). Therefore every $\underline{\mathrm{Aut}}_0(\mathcal{A}\otimes\mathcal{J}_X)$ torsor is trivial. □

In what follows we will use $\nabla^{can}_{\mathcal{E}}$ to denote the canonical flat connection on $\mathcal{J}(\mathcal{E})$. A choice of $\sigma \in \mathrm{Isom}_0(\mathcal{A}\otimes\mathcal{J}_X, \mathcal{J}(\mathcal{A}))$ induces the flat connection $\sigma^{-1}\circ\nabla^{can}_{\mathcal{A}}\circ\sigma$ on $\mathcal{A}\otimes\mathcal{J}_X$.

A choice of $\nabla \in \mathcal{C}(\mathcal{A})(X)$ give rise to the connection $\nabla\otimes\mathtt{Id} + \mathtt{Id}\otimes\nabla^{can}_{\mathcal{O}}$ on $\mathcal{A}\otimes\mathcal{J}_X$.

**Lemma 8.** (1) *For any* $\sigma \in \mathrm{Isom}_0(\mathcal{A}\otimes\mathcal{J}_X, \mathcal{J}(\mathcal{A}))$, $\nabla \in \mathcal{C}(\mathcal{A})(X)$, *the difference*

$$\sigma^{-1}\circ\nabla^{can}_{\mathcal{A}}\circ\sigma - (\nabla\otimes\mathtt{Id} + \mathtt{Id}\otimes\nabla^{can}_{\mathcal{O}}) \tag{3.0.3}$$

*is* $\mathcal{J}_X$*-linear.*

(2) *There exists a unique* $F \in \Gamma(X; \Omega^1_X \otimes \mathcal{A}_0 \otimes \mathcal{J}_X)$ *(depending on* $\sigma$ *and* $\nabla$*) such that* (3.0.3) *is equal to* $\mathrm{ad}(F)$.

(3) *Moreover,* $F$ *satisfies*

$$(\nabla\otimes\mathtt{Id} + \mathtt{Id}\otimes\nabla^{can}_{\mathcal{O}})F + \frac{1}{2}[F,F] + \theta = 0 \tag{3.0.4}$$

*Proof.* We leave the verification of the first claim to the reader. If follows that (3.0.3) is a global section of $\Omega^1_X \otimes \mathtt{Der}_{\mathcal{J}_X}(\mathcal{A}\otimes\mathcal{J}_X)$. Since the map $\mathcal{A}_0\otimes\mathcal{J}_X \xrightarrow{\mathrm{ad}\otimes\mathtt{Id}} \mathtt{Der}_{\mathcal{J}_X}(\mathcal{A}\otimes\mathcal{J}_X)$ is an isomorphism the second claim follows.

We have

$$\sigma^{-1}\circ\nabla^{can}\circ\sigma = \nabla\otimes\mathtt{Id} + \mathtt{Id}\otimes\nabla^{can}_{\mathcal{O}} + \mathrm{ad}\,F \tag{3.0.5}$$

where $F \in \Gamma(X; \Omega^1_X \otimes \mathcal{A}\otimes\mathcal{J}_X)$.

Since $(\sigma^{-1}\circ\nabla^{can}\circ\sigma)^2 = 0$ the element $(\nabla^{\mathcal{A}}\otimes\mathtt{Id} + \mathtt{Id}\otimes\nabla^{can})F + \frac{1}{2}[F,F] + \theta$ must be central; it also must lie in $\mathcal{A}_0\otimes\mathcal{J}$. Therefore, it vanishes, i.e. the formula (3.0.4) holds. □

## 4. Deformations of Azumaya algebras.

4.1. **Review of formal deformation theory.** Consider a DGLA $\mathcal{L}^\bullet$ with the differential $\delta$. A Maurer-Catan element of $\mathcal{L}^\bullet$ is by definition an element $\lambda$ of $\mathcal{L}^1$ satisfying

$$\delta\lambda + \frac{1}{2}[\lambda,\lambda] = 0 \tag{4.1.1}$$

Now assume that $\mathcal{L}^0$ is nilpotent. Then $\exp(\mathcal{L}^0)$ is an algebraic group over the ring of scalars $k$. This group acts on the set of Maurer-Cartan elements via

$$e^X(\lambda) = \mathrm{Ad}_{e^X}(\lambda) - \sum_{n=0}^{\infty}\frac{1}{(n+1)!}\mathrm{ad}^n_X(\delta X) \tag{4.1.2}$$

Informally,

$$\delta + e^X(\lambda) = \mathrm{Ad}_{e^X}(\delta+\lambda).$$

We call two Maurer-Cartan elements equivalent if they are in the same orbit of the above action.

**Theorem 9.** *([GM]) Let $\mathfrak{a}$ be an Artinian algebra with the maximal ideal $\mathfrak{m}$. A quasi-isomorphism of DGLAs $\mathcal{L}_1^\bullet \to \mathcal{L}_2^\bullet$ induces a bijection between the sets of equivalence classes of Maurer-Cartan elements of $\mathcal{L}_1^\bullet \otimes \mathfrak{m}$ and of $\mathcal{L}_2^\bullet \otimes \mathfrak{m}$.*

Given a DGLA $\mathcal{L}^\bullet$ and an Artinian algebra $\mathfrak{a}$ with the maximal ideal $\mathfrak{m}$, denote by $\mathrm{MC}(\mathcal{L}^\bullet)$ the set of equivalence classes of Maurer-Cartan elements of $\mathcal{L}^\bullet \otimes \mathfrak{m}$.

**4.2.** Let $\mathcal{A}$ be an Azymaya algebra on $X$. Let $\mathrm{Def}(\mathcal{A})$ denote the formal deformation theory of $\mathcal{A}$ as a sheaf of associative $\mathbb{C}$-algebras, i.e. the groupoid-valued functor of (commutative) Artin $\mathbb{C}$-algebras which associates to an Artin algebra $\mathfrak{a}$ the groupoid whose objects are pairs $(\widetilde{\mathcal{A}}, \phi)$ consisting of a flat $\mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{a}$-algebra $\widetilde{\mathcal{A}}$ and a map $\phi : \widetilde{\mathcal{A}} \to \mathcal{A}$ which induces an isomorphism $\widetilde{\mathcal{A}} \otimes_{\mathfrak{a}} \mathbb{C} \to \mathcal{A}$. The morphism in $\mathrm{Def}(\mathcal{A})(\mathfrak{a})$ are morphisms of $\mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{a}$-algebras which commute with the respective structure maps to $\mathcal{A}$.

Theorem 10 is the main result of this note.

**Theorem 10.** *Suppose that $\mathcal{A}$ is an Azumaya algebra on $X$. There exists a canonical equivalence* $\mathrm{Def}(\mathcal{A}) \cong \mathrm{Def}(\mathcal{O}_X)$.

The proof of Theorem 10 will be given in 4.8. The main technical ingredient in the proof is Theorem 20.

**4.3. Hochschild cochains.** Recall that the sheaf $C^n(\mathcal{A})$ of Hochschild cochains of degree $n$ is defined by

$$C^n(\mathcal{A}) := \underline{\mathrm{Hom}}_{\mathbb{C}}(\mathcal{A}^{\otimes_{\mathbb{C}} n}, \mathcal{A}) \ .$$

In the case of Azumaya algebras, we will always consider the subcomplex of local sections of $C^n(\mathcal{A})$, i.e. of multidifferential operators. The link from the deformation theory to the Hochschild theory is provided by the following

**Proposition 11.** [G] *For every algebra $A$, there exists a canonical equivalence* $\mathrm{Def}(A) \cong \mathrm{MC}(C^\bullet(A)[1]))$.

In particular, for an Azumaya algebra $\mathcal{A}$ there exists a canonical equivalence $\mathrm{Def}(\mathcal{A}) \cong \mathrm{MC}(\Gamma(X; C^\bullet(\mathcal{A})[1]))$.

*Proof.* By definition, an element of degree one in the DGLA $C^\bullet(A)[1] \otimes \mathfrak{m}$ is a map $\lambda : A \otimes A \to A \otimes \mathfrak{m}$. Put $a * b = ab + \lambda(a, b)$. Extend $*$ a binary $\mathfrak{a}$-linear operation on $A \otimes \mathfrak{a}$. Modulo $\mathfrak{m}$, this operation is the multiplication in $A$. Its associativity is equivalent to the Maurer-Cartan equation (1.0.1). Two Maurer-Cartan elements are equivalent if and only if there is a map $X : A \to A \otimes \mathfrak{m}$ such that, if one extends it to an $\mathfrak{a}$-linear map $X : A \otimes \mathfrak{m} \to A \otimes \mathfrak{a}$, its exponential $\mathrm{expad}(X)$ is an isomorphism of the corresponding two associative algebra structures on $A \otimes \mathfrak{a}$. On the other hand, any such isomorphism of algebra structures (which is identical modulo $\mathfrak{m}$ is of the form $\mathrm{expad}(X)$ for some $X$. □

Let

$$C^n(\mathcal{J}(\mathcal{A})) := \underline{\mathrm{Hom}}^{cont}_{\mathcal{O}_X}(\mathcal{J}(\mathcal{A})^{\otimes_{\mathcal{O}_X} n}, \mathcal{J}(\mathcal{A}))$$

There exists a canonical map $j^\infty : C^n(\mathcal{A}) \to C^n(\mathcal{J}(\mathcal{A}))$ which to a multidifferential operator associates its linearization. The canonical flat connection on $\mathcal{J}(\mathcal{A})$ induces a flat connection, denoted $\nabla^{can}$, on $C^n(\mathcal{J}(\mathcal{A}))$. The de Rham complex $\mathtt{DR}(C^n(\mathcal{J}(\mathcal{A})))$ satisfies $H^i(\mathtt{DR}(C^n(\mathcal{J}(\mathcal{A})))) = 0$ for $i \neq 0$ while the map $j^\infty$ induces an isomorphism $C^n(\mathcal{A}) \stackrel{\sim}{=} H^0(\mathtt{DR}(C^n(\mathcal{J}(\mathcal{A}))))$.

The Hochschild differential, denoted $\delta$ and the Gerstenhaber bracket endow $C^\bullet(\mathcal{A})[1]$ (respectively, $C^\bullet(\mathcal{J}(\mathcal{A}))[1]$) with a structure of a DGLA. The connection $\nabla^{can}$ acts by derivations of the Gerstenhaber bracket on $C^\bullet(\mathcal{J}(\mathcal{A}))[1]$. Since it acts by derivations on $\mathcal{J}(\mathcal{A})$ the induced connection on $C^\bullet(\mathcal{J}(\mathcal{A}))[1]$ commutes with the Hochschild differential. Hence, the graded Lie algebra $\Omega^\bullet_X \otimes C^\bullet(\mathcal{J}(\mathcal{A}))[1]$ equipped with the differential $\nabla^{can} + \delta$ is DGLA.

**Proposition 12.** *The map*

$$j^\infty : C^\bullet(\mathcal{A})[1] \to \Omega^\bullet \otimes C^\bullet(\mathcal{J}(\mathcal{A}))[1] \tag{4.3.1}$$

*is a quasi-isomorphism of DGLA.*

*Proof.* It is clear that the map (4.3.1) is a morphism of DGLA.

Let $F_i C^\bullet(?) = C^{\geq -i}(?)$. Then, $F_\bullet C^\bullet(?)$ is a filtered complex and the differential induced on $Gr^F_\bullet C^\bullet(?)$ is trivial. Consider $\Omega^\bullet_X$ as equipped with the trivial filtration. Then, the map (4.3.1) is a morphism of filtered complexes with respect to the induced filtrations on the source and the target. The induced map of the associated graded objects a quasi-isomorphism, hence, so is (4.3.1). □

**Corollary 13.** *The map*

$$\mathrm{MC}(\Gamma(X; C^\bullet(\mathcal{A})[1])) \to \mathrm{MC}(\Gamma(X; \Omega^\bullet_X \otimes C^\bullet(\mathcal{J}(\mathcal{A}))[1]))$$

*induced by* (4.3.1) *is an equivalence.*

**4.4. The cotrace map.** Let $\overline{C}^n(\mathcal{J}_X)$ denote the sheaf of normalized Hochschild cochains. It is a subsheaf of $C^n(\mathcal{J}_X)$ whose stalks are given by

$$\overline{C}^n(\mathcal{J}_X)_x = \mathrm{Hom}_{\mathcal{O}_{X,x}}((\mathcal{J}_{X,x}/\mathcal{O}_{X,x}\cdot 1)^{\otimes n}, \mathcal{J}_{X,x}) \subset \mathrm{Hom}\,\mathcal{O}_{X,x}(\mathcal{J}_{X,x}^{\otimes n}, \mathcal{J}_{X,x}) = C^n(\mathcal{J}_X)_x$$

The sheaf $\overline{C}^\bullet(\mathcal{A})[1]$ (respectively, $\overline{C}^\bullet(\mathcal{J}_X)[1]$ is actually a sub-DGLA of $C^\bullet(\mathcal{A})[1]$ (respectively, $C^\bullet(\mathcal{J}_X)[1]$ ) and the inclusion map is a quasi-isomorphism.

The flat connection $\nabla^{can}$ preserves $\overline{C}^\bullet(\mathcal{J}_X)[1]$, and the (restriction to $\overline{C}^\bullet(\mathcal{O}_X)[1]$ of the) map $j^\infty$ is a quasi-isomorphism of DGLA.

Consider now the map

$$\mathrm{cotr} : \overline{C}^\bullet(\mathcal{J}_X)[1] \to C^\bullet(\mathcal{A}\otimes\mathcal{J}_X)[1] \tag{4.4.1}$$

defined as follows:

$$\mathrm{cotr}(D)(a_1\otimes j_1, \ldots, a_n\otimes j_n) = a_0\ldots a_n D(j_1,\ldots,j_n). \tag{4.4.2}$$

**Proposition 14.** *The map* cotr *is a quasiisomorphism of DGLAs.*

*Proof.* It is easy to see that cotr is a morphism of DGLAs. Since the fact that this is a quasiisomorphism is local it is enough to verify it when $\mathcal{A} = \mathtt{Mat}_n(\mathcal{O}_X)$. In this case it is a well-known fact (cf. [Lo], section 1.5.6). ☐

**4.5. Comparison of deformation complexes.** Let $\sigma \in \mathrm{Isom}_0(\mathcal{A}\otimes\mathcal{J}_X, \mathcal{J}(\mathcal{A}))$, $\nabla \in \mathcal{C}(\mathcal{A})$. The isomorphisms of algebras $\sigma$ induces the isomorphism of DGLA

$$\sigma_* : C^\bullet(\mathcal{A}\otimes\mathcal{J}_X)[1] \to C^\bullet(\mathcal{J}(\mathcal{A}))[1]$$

which is horizontal with respect to the flat connection (induced by) $\nabla^{can}$ on $C^\bullet(\mathcal{J}(\mathcal{A}))[1]$ and the induced flat connection given by (3.0.5). Therefore, it induces the isomorphism of DGLA (the respective de Rham complexes)

$$\sigma_* : \Omega^\bullet_X\otimes C^\bullet(\mathcal{A}\otimes\mathcal{J}_X)[1] \to \Omega^\bullet_X\otimes C^\bullet(\mathcal{J}(\mathcal{A}))[1]. \tag{4.5.1}$$

Here, the differential in $\Omega^\bullet_X\otimes C^\bullet(\mathcal{A}\otimes\mathcal{J}_X)[1]$ is $\nabla\otimes\mathtt{Id} + \mathtt{Id}\otimes\nabla^{can} + \mathrm{ad}\,F + \delta$ and the differential in $\Omega^\bullet_X\otimes C^\bullet(\mathcal{J}(\mathcal{A}))[1]$ is $\nabla^{can}+\delta$.

Let $\iota_G$ denote the adjoint action of $G\in\Gamma(X;\Omega^k_X\otimes C^0(\mathcal{A}\otimes\mathcal{J}_X))$ (recall that $C^0(\mathcal{A}\otimes\mathcal{J}_X) = \mathcal{A}\otimes\mathcal{J}_X$). Thus, $\iota_G$ is a map $\Omega^p_X\otimes C^q(\mathcal{A}\otimes\mathcal{J}_X)\to\Omega^{p+k}_X\otimes C^{q-1}(\mathcal{A}\otimes\mathcal{J}_X)$.

**Lemma 15.** *For any $H, G \in \Gamma(X; \Omega^\bullet_X \otimes C^0(\mathcal{A} \otimes \mathcal{J}_X))$ we have:*

$$[\delta, \iota_H] = \operatorname{ad} H \tag{4.5.2}$$

$$[\operatorname{ad} H, \iota_G] = \iota_{[H,G]} \tag{4.5.3}$$

$$[\iota_H, \iota_G] = 0 \tag{4.5.4}$$

$$[\nabla \otimes \mathtt{Id} + \mathtt{Id} \otimes \nabla^{can}_{\mathcal{O}}, \iota_H] = \iota_{(\nabla \otimes \mathtt{Id} + \mathtt{Id} \otimes \nabla^{can}_{\mathcal{O}})H} \tag{4.5.5}$$

*Proof.* Direct calculation. □

Let $\exp(t\iota_F) = \sum_{k=0}^{\infty} \frac{1}{n!} t^k (\iota_F)^{\circ k} = \mathtt{Id} + t\iota_F + \frac{t^2}{2}\iota_F \circ \iota_F + \cdots$. Note that this is a polynomial in $t$ since $\Omega^p_X = 0$ for $p > \dim X$. Since $\iota_F$ is a derivation, the operation $\exp(\iota_F)$ is an automorphism of the graded Lie algebra $\Omega^\bullet_X \otimes C^\bullet(\mathcal{A} \otimes \mathcal{J}_X)[1]$. The automorphism $\exp(\iota_F)$ does not commute with the differential.

**Lemma 16.**

$$\begin{aligned} &\exp(\iota_F) \circ (\nabla \otimes \mathtt{Id} + \mathtt{Id} \otimes \nabla^{can}_{\mathcal{O}} + \delta + \operatorname{ad} F) \circ \exp(-\iota_F) = \\ &\qquad\qquad \nabla \otimes \mathtt{Id} + \mathtt{Id} \otimes \nabla^{can}_{\mathcal{O}} + \delta + \iota_\theta. \end{aligned}$$

*Proof.* Consider the following polynomial in $t$: $p(t) = \exp(t\iota_F) \circ \delta \circ \exp(-t\iota_F)$. Then using the identities from the Lemma 15 we obtain $p'(t) = \exp(t\iota_F) \circ [\iota_F, \delta] \circ \exp(-t\iota_F) = -\exp(t\iota_F) \circ \operatorname{ad} F \circ \exp(-t\iota_F)$, $p''(t) = -\exp(t\iota_F) \circ [\iota_F, \operatorname{ad} F] \circ \exp(-t\iota_F) = \exp(t\iota_F) \circ \iota_{[F,F]} \circ \exp(-t\iota_F)$, and $p^{(n)} = 0$ for $n \geq 3$. Therefore $p(t) = \delta - t \operatorname{ad} F + \frac{t^2}{2}\iota_{[F,F]}$. Setting $t = 1$ we obtain

$$\exp(\iota_F) \circ \delta \circ \exp(-\iota_F) = \delta - \operatorname{ad} F + \frac{1}{2}\iota_{[F,F]}.$$

Similarly we obtain

$$\exp(\iota_F) \circ \operatorname{ad} F \circ \exp(-\iota_F) = \operatorname{ad} F - \iota_{[F,F]}$$

and

$$\begin{aligned} &\exp(\iota_F) \circ (\nabla \otimes \mathtt{Id} + \mathtt{Id} \otimes \nabla^{can}_{\mathcal{O}}) \circ \exp(-\iota_F) = \\ &\qquad\qquad (\nabla \otimes \mathtt{Id} + \mathtt{Id} \otimes \nabla^{can}_{\mathcal{O}}) - \iota_{(\nabla \otimes \mathtt{Id} + \mathtt{Id} \otimes \nabla^{can}_{\mathcal{O}})F} \end{aligned}$$

Adding these formulas up and using the identity (3.0.4) we obtain the desired result. □

**Lemma 17.** *The map*

$$\mathtt{Id} \otimes \operatorname{cotr} : \Omega^\bullet_X \otimes \overline{C}^\bullet(\mathcal{J}_X)[1] \to \Omega^\bullet_X \otimes C^\bullet(\mathcal{A} \otimes \mathcal{J}_X)[1] \ . \tag{4.5.6}$$

*is a quasiisomorphism of DGLA, where the source (respectively, the target) is equipped with the differential $\nabla^{can}_{\mathcal{O}} + \delta$ (respectively, $\nabla \otimes \mathtt{Id} + \mathtt{Id} \otimes \nabla^{can} + \delta + \iota_\theta$).*

*Proof.* It is easy to see that $\mathtt{Id}\otimes\mathrm{cotr}$ is a morphism of graded Lie algebras, which satisfies $(\nabla\otimes\mathtt{Id}+\mathtt{Id}\otimes\nabla^{can}_{\mathcal{O}})\circ(\mathtt{Id}\otimes\mathrm{cotr}) = (\mathtt{Id}\otimes\mathrm{cotr})\circ\nabla^{can}_{\mathcal{O}}$ and $\delta\circ(\mathtt{Id}\otimes\mathrm{cotr}) = (\mathtt{Id}\otimes\mathrm{cotr})\circ\delta$. Since the domain of $(\mathtt{Id}\otimes\mathrm{cotr})$ is the normalized complex, we also have $\iota_\theta\circ(\mathtt{Id}\otimes\mathrm{cotr}) = 0$. This implies that $(\mathtt{Id}\otimes\mathrm{cotr})$ is a morphism of DGLA.

Introduce filtration on $\Omega^\bullet_X$ by $F_i\Omega^\bullet_X = \Omega^{\geq -i}_X$ (the "stupid" filtration) and consider the complexes $\overline{C}^\bullet(\mathcal{J}_X)[1]$ and $C^\bullet(\mathcal{A}\otimes\mathcal{J}_X)[1]$ equipped with the trivial filtration. The map (4.5.6) is a morphism of filtered complexes with respect to the induced filtrations on the source and the target. The differentials induced on the associated graded complexes are $\delta$ (or, more precisely, $\mathtt{Id}\otimes\delta$) and the induced map of the associated graded objects is $\mathtt{Id}\otimes\mathrm{cotr}$ which is a quasi-isomorphism in virtue of Proposition 14. Therefore, the map (4.5.6) is a quasiisomorphism as claimed. □

**Proposition 18.** *For a any choice of* $\sigma\in\mathrm{Isom}_0(\mathcal{A}\otimes\mathcal{J}_X,\mathcal{J}(\mathcal{A}))$, $\nabla\in\mathcal{C}(\mathcal{A})$, *the composition* $\Phi_{\sigma,\nabla} := \sigma_*\circ\exp(\iota_{F_{\sigma,\nabla}})\circ(\mathtt{Id}\otimes\mathrm{cotr})$ *(where $F$ is as in Lemma 8),*

$$\Phi_{\sigma,\nabla} : \Omega^\bullet_X\otimes\overline{C}^\bullet(\mathcal{J}_X)[1]\to\Omega^\bullet_X\otimes C^\bullet(\mathcal{J}(\mathcal{A}))[1] \tag{4.5.7}$$

*is a quasi-isomorphism of DGLA.*

*Proof.* This is a direct consequence of the Lemmata 16, 4.5.6. □

**4.6. Independence of choices.** According to Proposition 18, for any choice of $\sigma\in\mathrm{Isom}_0(\mathcal{A}\otimes\mathcal{J}_X,\mathcal{J}(\mathcal{A}))$, $\nabla\in\mathcal{C}(\mathcal{A})$ we have a quasi-isomorphism of DGLA $\Phi_{\sigma,\nabla}$.

**Proposition 19.** *The image of $\Phi_{\sigma,\nabla}$ in the derived category is independent of the choices made.*

*Proof.* For $i=0,1$ suppose given $\sigma_i\in\mathrm{Isom}_0(\mathcal{A}\otimes\mathcal{J}_X,\mathcal{J}(\mathcal{A}))$, $\nabla_i\in\mathcal{C}(\mathcal{A})$. Let $\Phi_i=\Phi_{\sigma_i,\nabla_i}$. The goal is to show that $\Phi_0=\Phi_1$ in the derived category.

There is a unique $\theta_i\in\Gamma(X;\Omega^2_X\otimes\mathcal{A}_0\otimes\mathcal{J}_X))$ such that $\nabla_i^2=\mathrm{ad}(\theta_i)$. By Lemma 8 there exist unique $F_i\in\Gamma(X;\Omega^1_X\otimes\mathcal{A}_0\otimes\mathcal{J}_X)$ such that

$$\mathrm{ad}(F_i)=\sigma_i^{-1}\circ\nabla^{can}_{\mathcal{A}}\circ\sigma_i-(\nabla_i\otimes\mathtt{Id}+\mathtt{Id}\otimes\nabla^{can}_{\mathcal{O}})$$

Let $I:=[0,1]$. Let $\epsilon_i : X\to I\times X$ denote the map $x\mapsto(i,x)$. Let $\mathtt{pr} : I\times X\to X$ denote the projection on the second factor.

It follows from Lemma 5, the isomorphism (3.0.2) and the isomorphism $\mathcal{A}_0\otimes\mathcal{J}_X\to\mathtt{Der}_{\mathcal{J}_{X,0}}(\mathcal{A}\otimes\mathcal{J}_X)$ that there exists a unique $f\in\Gamma(X;\mathcal{A}_0\otimes\mathcal{J}_X)$ such that $\sigma_1=\sigma_0\circ\exp(\mathrm{ad}(f))$. For $t\in I$ let $\sigma_t=\sigma_0\circ\exp(\mathrm{ad}(tf))$. Let $\widetilde{\sigma}$ denote the isomorphism $\mathtt{pr}^*(\mathcal{A}\otimes\mathcal{J}_X)\to\mathtt{pr}^*\mathcal{J}(\mathcal{A})$ which restricts to $\sigma(t)$ on $\{t\}\times X$. In particular, $\sigma_i=\epsilon_i^*(\widetilde{\sigma})$.

For $t \in I$ let $\nabla_t = t\nabla_0 + (1-t)\nabla_1$. Let $\widetilde{\nabla}$ denote the connection on $\mathtt{pr}^*\mathcal{A}$ which restricts to $\nabla_t$ on $\{t\} \times X$. In particular, $\nabla_i = \epsilon_i^*(\widetilde{\nabla})$.

Suppose that

(1) $\widetilde{\sigma} : \mathtt{pr}^*(\mathcal{A} \otimes \mathcal{J}_X) \to \mathtt{pr}^*\mathcal{J}(\mathcal{A})$ is an isomorphism of $\mathtt{pr}^*\mathcal{J}_X$-algebras which reduces to the identity map on $\mathtt{pr}^*\mathcal{A}$ modulo $\mathcal{J}_0$ and satisfies $\sigma_i = \epsilon_i^*(\widetilde{\sigma})$;

(2) $\widetilde{\nabla} \in \mathcal{C}(\mathtt{pr}^*\mathcal{A})$ satisfies $\nabla_i = \epsilon_i^*(\widetilde{\nabla})$.

(Examples of such are constructed above.)

Then, there exists a unique $\widetilde{F} \in \Gamma(I \times X; \Omega^1_{I\times X} \otimes \mathtt{pr}^*(\mathcal{A}_0 \otimes \mathcal{J}_X))$ such that

$$\widetilde{\sigma}^{-1} \circ \mathtt{pr}^*(\nabla^{can}_{\mathcal{A}}) \circ \widetilde{\sigma} = \widetilde{\nabla} \otimes \mathtt{Id} + \mathtt{Id} \otimes \mathtt{pr}^*(\nabla^{can}_{\mathcal{O}}) + \mathrm{ad}(\widetilde{F})$$

It follows from the uniqueness that $\epsilon^*(\widetilde{F}) = F_i$.

There exists a unique $\widetilde{\theta} \in \Gamma(I \times X; \Omega^2_{I\times X} \otimes \mathtt{pr}^*(\mathcal{A}_0 \otimes \mathcal{J}_X))$ such that $\widetilde{\nabla}^2 = \mathrm{ad}(\widetilde{\theta})$. It follows from the uniqueness that $\epsilon_i^*(\widetilde{\theta}) = \theta_i$.

The composition $\widetilde{\Phi} := \widetilde{\sigma}_* \circ \exp(\iota_{\widetilde{F}}) \circ (\mathtt{Id} \otimes \mathrm{cotr})$,

$$\widetilde{\Phi} : \Omega^\bullet_{I\times X} \otimes \mathtt{pr}^*\overline{C}^\bullet(\mathcal{J}_X))[1] \to \Omega^\bullet_{I\times X} \otimes \mathtt{pr}^*C^\bullet(\mathcal{J}(\mathcal{A}))[1] \tag{4.6.1}$$

is a quasi-isomorphism of DGLA, where the differential on source (respectively, target) is $\mathtt{pr}^*(\nabla^{can}_{\mathcal{O}}) + \delta$ (respecively, $\mathtt{pr}^*(\widetilde{\nabla}^{can}_{\mathcal{A}}) + \delta + \iota_{\widetilde{\theta}}$).

The map (4.6.1) induces the map of direct images under the projection $\mathtt{pr}$

$$\widetilde{\Phi} : \mathtt{pr}_*\Omega^\bullet_{I\times X} \otimes \overline{C}^\bullet(\mathcal{J}_X)[1] \to \mathtt{pr}_*\Omega^\bullet_{I\times X} \otimes C^\bullet(\mathcal{J}(\mathcal{A}))[1] \tag{4.6.2}$$

which is a quasi-isomorphism (since all higher direct images vanish).

The pull-back of differential forms $\mathtt{pr}^* : \Omega^\bullet \to \mathtt{pr}_*\Omega^\bullet_{I\times X}$ is a quasi-isomorphism of commutative DGA inducing the quasi-isomorphism of DGLA $\mathtt{pr}^* \otimes \mathtt{Id} : \Omega^\bullet_X \otimes \overline{C}^\bullet(\mathcal{J}_X)[1] \to \mathtt{pr}_*\Omega^\bullet_{I\times X} \otimes \overline{C}^\bullet(\mathcal{J}_X)[1]$ (with differentials $\nabla^{can}_{\mathcal{O}} + \delta$ and $\mathtt{pr}^*(\nabla^{can}_{\mathcal{O}}) + \delta$ respectively).

The diagram of quasi-isomorphisms of DGLA

(4.6.3)

$$\begin{tikzcd}
& \Omega^\bullet_X \otimes C^\bullet(\mathcal{J}(\mathcal{A}))[1] \\
\Omega^\bullet_X \otimes C^\bullet(\mathcal{J}_X)[1] \arrow[ur, "\Phi_0"] \arrow[r, "\widetilde{\Phi}\circ(\mathtt{pr}^*\otimes\mathtt{Id})"] \arrow[dr, "\Phi_1"'] & \mathtt{pr}_*\Omega^\bullet_{I\times X} \otimes C^\bullet(\mathcal{J}(\mathcal{A}))[1] \arrow[u, "\epsilon_0^*\otimes\mathtt{Id}"'] \arrow[d, "\epsilon_1^*\otimes\mathtt{Id}"'] \\
& \Omega^\bullet_X \otimes C^\bullet(\mathcal{J}(\mathcal{A}))[1]
\end{tikzcd}$$

is commutative. Since $\epsilon_i^* \circ \mathtt{pr}^* = \mathtt{Id}$ for $i = 0, 1$ and $\mathtt{pr}^* \otimes \mathtt{Id}$ is a quasiisomorphism, $\epsilon_0^* \otimes \mathtt{Id}$ and $\epsilon_1^* \otimes \mathtt{Id}$ represent the same morphism in the derived category. Hence so do $\Phi_0$ and $\Phi_1$. $\square$

**4.7. The main technical ingredient.** To each pair $(\sigma, \nabla)$ with $\sigma \in \mathrm{Isom}_0(\mathcal{A}\otimes\mathcal{J}_X, \mathcal{J}(\mathcal{A}))$ and $\nabla \in \mathcal{C}(\mathcal{A})$ we associated the quasi-isomorphism of DGLA (4.5.7) (Proposition 18). According to Proposition 19 all of these give rise to the same isomorphism in the derived category. We summarize these findings in the following theorem.

**Theorem 20.** *Suppose that $\mathcal{A}$ is an Azumaya algebra on $X$. There exists a canonical isomorphism in the derived category of DGLA $\Omega^\bullet_X \otimes \overline{C}^\bullet(\mathcal{J}_X)[1] \cong \Omega^\bullet_X \otimes C^\bullet(\mathcal{J}(\mathcal{A}))[1]$.*

**4.8. The proof of Theorem 10.** The requisite equivalence is the composition of the equivalences

$$\begin{aligned}
\mathrm{Def}(\mathcal{O}_X) &\cong \mathrm{MC}(\Gamma(X; \overline{C}^\bullet(\mathcal{O}_X)[1])) \\
&\cong \mathrm{MC}(\Gamma(X; \Omega^\bullet_X \otimes \overline{C}^\bullet(\mathcal{J}_X)[1])) \\
&\cong \mathrm{MC}(\Gamma(X; \Omega^\bullet_X \otimes C^\bullet(\mathcal{J}(\mathcal{A}))[1])) \\
&\cong \mathrm{MC}(\Gamma(X; C^\bullet(\mathcal{A})[1])) \\
&\cong \mathrm{Def}(\mathcal{A})
\end{aligned}$$

The first and the last equivalences are those of Proposition 11, the second and the fourth are those of Corollary 13, and the third one is induced by the canonical isomorphism in the derived category of Theorem 10.

## 5. Holomorphic case

**5.1. Complex distributions.** Let $\mathcal{T}_X$ denote the sheaf of *real valued* vector fields on $X$ and let $\mathcal{T}_X^{\mathbb{C}} := \mathcal{T}_X \otimes_{\mathbb{R}} \mathbb{C}$.

**Definition 21.** A *(complex) distribution* on $X$ is a sub-bundle of $\mathcal{T}_X^{\mathbb{C}}$[1]

For a distribution $\mathcal{P}$ on $X$ we denote by $\mathcal{P}^\perp \subseteq \Omega^1_X$ the annihilator of $\mathcal{P}$ (with respect to the canonical duality pairing).

**Definition 22.** A distribution $\mathcal{P}$ of rank $r$ on $X$ is called *integrable* if, locally on $X$, there exist functions $f_1, \ldots, f_r \in \mathcal{O}_X$ such that $df_1, \ldots, df_r$ form a local frame for $\mathcal{P}^\perp$.

**Lemma 23.** *An integrable distribution is* involutive*, i.e. it is a Lie subalgebra of $\mathcal{T}_X^{\mathbb{C}}$ (with respect to the Lie bracket of vector fields).*

[1]A sub-bundle is an $\mathcal{O}_X$-submodule which is a direct summand locally on $X$

**5.2. Differential calculus on $X/\mathcal{P}$.** In this subsection we briefly review relevant definitions and results of the differential calculus in the presence of integrable complex distribution. We refer the reader to [Ko], [R] and [FW] for details and proofs. Suppose that $\mathcal{P}$ is an integrable distribution. Let $F_\bullet\Omega^\bullet_X$ denote the filtration by the powers of the differential ideal generated by $\mathcal{P}^\perp$, i.e. $F_{-i}\Omega^j_X = \bigwedge^i \mathcal{P}^\perp \wedge \Omega^{j-i}_X \subseteq \Omega^j_X$. Let $\overline{\partial}$ denote the differential in $Gr^F\Omega^\bullet_X$. The wedge product of differential forms induces a structure of a commutative DGA on $(Gr^F\Omega^\bullet_X, \overline{\partial})$.

**Lemma 24.** *The complex $Gr^F_{-i}\Omega^\bullet_X$ satisfies $H^j(Gr^F_{-i}\Omega^\bullet_X, \overline{\partial}) = 0$ for $j \neq i$.*

Let $\Omega^i_{X/\mathcal{P}} := H^i(Gr^F_{-i}\Omega^\bullet_X, \overline{\partial})$, $\mathcal{O}_{X/\mathcal{P}} := \Omega^0_{X/\mathcal{P}}$. We have $\mathcal{O}_{X/\mathcal{P}} := \Omega^0_{X/\mathcal{P}} \subset \mathcal{O}_X$, $\Omega^1_{X/\mathcal{P}} \subset \mathcal{P}^\perp \subset \Omega^1_X$ and, more generally, $\Omega^i_{X/\mathcal{P}} \subset \bigwedge^i \mathcal{P}^\perp \subset \Omega^i_X$. The wedge product of differential forms induces a structure of a graded-commutative algebra on $\Omega^\bullet_{X/\mathcal{P}} := \oplus_i \Omega^i_{X/\mathcal{P}}[-i] = H^\bullet(Gr^F\Omega^\bullet_X, \overline{\partial})$.

If $f_1, \ldots, f_r$ are as in 22, then $\Omega^1_{X/\mathcal{P}} = \sum_{i=1}^r \mathcal{O}_{X/\mathcal{P}} \cdot df_i$, in particular, $\Omega^1_{X/\mathcal{P}}$ is a locally free $\mathcal{O}_{X/\mathcal{P}}$-module. Moreover, the multiplication induces an isomorphism $\bigwedge^i_{\mathcal{O}_{X/\mathcal{P}}} \Omega^1_{X/\mathcal{P}} \to \Omega^i_{X/\mathcal{P}}$.

The de Rham differential $d$ restricts to the map $d : \Omega^i_{X/\mathcal{P}} \to \Omega^{i+1}_{X/\mathcal{P}}$ and the complex $\Omega^\bullet_{X/\mathcal{P}} := (\Omega^i_{X/\mathcal{P}}, d)$ is a commutative DGA. Moreover, the inclusion $\Omega^\bullet_{X/\mathcal{P}} \to \Omega^\bullet_X$ is a quasi-isomorphism.

*Example* 25. Suppose that $\overline{\mathcal{P}} = \mathcal{P}$. Then, $\mathcal{P} = \mathcal{D} \otimes_{\mathbb{R}} \mathbb{C}$, where $\mathcal{D}$ a subbundle of $\mathcal{T}_X$. Then, $\mathcal{D}$ is an integrable real distribution which defines a foliation on $X$ and $\Omega^\bullet_{X/\mathcal{P}}$ is the complex of basic forms.

*Example* 26. Suppose that $\bar{\mathcal{P}} \cap \mathcal{P} = 0$ and $\bar{\mathcal{P}} \oplus \mathcal{P} = \mathcal{T}^{\mathbb{C}}_X$. In this case again $\mathcal{P}$ is integrability is equivalent to involutivity, by Newlander-Nirenberg theorem. $\Omega^\bullet_{X/\mathcal{P}}$ in this case is a holomorphic de Rham complex.

**5.3. $\overline{\partial}$-operators.** Suppose that $\mathcal{E}$ is a vector bundle on $X$, i.e. a locally free $\mathcal{O}_X$-module of finite rank. A connection along $\mathcal{P}$ on $\mathcal{E}$ is, by definition, a map $\nabla^{\mathcal{P}} : \mathcal{E} \to \Omega^1_X/\mathcal{P}^\perp \otimes_{\mathcal{O}_X} \mathcal{E}$ which satisfies the Leibniz rule $\nabla^{\mathcal{P}}(fe) = f\nabla^{\mathcal{P}}(e) + \overline{\partial}f \cdot e$. A connection along $\mathcal{P}$ gives rise to the $\mathcal{O}_X$-linear map $\nabla^{\mathcal{P}}_{(\bullet)} : \mathcal{P} \to \underline{\mathrm{End}}_{\mathbb{C}}(\mathcal{E})$ defined by $\mathcal{P} \ni \xi \mapsto \nabla^{\mathcal{P}}_\xi$, with $\nabla^{\mathcal{P}}_\xi(e) = \iota_\xi \nabla^{\mathcal{P}}(e)$.

Conversely, an $\mathcal{O}_X$-linear map $\nabla^{\mathcal{P}}_{(\bullet)} : \mathcal{P} \to \underline{\mathrm{End}}_{\mathbb{C}}(\mathcal{E})$ which satisfies the Leibniz rule $\nabla^{\mathcal{P}}_\xi(fe) = f\nabla^{\mathcal{P}}_\xi(e) + \overline{\partial}f \cdot e$ determines a connection along $\mathcal{P}$. In what follows we will not distinguish between the two avatars of a connection along $\mathcal{P}$ described above. Note that, as a consequence of the $\overline{\partial}$-Leibniz rule a connection along $\mathcal{P}$ is $\mathcal{O}_{X/\mathcal{P}}$-linear.

A connection along $\mathcal{P}$ on $\mathcal{E}$ is called flat if the corresponding map $\nabla^{\mathcal{P}}_{(\bullet)} : \mathcal{P} \to \underline{\mathrm{End}}_{\mathbb{C}}(\mathcal{E})$ is a morphism of Lie algebras. We will refer to a flat connection along $\mathcal{P}$ on $\mathcal{E}$ as a $\overline{\partial}$-operator on $\mathcal{E}$.

*Example* 27. The differential $\overline{\partial}$ in $Gr^F\Omega^\bullet_X$ gives rise to canonical $\overline{\partial}$-operators on $\bigwedge^i \mathcal{P}^\perp$, $i = 0, 1, \ldots$.

*Example* 28. The adjoint action of $\mathcal{P}$ on $\mathcal{T}^{\mathbb{C}}_X$ preserves $\mathcal{P}$, hence descends to an action of the Lie algebra $\mathcal{P}$ on $\mathcal{T}^{\mathbb{C}}_X/\mathcal{P}$. The latter action is easily seen to be a connection along $\mathcal{P}$, i.e. a canonical $\overline{\partial}$-operator on $\mathcal{T}^{\mathbb{C}}_X/\mathcal{P}$ which is easily seen to coincide with the one induced on $\mathcal{T}^{\mathbb{C}}_X/\mathcal{P}$ via the duality pairing between the latter and $\mathcal{P}^\perp$. In the situation of Example 25 this connection is known as the Bott connection.

*Example* 29. Suppose that $\mathcal{F}$ is a locally free $\mathcal{O}_{X/\mathcal{P}}$-module of finite rank. Then, $\mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{F}$ is a locally free $\mathcal{O}_X$-module of rank $\mathrm{rk}_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{F}$ and is endowed in a canonical way with the $\overline{\partial}$-operator, namely, $\mathtt{Id}\otimes\overline{\partial}$.

A connection on $\mathcal{E}$ along $\mathcal{P}$ extends uniquely to a derivation of the graded $Gr^F_0\Omega^\bullet_X$-module $Gr^F_0\Omega^\bullet_X \otimes_{\mathcal{O}_X} \mathcal{E}$. A $\overline{\partial}$-operator $\overline{\partial}_{\mathcal{E}}$ satisfies $\overline{\partial}^2_{\mathcal{E}} = 0$. The complex $(Gr^F_0\Omega^\bullet_X\otimes_{\mathcal{O}_X}\mathcal{E}, \overline{\partial}_{\mathcal{E}})$ is referred to as the (corresponding) $\overline{\partial}$-complex. Since $\overline{\partial}_{\mathcal{E}}$ is $\mathcal{O}_{X/\mathcal{P}}$-linear, the sheaves $H^i(\mathcal{E}, \overline{\partial}_{\mathcal{E}})$ are $\mathcal{O}_{X/\mathcal{P}}$-modules.

**Lemma 30.** *Suppose that $\mathcal{E}$ is a vector bundle and $\overline{\partial}_{\mathcal{E}}$ is a $\overline{\partial}$-operator on $\mathcal{E}$. Then, $H^i(Gr^F_0 \otimes_{\mathcal{O}_X} \mathcal{E}, \overline{\partial}_{\mathcal{E}}) = 0$ for $i \neq 0$, i.e. the $\overline{\partial}$-complex is a resolution of $\ker(\overline{\partial}_{\mathcal{E}})$. Moreover, $\ker(\overline{\partial}_{\mathcal{E}})$ is locally free over $\mathcal{O}_{X/\mathcal{P}}$ of rank $\mathrm{rk}_{\mathcal{O}_X} \mathcal{E}$ and the map $\mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \ker(\overline{\partial}_{\mathcal{E}}) \to \mathcal{E}$ (the $\mathcal{O}_X$-linear extension of the inclusion $\ker(\overline{\partial}_{\mathcal{E}}) \to \mathcal{E}$) is an isomorphism.*

*Remark* 31. In the notations of Example 29 and Lemma 30, the assignments $\mathcal{F} \mapsto (\mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{F}, \mathtt{Id}\otimes\overline{\partial})$ and $(\mathcal{E}, \overline{\partial}_{\mathcal{E}}) \mapsto \ker(\overline{\partial}_{\mathcal{E}})$ are mutually inverse equivalences of suitably defined categories.

By the very definition, the kernel of the canonical $\overline{\partial}$-operator on $\mathcal{T}^{\mathbb{C}}_X/\mathcal{P}$ coincides with $\left(\mathcal{T}^{\mathbb{C}}_X/\mathcal{P}\right)^{\mathcal{P}}$ (the subsheaf of $\mathcal{P}$-invariant sections, see Example 28). We denote this subsheaf by $\mathcal{T}_{X/\mathcal{P}}$.

The duality pairing restricts to a non-degenerate $\mathcal{O}_{X/\mathcal{P}}$-bilinear pairing between $\Omega^1_{X/\mathcal{P}}$ and $\mathcal{T}_{X/\mathcal{P}}$ giving rise to a faithful action of $\mathcal{T}_{X/\mathcal{P}}$ on $\mathcal{O}_{X/\mathcal{P}}$ by derivations by the usual formula $\xi(f) = \iota_\xi df$, for $\xi \in \mathcal{T}_{X/\mathcal{P}}$ and $f \in \mathcal{O}_{X/\mathcal{P}}$.

Let $\mathtt{pr}_i : X \times X \to X$ denote the projection on the $i^{\text{th}}$ factor and let $\Delta_X : X \to X \times X$ denote the diagonal embeding. The latter satisfies $\Delta^*(\mathcal{P} \times \mathcal{P}) = \mathcal{P}$. Therefore, the induced map $\Delta_X^* : \mathcal{O}_{X\times X} \to \mathcal{O}_X$ satisfies

$$\operatorname{Im}(\Delta_X^*|_{\mathcal{O}_{X\times X/\mathcal{P}\times\mathcal{P}}}) \subset \mathcal{O}_{X/\mathcal{P}} \ .$$

Let $\Delta_{X/\mathcal{P}}^* := \Delta_X^*|_{\mathcal{O}_{X\times X/\mathcal{P}\times\mathcal{P}}}$, $\mathcal{I}_{X/\mathcal{P}} := \ker(\Delta_{X/\mathcal{P}}^*)$.

For a locally-free $\mathcal{O}_{X/\mathcal{P}}$-module of finite rank $\mathcal{E}$ let

$$\mathcal{J}^k(\mathcal{E}) := (\mathtt{pr}_1)_* \left( \mathcal{O}_{X\times X/\mathcal{P}\times\mathcal{P}} / \mathcal{I}_{X/\mathcal{P}}^{k+1} \otimes_{\mathtt{pr}_2^{-1}\mathcal{O}_{X/\mathcal{P}}} \mathtt{pr}_2^{-1}\mathcal{E} \right) \ ,$$

let $\mathcal{J}^k_{X/\mathcal{P}} := \mathcal{J}^k(\mathcal{O}_{X/\mathcal{P}})$. It is clear from the above definition that $\mathcal{J}^k_{X/\mathcal{P}}$ is, in a natural way, a commutative algebra and $\mathcal{J}^k(\mathcal{E})$ is a $\mathcal{J}^k_{X/\mathcal{P}}$-module.

We regard $\mathcal{J}^k(\mathcal{E})$ as $\mathcal{O}_{X/\mathcal{P}}$-modules via the pull-back map $\mathtt{pr}_1^* : \mathcal{O}_{X/\mathcal{P}} \to (\mathtt{pr}_1)_*\mathcal{O}_{X\times X/\mathcal{P}\times\mathcal{P}}$ (the restriction to $\mathcal{O}_{X/\mathcal{P}}$ of the map $\mathtt{pr}_1^* : \mathcal{O}_X \to (\mathtt{pr}_1)_*\mathcal{O}_{X\times X}$) with the quotient map $(\mathtt{pr}_1)_*\mathcal{O}_{X\times X/\mathcal{P}\times\mathcal{P}} \to \mathcal{J}^k_{X/\mathcal{P}}$.

For $0 \leq k \leq l$ the inclusion $\mathcal{I}^{l+1}_{X/\mathcal{P}} \to \mathcal{I}^{k+1}_{X/\mathcal{P}}$ induces the surjective map $\pi_{l,k} : \mathcal{J}^l(\mathcal{E}) \to \mathcal{J}^k(\mathcal{E})$. The sheaves $\mathcal{J}^k(\mathcal{E})$, $k = 0, 1, \ldots$ together with the maps $\pi_{l,k}$, $k \leq l$ form an inverse system. Let $\mathcal{J}(\mathcal{E}) = \mathcal{J}^\infty(\mathcal{E}) := \varprojlim \mathcal{J}^k(\mathcal{E})$. Thus, $\mathcal{J}(\mathcal{E})$ carries a natural topology.

Let $j^k : \mathcal{E} \to \mathcal{J}^k(\mathcal{E})$ denote the map $e \mapsto 1 \otimes e$, $j^\infty := \varprojlim j^k$.

Let

$$\begin{aligned} d_1 : \mathcal{O}_{X\times X/\mathcal{P}\times\mathcal{P}} \otimes_{\mathtt{pr}_2^{-1}\mathcal{O}_{X/\mathcal{P}}} \mathtt{pr}_2^{-1}\mathcal{E} \to \\ \to \mathtt{pr}_1^{-1}\Omega^1_{X/\mathcal{P}} \otimes_{\mathtt{pr}_1^{-1}\mathcal{O}_{X/\mathcal{P}}} \mathcal{O}_{X\times X/\mathcal{P}\times\mathcal{P}} \otimes_{\mathtt{pr}_2^{-1}\mathcal{O}_{X/\mathcal{P}}} \mathtt{pr}_2^{-1}\mathcal{E} \end{aligned}$$

denote the exterior derivative along the first factor. It satisfies

$$\begin{aligned} d_1(\mathcal{I}^{k+1}_{X/\mathcal{P}} \otimes_{\mathtt{pr}_2^{-1}\mathcal{O}_{X/\mathcal{P}}} \mathtt{pr}_2^{-1}\mathcal{E}) \subset \\ \mathtt{pr}_1^{-1}\Omega^1_X \otimes_{\mathtt{pr}_1^{-1}\mathcal{O}_{X/\mathcal{P}}} \mathcal{I}^k_{X/\mathcal{P}} \otimes_{\mathtt{pr}_2^{-1}\mathcal{O}_{X/\mathcal{P}}} \mathtt{pr}_2^{-1}\mathcal{E} \end{aligned}$$

for each $k$ and, therefore, induces the map

$$d_1^{(k)} : \mathcal{J}^k(\mathcal{E}) \to \Omega^1_{X/\mathcal{P}} \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}^{k-1}(\mathcal{E})$$

The maps $d_1^{(k)}$ for different values of $k$ are compatible with the maps $\pi_{l,k}$ giving rise to the *canonical flat connection*

$$\nabla^{can}_{\mathcal{E}} : \mathcal{J}(\mathcal{E}) \to \Omega^1_{X/\mathcal{P}} \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}(\mathcal{E})$$

which extends to the flat connection

$$\nabla^{can}_{\mathcal{E}} : \mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}(\mathcal{E}) \to \Omega^1_X \otimes_{\mathcal{O}_X} (\mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}(\mathcal{E}))$$

Here and below by abuse of notation we write $(\bullet) \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}_{X/\mathcal{P}}(\mathcal{E})$ for $\varprojlim (\bullet) \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}^k(\mathcal{E})$.

**5.4. Hochschild cochains.** Suppose that $\mathcal{A}$ is an $\mathcal{O}_{X/\mathcal{P}}$-Azumaya algebra , i.e. a sheaf of algebras on $X$ locally isomorphic to $\mathtt{Mat}_n(\mathcal{O}_{X/\mathcal{P}})$.

For $n > 0$ let $C^n(\mathcal{A})$ denote the sheaf of multidifferential operators $\mathcal{A}^{\times n} \to \mathcal{A}$; Let $C^0(\mathcal{A}) = \mathcal{A}$. The subsheaf of normalized cochains $\overline{C}^n(\mathcal{A})$ consists of those multidifferential operators which yield zero whenever one of the arguments is in $\mathbb{C} \cdot 1 \subset \mathcal{A}$.

With the Gerstenhaber bracket $[\ ,\ ]$ and the Hochschild differential, denoted $\delta$, defined in the standard fashion, $C^\bullet(\mathcal{A})[1]$ and $\overline{C}^\bullet(\mathcal{A})[1]$ are DGLA and the inclusion $\overline{C}^\bullet(\mathcal{A})[1] \to C^\bullet(\mathcal{A})[1]$ is a quasi-isomorphism of such.

For $n > 0$ let $C^n(\mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}(\mathcal{A}))$ denote the sheaf of continuous $\mathcal{O}_X$-multilinear maps $(\mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}(\mathcal{A}))^{\times n} \to \mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}(\mathcal{A})$. The canonical flat connection $\nabla^{can}_{\mathcal{A}}$ on $\mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}(\mathcal{A})$ induces the flat connection, still denoted $\nabla^{can}_{\mathcal{A}}$, on $C^n(\mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}(\mathcal{A}))$. Equipped with the Gerstenhaber bracket and the Hochschild differential $\delta$, $C^\bullet(\mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}(\mathcal{A}))[1]$ is a DGLA. Just as in the case $\mathcal{P} = 0$ (see 4.3) we have the DGLA $\Omega^\bullet_X \otimes_{\mathcal{O}_X} C^\bullet(\mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}(\mathcal{A}))[1]$ with the differential $\nabla^{can} + \delta$.

**Lemma 32.** *The de Rham complex* $\mathtt{DR}(C^n(\mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}(\mathcal{A}))) := (\Omega^\bullet_X \otimes_{\mathcal{O}_X} C^n(\mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}(\mathcal{A})), \nabla^{can}_{\mathcal{A}})$ *satisfies*

1. $H^i\mathtt{DR}(C^n(\mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}(\mathcal{A}))) = 0$ *for* $i \neq 0$
2. *The map* $j^\infty : C^n(\mathcal{A}) \to C^n(\mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}(\mathcal{A}))$ *is an isomorphism onto* $H^0\mathtt{DR}(C^n(\mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}(\mathcal{A})))$.

**Corollary 33.** *The map* $j^\infty : C^\bullet(\mathcal{A}) \to \Omega^\bullet_X \otimes_{\mathcal{O}_X} C^\bullet(\mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}(\mathcal{A}))[1]$ *is a quasi-isomorphism of DGLA.*

**5.5. Azumaya.** Suppose that $\mathcal{A}$ is an $\mathcal{O}_{X/\mathcal{P}}$-Azumaya algebra.

The sheaves $\mathcal{J}(\mathcal{A})$ and $\mathcal{A} \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}_{X/\mathcal{P}}$ have canonical structures of sheaves of (central) $\mathcal{J}_{X/\mathcal{P}}$-algebras locally isomorphic to $\mathtt{Mat}_n(\mathcal{J}_{X/\mathcal{P}})$ and come equipped with projections to $\mathcal{A}$.

Let $\underline{\mathrm{Isom}}_0(\mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{A} \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}_{X/\mathcal{P}}, \mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}(\mathcal{A}))$ denote the sheaf of (locally defined) $\mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}_{X/\mathcal{P}}$-algebra isomorphisms $\mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{A} \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}_{X/\mathcal{P}} \to \mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}(\mathcal{A})$ which induce the identity map on $\mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{A}$. Let $\underline{\mathrm{Aut}}_0(\mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{A} \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}_{X/\mathcal{P}})$ denote the sheaf of (locally defined) $\mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}_{X/\mathcal{P}}$-algebra automorphisms of $\mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{A} \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{J}_{X/\mathcal{P}}$ which induce the identity map on $\mathcal{O}_X \otimes_{\mathcal{O}_{X/\mathcal{P}}} \mathcal{A}$.

Just as in Corollary 7 we may conclude that

$$\begin{aligned}\mathrm{Isom}_0(\mathcal{O}_X\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{A}\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{J}_{X/\mathcal{P}},\mathcal{O}_X\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{J}(\mathcal{A})) :=\\ \Gamma(X;\underline{\mathrm{Isom}}_0(\mathcal{O}_X\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{A}\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{J}_{X/\mathcal{P}},\mathcal{O}_X\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{J}(\mathcal{A})))\end{aligned}$$

is non-empty.

A choice of $\sigma\in\mathrm{Isom}_0(\mathcal{O}_X\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{A}\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{J}_{X/\mathcal{P}},\mathcal{O}_X\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{J}(\mathcal{A}))$ and $\nabla\in\mathcal{C}(\mathcal{O}_X\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{A})$ give rise to a unique $F\in\Gamma(X;\Omega^1_X\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{A}_0\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{J}_{X/\mathcal{P}})$ and $\theta\in\Gamma(X;\Omega^2_X\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{A}_0)$ such that $\nabla^2=\mathrm{ad}\,\theta$ and the equation (3.0.5) holds. Such a $\sigma$ provides us with the isomorphism of DGLA

$$\begin{aligned}(5.5.1)\quad \sigma_*:\Omega^\bullet_X\otimes_{\mathcal{O}_X}C^\bullet(\mathcal{O}_X\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{A}\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{J}_{X/\mathcal{P}})[1]\to\\ \to\Omega^\bullet_X\otimes_{\mathcal{O}_X}C^\bullet(\mathcal{O}_X\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{J}(\mathcal{A}))[1]\end{aligned}$$

where the latter is equipped with the differential $\nabla^{can}_{\mathcal{A}}+\delta$ and the former is equipped with the differential $\nabla\otimes\mathtt{Id}+\mathtt{Id}\otimes\nabla^{can}_{\mathcal{O}}+\mathrm{ad}(F)+\delta$.

The operator $\exp(\iota_F)$ is an automorphism of the graded Lie algebra $\Omega^\bullet_X\otimes_{\mathcal{O}_X}C^\bullet(\mathcal{O}_X\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{A}\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{J}_{X/\mathcal{P}})[1]$. It does not commute with the differential $\nabla\otimes\mathtt{Id}+\mathtt{Id}\otimes\nabla^{can}_{\mathcal{O}}+\mathrm{ad}(F)+\delta$. Instead, the formula of Lemma 16 holds. Hence, the composition $\sigma_*\circ\exp(\iota_F)$ is a quasi-isomorphism of DGLA as in (5.5.1) but with the source equipped with the differential $\nabla\otimes\mathtt{Id}+\mathtt{Id}\otimes\nabla^{can}_{\mathcal{O}}+\delta+\iota_\theta$.

The cotrace map

$$\begin{aligned}\mathrm{cotr}:\overline{C}^\bullet(\mathcal{O}_X\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{J}_{X/\mathcal{P}})[1]\to\\ \to C^\bullet(\mathcal{O}_X\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{A}\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{J}_{X/\mathcal{P}})[1]\end{aligned}$$

defined as in (4.4.2) gives rise to the quasi-isomorphism of DGLA

$$\begin{aligned}\mathtt{Id}\otimes\mathrm{cotr}:\Omega^\bullet_X\otimes_{\mathcal{O}_X}\overline{C}^\bullet(\mathcal{O}_X\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{J}_{X/\mathcal{P}})[1]\to\\ \to\Omega^\bullet_X\otimes_{\mathcal{O}_X}C^\bullet(\mathcal{O}_X\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{A}\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{J}_{X/\mathcal{P}})[1]\end{aligned}$$

where the source (respectively, the target) is equipped with the differential $\nabla^{can}_{\mathcal{O}}+\delta$ (respectively, $\nabla\otimes\mathtt{Id}+\mathtt{Id}\otimes\nabla^{can}_{\mathcal{O}}+\delta+\iota_\theta$).

The proof of 19 shows that the image of the composition $\sigma_*\circ\exp(\iota_F)\circ(\mathtt{Id}\otimes\mathrm{cotr})$ in the derived category does not depend on the choices made. We summarize the above in the following theorem.

**Theorem 34.** *Suppose that $\mathcal{P}$ is an integrable (complex) distribution on $X$ and $\mathcal{A}$ is an $\mathcal{O}_{X/\mathcal{P}}$-Azumaya algebra. There is a canonical isomorphism in the derived category of DGLA $\Omega^\bullet_X\otimes_{\mathcal{O}_X}\overline{C}^\bullet(\mathcal{O}_X\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{J}_{X/\mathcal{P}})[1]\cong\Omega^\bullet_X\otimes_{\mathcal{O}_X}C^\bullet(\mathcal{O}_X\otimes_{\mathcal{O}_{X/\mathcal{P}}}\mathcal{J}(\mathcal{A}))[1]$.*

**Corollary 35.** *Under the assumptions of Theorem 34, there is a canonical isomorphism in the derived category of DGLA* $\overline{C}^{\bullet}(\mathcal{O}_{X/\mathcal{P}}) \cong C^{\bullet}(\mathcal{A})$.

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Department of Mathematics, University of Arizona
*E-mail address*: `bressler@math.arizona.edu`

Department of Mathematics, University of Colorado
*E-mail address*: `Alexander.Gorokhovsky@colorado.edu`

Department of Mathematics, University of Copehagen
*E-mail address*: `rnest@math.ku.dk`

Department of Mathematics, Northwestern University
*E-mail address*: `tsygan@math.northwestern.edu`