Chern classes in Alexander-Spanier cohomology

Alexander Gorokhovsky
Department of Mathematics,
Ohio State University,
Columbus, OH 43210
sasha@math.ohio-state.edu

June 29, 1998

Abstract

In this article we construct explicit cocycles in the Alexander-Spanier cohomological complex, representing the Chern character of an element in K-theory

1 Introduction.

In recent years explicit formulas for cocycles representing the Chern character of an element in K-homology were obtained in several papers ([CST], [MW]). This led in particular to a solution of the problem of finding explicit formulas for the rational Pontrjagin classes of topological manifolds. One of the distinctive features of these approaches, based on [CM], is the use of the Alexander-Spanier (co)homology, so the cocycles obtained live in the Alexander-Spanier homological complex.

In connection with this a "dual" problem naturally arises, that of finding explicit formulas, representing the Chern character of an element in K-theory in the Alexander-Spanier cohomology. In this paper we construct such cocycles. In conjunction with the formulas from [CST], [MW] this allows to write down explicitly index pairing in terms of the Alexander-Spanier (co)homology.

Let us briefly outline the main steps of our construction. Alexander-Spanier n-cochains are continuous functions in the neighborhood of the diagonal in X^{n+1}. Now, for example, if we consider an element in $K^0(X)$ represented by an idempotent $e(x) \in M_N(C(X))$, our cocycle computed at the point $(x_0, x_1, \ldots, x_n) \in X^{n+1}$ is an integral of the differential form representing the Chern character of the universal bundle over the Grassmanian over the canonical simplex with the vertices at $e(x_0)$, $e(x_1)$, ... , $e(x_n)$. It follows easily from Stokes' theorem that the cochain constructed in this way is a cocycle. To verify that our cocycle represents the Chern character components in the smooth case we compute its image under the canonical projection to de Rham cohomology. This projection effectively amounts to averaging our form over smaller and smaller simplices, so in the limit one should get the original form. Finally, for the case of a general compact topological space the result follows by the argument which uses functoriality and homotopy invariance of classes of our cocycles.

The paper is organized as follows : in section 2 we recall the main facts about the Alexander-Spanier complex, and in sections 3 and 4 we construct cocycles representing the Chern character of elements in $K^{-1}(X)$ and $K^0(X)$ respectively.

2 Preliminaries.

We will describe a version of the Alexander-Spanier cohomology (with complex coefficients), following [CM], [MW]. All the proofs can be found in these papers. Let us recall the main definitions and facts concerning this cohomology. Let X be a compact separable topological space. Let $Cov^f(X)$ denote the set of all finite open coverings of X, and let $\mathfrak{U} \in Cov^f(X)$. Let $\mathfrak{U}^n$ denote the neighborhood of the diagonal in X^n given by $\cup_{U \in \mathfrak{U}} U^n$. Then Alexander-Spanier n-cocycles (corresponding to $\mathfrak{U}$) are continuous functions on $\mathfrak{U}^{n+1}$. The space of n-cocycles is denoted by $C^n(X, \mathfrak{U})$. The boundary operator $\partial : C^n(X, \mathfrak{U}) \to C^{n+1}(X, \mathfrak{U})$ is defined by the formula (here $\phi \in C^n(X, \mathfrak{U})$)

$$\partial\phi(x_0, x_1, \ldots, x_{n+1}) = \sum_{j=0}^{n+1} (-1)^j \phi(x_0, \ldots, \hat{x}_j, \ldots, x_{n+1}) \tag{1}$$

Then $\partial^2 = 0$ and the cohomology of the complex $(C^n(X, \mathfrak{U}), \partial)$ is called Alexander-Spanier cohomology of the covering $\mathfrak{U}$; it is denoted $H^n(X, \mathfrak{U})$. If

$\mathfrak{V}$ is a refinement of $\mathfrak{U}$, there is an obvious map (restriction) from $C^n(X, \mathfrak{U})$ to $C^n(X, \mathfrak{U})$, which commutes with the boundary from (1) , and hence defines a map $H^n(X, \mathfrak{U}) \to H^n(X, \mathfrak{V})$. Then Alexander-Spanier cohomology is defined as a direct limit over finite covers:

$$H^n(X) = \varinjlim H^n(X, \mathfrak{U})$$

The cup-product on the Alexander-Spanier complex is given by the formula

$$\phi \cup \psi(x_0, \ldots, x_{m+n}) = \phi(x_0, \ldots, x_n)\psi(x_n, \ldots, x_{m+n})$$

where ϕ and ψ are n and m cochains respectively.

We will also describe, following [MW], the dual theory — Alexander-Spanier homology. Again, we fix a cover $\mathfrak{U}$ of X. Let $C_n(X, \mathfrak{U})$ be the space of measures with (compact) support in $\mathfrak{U}^{n+1}$ — chains for the Alexander-Spanier homology complex. This is the dual space for $C^n(X, \mathfrak{U})$ endowed with a topology of uniform convergence on the compacts. The boundary operator, which we also denote ∂, is defined as transposed to the operator given by (1):

$$\partial\mu(\phi) := \mu(\partial\phi)$$

Here $\mu \in C_n(X, \mathfrak{U})$, $\phi \in C^n(X, \mathfrak{U})$. The homology of $(C_n(X, \mathfrak{U}), \partial)$ is called Alexander-Spanier homology of $\mathfrak{U}$ and is denoted $H_n(X, \mathfrak{U})$. Now, if $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, there is a map of complexes $C_*(X, \mathfrak{V}) \to C_*(X, \mathfrak{U})$ (extension by zero). It induces a map $H_n(X, \mathfrak{V}) \to H_n(X, \mathfrak{U})$. Alexander-Spanier homology of X is then defined as an inverse limit over finite covers:

$$H_n(X) = \varprojlim H_n(X, \mathfrak{U})$$

It can be shown that Alexander-Spanier (co)homology is isomorphic to the Čech (co)homology, and hence to the de Rham (co)homology if X is a smooth manifold. In the latter case one can consider the smooth Alexander-Spanier complex, where n-cochains are given by smooth functions in the neighborhood of the diagonal in X^{n+1}, and the coboundary operator is given by (1). We will now describe a canonical morphism λ from the smooth Alexander-Spanier complex to the de Rham cohomological complex, which induces an isomorphism in cohomology. Let ϕ be a smooth Alexander-Spanier n-cocycle.

Let $x_0 \in X$, and $v_1, \ldots, v_n \in T_{x_0}X$. Consider any n curves $x_j(\epsilon)$, $j = 1, \ldots, n$ with the properties

$$x_j(0) = x_0$$
$$\frac{d}{d\epsilon} x_j|_{\epsilon=0} = v_j$$

Then the differential n-form $\lambda(\phi)$ is defined by the equation:

$$\lambda(\phi)_{x_0}(v_1, \ldots, v_n) = \frac{1}{n!} \sum_{\sigma \in S_n} sgn(\sigma) \frac{\partial}{\partial \epsilon_1} \ldots \frac{\partial}{\partial \epsilon_n} \phi\left(x_0, x_{\sigma(1)}(\epsilon_1), \ldots, x_{\sigma(n)}(\epsilon_n)\right)|_{\epsilon_j=0} \quad (2)$$

3 The odd case.

In this section we will give formulas for the Alexander-Spanier cochain describing the odd Chern character $Ch : K^{-1}(X) \to H^{odd}(X)$, where X is a compact topological space. Corresponding formulas for the de Rham cohomology in the smooth case are well known; a good exposition can be found in the first section of [Ge]. An element in K^{-1} can be represented by a continuous map U from X to $\mathbf{U}(N)$ – the group of unitary $N \times N$ matrices, for some N (note that since we are working with complex coefficients, $K^{-1}(X) = K^1(X)$).

Now, let $x_0, x_1, \ldots, x_n$ be $n+1$ sufficiently close points in X, so that $\|U(x_k) - U(x_l)\| < \rho < 1$ for some ρ, and where $\|\cdot\|$ is the operator norm.

Let $t_1, t_2, \ldots, t_n$ be nonnegative numbers with $t_1 + t_2 \cdots + t_n \leq 1$. Put $t_0 = 1 - \sum_{j=0}^{n} t_j$. Define

$$U(t_1, t_2, \ldots, t_n) = \sum_{j=0}^{n} t_j U(x_j) = U(x_0) + \delta \quad (3)$$

where

$$\delta = \sum_{j=0}^{n} t_j (U(x_j) - U(x_0)) = \sum_{j=0}^{n} t_j \delta_j \quad (4)$$
$$\delta_j = U(x_j) - U(x_0) \quad (5)$$

Since $\|\delta\| < 1$, $U(t_1, t_2, \ldots, t_n)$ is always invertible.

We can now consider on the n-simplex

$$\Delta^n = \{t_j, j = 1, \dots, n | \sum_{j=1}^{n} t_j \leq 1\} = \{t_j, j = 0, \dots, n | \sum_{j=0}^{n} t_j = 1\}$$

a matrix-valued 1-form $U(t_1, \dots, t_n)^{-1} dU(t_1, \dots, t_n)$, where

$$dU(t_1, \dots, t_n) = \sum_{j=1}^{n} \frac{\partial U(t_1, \dots, t_n)}{\partial t_j} dt_j$$

This form can be rewritten as

$$\begin{aligned} U(t_1, t_2, \dots, t_n)^{-1} dU(t_1, t_2, \dots, t_n) &= (U(x_0) + \delta)^{-1} d(U(x_0) + \delta) \\ &= \sum_{j=1}^{n} \sum_{k=0}^{\infty} (-1)^k (U(x_0)^{-1} \delta)^k U(x_0)^{-1} \delta_j dt_j \end{aligned} \quad (6)$$

From this one can construct for n odd an Alexander-Spanier cochain

$$Ch^n(x_0, \dots, x_n) = c_n \int_{\Delta^n} Tr\ (U(t_1, t_2, \dots, t_n)^{-1} dU(t_1, t_2, \dots, t_n))^n \quad (7)$$

Here the n-th power is taken with respect to the exterior product and $c_n = \frac{(-1)^{(n-1)/2}((n-1)/2)!}{(2\pi i)^{(n+1)/2} n!}$.

For example, for $n = 1$, $\delta = t_1 \delta_1 = t_1(U(x_1) - U(x_0))$,

$$\begin{aligned} U(t_1)^{-1} dU(t_1) &= \sum_{k=0}^{\infty} (-1)^k (U(x_0)^{-1} \delta)^k U(x_0)^{-1} \delta_1 dt_1 \\ &= \sum_{k=0}^{\infty} (-1)^k (U(x_0)^{-1} \delta_1)^{k+1} t_1^k dt_1 \end{aligned} \quad (8)$$

by (6), and

$$\begin{aligned} Ch^1(x_0, x_1) &= \frac{1}{2\pi i} \int_0^1 Tr \sum_{k=0}^{\infty} (-1)^k (U(x_0)^{-1} \delta_1)^{k+1} t_1^k dt_1 \\ &= \frac{1}{2\pi i} Tr \sum_{k=0}^{\infty} \frac{(-1)^k (U(x_0)^{-1} \delta_1)^{k+1}}{k+1} = \frac{1}{2\pi i} Tr \log(1 + U(x_0)^{-1} \delta_1) \\ &= \frac{1}{2\pi i} \log\det\big((1 + U(x_0)^{-1} \delta_1\big) = \frac{1}{2\pi i} \log\det\big(U(x_0)^{-1} U(x_1)\big) \end{aligned} \quad (9)$$

Remark 1. Note that this function is well-defined and continuous in the neighborhood of the diagonal given by condition $\|\delta_1\| < \rho < 1$, while the function $\log\det U(x_0)$ is not well defined. This explains why we *cannot* write that

$$\begin{aligned} Ch^1(x_0, x_1) &= \frac{1}{2\pi i}\left(-\log\det U(x_0) + \log\det U(x_1)\right) \\ &= \partial\left(\frac{1}{2\pi i}(\log\det U)(x_0, x_1)\right) \end{aligned}$$

We can now formulate the main result of this section:

Theorem 2. *Cochains Ch^n, defined by (7), give components for the Chern character of $U(x)$.*

To prove this theorem we need several lemmas.

Lemma 3. *Formula (7) defines an Alexander-Spanier cocycle.*

Proof. First, notice that Ch^n is clearly a continuous function (in the neighborhood of the diagonal). Now suppose that we are given $n+2$ points $x_0, \ldots, x_{n+1}$. Similarly to (3) we can construct an operator over $\Delta^{n+1} = \{t_j, j = 0, \ldots, n+1 \mid \sum_{j=0}^{n+1} t_j = 1, t_j \geq 0\}$

$$V(t_1, t_2, \ldots, t_{n+1}) = \sum_{j=0}^{n+1} t_j U(x_j) \tag{10}$$

We can consider for any n a form $Tr(V^{-1}dV)^n$ on Δ^{n+1}. This form is 0 for even n — indeed,

$$\begin{aligned} Tr(V^{-1}dV)^n &= Tr V^{-1}dV(V^{-1}dV)^{n-1} \\ &= (-1)^{n-1}Tr(V^{-1}dV)^{n-1}V^{-1}dV = -Tr(V^{-1}dV)^n \end{aligned}$$

For n odd this form is closed, since, using the equality $d(V^{-1}) = -V^{-1}dVV^{-1}$ we get:

$$dTr(V^{-1}dV)^n = -Tr(V^{-1}dV)^{n+1} = 0$$

by the previous computation.

Now, the restriction of V to the face given by equation $t_j = 0$ coincides (after an obvious renumeration of variables) with the operators constructed by the formula (3) from the points $x_0, \dots, \hat{x}_j, \dots, x_{n+1}$. From this we conclude that

$$\begin{aligned}\partial Ch^n(x_0, \dots, x_{n+1}) &= \sum_{j=0}^{n+1} (-1)^j Ch^n(x_0, \dots, \hat{x}_j, \dots, x_{n+1}) \\ &= \frac{(-1)^{(n-1)/2}((n-1)/2)!}{(2\pi i)^{(n+1)/2} n!} \int_{\partial\Delta^{n+1}} Tr(V^{-1}dV)^n \\ &= \frac{(-1)^{(n-1)/2}((n-1)/2)!}{(2\pi i)^{(n+1)/2} n!} \int_{\Delta^{n+1}} dTr(V^{-1}dV)^n = 0 \quad (11)\end{aligned}$$

by Stokes' theorem. □

Lemma 4. *The cohomology class of the cocycle defined by (7) depends only on the class $[U] \in K^{-1}(X)$ and not on the particular representative U.*

Proof. Let $U_0(x)$, $U_1(x)$ be two maps into $\mathbf{U}(N)$ representing the same element in $K^{-1}(X)$. By taking N big enough one can suppose that $U_0(x)$ and $U_1(x)$ are homotopic via piecewise-smooth family of maps $U_\tau(x)$, $0 \leq \tau \leq 1$. Using the formula (3) we define homotopy $U_\tau(t_1, \dots, t_n)$ between $U_0(t_1, \dots, t_n)$ and $U_1(t_1, \dots, t_n)$.

We will now show that cocycles corresponding to the smoothly homotopic unitaries differ by a coboundary. Define Ch^n_τ by the equation (7). Then (we write just U_τ for $U_\tau(t_1, \dots, t_n)$):

$$\frac{d}{d\tau} Ch^n_\tau = \frac{(-1)^{(n-1)/2}((n-1)/2)!}{(2\pi i)^{(n+1)/2} n!} \int_{\Delta^n} \frac{d}{d\tau} Tr(U_\tau^{-1} dU_\tau)^n \qquad (12)$$

But

$$\frac{d}{d\tau} Tr(U_\tau^{-1} dU_\tau)^n = n \left(dTrU_\tau^{-1} \frac{dU}{d\tau} (U_\tau^{-1} dU_\tau)^{n-1} \right)$$

Indeed,

$$\frac{d}{d\tau}Tr(U_\tau^{-1}dU_\tau)^n = nTr\left(\frac{d}{d\tau}(U_\tau^{-1}dU_\tau)\right)(U_\tau^{-1}dU_\tau)^{n-1}$$
$$= nTr\left(U_\tau^{-1}\frac{dU}{d\tau}U_\tau^{-1}dU_\tau(U_\tau^{-1}dU_\tau)^{n-1}\right) + nTr\left(U_\tau^{-1}d(\frac{dU}{d\tau})(U_\tau^{-1}dU_\tau)^{n-1}\right)$$
$$= nTr\left(U_\tau^{-1}dU_\tau U_\tau^{-1}\frac{dU}{d\tau}(U_\tau^{-1}dU_\tau)^{n-1}\right) + nTr\left(U_\tau^{-1}d(\frac{dU}{d\tau})(U_\tau^{-1}dU_\tau)^{n-1}\right) \tag{13}$$

and

$$d\ TrU_\tau^{-1}\frac{dU}{d\tau}(U_\tau^{-1}dU_\tau)^{n-1} = Tr\ d(U_\tau^{-1})\frac{dU}{d\tau}(U_\tau^{-1}dU_\tau)^{n-1}$$
$$+ TrU_\tau^{-1}d(\frac{dU}{d\tau})(U_\tau^{-1}dU_\tau)^{n-1} + TrU_\tau^{-1}\frac{dU}{d\tau}d(U_\tau^{-1}dU_\tau)^{n-1}$$
$$= Tr\ U_\tau^{-1}(dU_\tau)U_\tau^{-1}\frac{dU}{d\tau}(U_\tau^{-1}dU_\tau)^{n-1} + TrU_\tau^{-1}d(\frac{dU}{d\tau})(U_\tau^{-1}dU_\tau)^{n-1} \tag{14}$$

since $d(U_\tau^{-1}dU_\tau)^{n-1} = 0$ for odd n.

Hence, from (12)

$$\frac{d}{d\tau}Ch_\tau^n = \int_{\partial\Delta^n}\frac{(-1)^{(n-1)/2}((n-1)/2)!}{(2\pi i)^{(n+1)/2}n!}n\left(TrU_\tau^{-1}\frac{dU}{d\tau}(U_\tau^{-1}dU_\tau)^{n-1}\right) \tag{15}$$

Now notice that since the restriction of U_τ to the j-th face of the Δ^n given by equation $t_j = 0$ depends only on $U(x_k), k \neq j$, and not on $U(x_j)$, we can define an Alexander-Spanier $n-1$ cochain

$$T_\tau(x_0,\ldots,x_{n-1}) = \frac{(-1)^{(n-1)/2}((n-1)/2)!}{(2\pi i)^{(n+1)/2}n!}n\int_{\Delta^{n-1}}TrU_\tau^{-1}\frac{dU}{d\tau}(U_\tau^{-1}dU_\tau)^{n-1}$$

where $\Delta^{n-1}=\{(t_1,\ldots,t_{n-1}) \mid t_j \geq 0, \sum_{j=1}^{n-1}t_j \leq 1\} = \{(t_1,\ldots,t_n) \in \Delta^n \mid t_n = 0\}$. Then according to (15)

$$\frac{d}{d\tau}Ch_\tau^n = \partial T_\tau$$

and

$$Ch_1^n - Ch_0^n = \partial\int_0^1 T_\tau d\tau$$

which proves the Lemma. □

Lemma 5. *Let X be a compact smooth manifold and $U : X \to \mathbf{U}(N)$ be a smooth map. Then Ch^n given by the formula (7) represents the n-th component of the Chern character of $[U] \in K^{-1}(X)$.*

Proof. It is known that in the situation described in the Lemma the differential form

$$\Omega = \frac{(-1)^{(n-1)/2}((n-1)/2)!}{(2\pi i)^{(n+1)/2} n!} Tr(U^{-1}dU)^n \tag{16}$$

represents the n-th component of the Chern character (here the differential is taken with respect to the x variable). We will show that the canonical map from Alexander-Spanier to de Rham complex maps cocycle given by (7) in the differential form given by (16). We consider a point x_0 and n curves $x_j(\epsilon_j), j = 1, \ldots, n$ such that $x_j(0) = x_0$; let the tangent vector to x_j at x_0 be v_j. From (4) we have easily

$$\delta|_{\epsilon_k=0} = 0$$
$$\delta_j|_{\epsilon_k=0} = 0$$

Also

$$\frac{\partial \delta}{\partial \epsilon_k}|_{\epsilon_j=0} = t_k(\mathcal{L}_{v_k}U)(x_0)$$

and

$$\frac{\partial \delta_l}{\partial \epsilon_k}|_{\epsilon_j=0} = \begin{cases} (\mathcal{L}_{v_k}U)(x_0), & \text{if } k = l \\ 0, & \text{otherwise} \end{cases}$$

Here $\mathcal{L}_{v_k}$ denotes directional derivative. From this by differentiating (6) one gets

$$\frac{\partial U(t_1, t_2, \ldots, t_n)^{-1} dU(t_1, t_2, \ldots, t_n)}{\partial \epsilon_k}|_{\epsilon_j=0} = U^{-1}(x_0)(\mathcal{L}_{v_k}U)(x_0)dt_k$$

We now compute with $c_n = \frac{(-1)^{(n-1)/2}((n-1)/2)!}{(2\pi i)^{(n+1)/2} n!}$

$$
\begin{aligned}
&\frac{\partial}{\partial \epsilon_1} \cdots \frac{\partial}{\partial \epsilon_n} Ch^n\left(x_0, x_1(\epsilon_1), \ldots, x_n(\epsilon_n)\right)|_{\epsilon_j=0} \\
&\qquad = c_n \int_{\Delta^n} Tr \frac{\partial}{\partial \epsilon_1} \cdots \frac{\partial}{\partial \epsilon_n} (U(t_1, t_2, \ldots, t_n)^{-1} dU(t_1, t_2, \ldots, t_n))^n|_{\epsilon_j=0} \\
&= \sum_{\sigma \in S_n} sgn(\sigma) c_n \int_{\Delta^n} Tr U^{-1}(x_0)(\mathcal{L}_{v_{\sigma(1)}} U)(x_0) \ldots U^{-1}(x_0)(\mathcal{L}_{v_{\sigma(n)}} U)(x_0) dt_1 \ldots dt_n \\
&\quad = c_n \sum_{\sigma \in S_n} sgn(\sigma) \frac{1}{n!} Tr U^{-1}(x_0)(\mathcal{L}_{v_{\sigma(1)}} U)(x_0) \ldots U^{-1}(x_0)(\mathcal{L}_{v_{\sigma(n)}} U)(x_0) \\
&\qquad = \Omega_{x_0}(v_1, \ldots, v_n) \quad (17)
\end{aligned}
$$

The result is already antisymmetric in $v_1, \ldots, v_n$, and the assertion follows. $\square$

Now we can prove our Theorem 2.

Proof of Theorem 2. When X and U are smooth the result was just proved. For the general X and U, since any element in $K^{-1}(X)$ can be pulled back from some smooth manifold, and our construction is manifestly functorial with respect to pull-backs of unitaries, we find that for any class in $K^{-1}(X)$ for *some* representative our formula represents the components of the Chern character. But then by the Lemma 4 this is true for *all* representatives. $\square$

4 The even case.

Let X be a compact topological space and $E \to X$ be a complex vector bundle. Suppose that E is embedded as a subbundle into a trivial bundle with total space $X \times \mathbb{C}^N$. This allows us to represent E by a self-adjoint projector $e(x)$ in $M_N(C(X))$ — the algebra of $N \times N$ matrices of continuous functions on X. We will construct here an Alexander-Spanier cochain representing the Chern character of E.

Let $x_0, x_1, \ldots, x_n$ be $n+1$ points in X sufficiently close to each other: namely, we require that $\|e(x_i) - e(x_j)\| < \rho < 1/2$ (with respect to the usual operator norm). We will now define the function $Ch^n(E)$, representing the n-th component of the Chern character of E (n is even here). Let $t_1, t_2, \ldots, t_n$

be positive numbers with $t_1+t_2+\cdots+t_n \leq 1$. Put $t_0 = 1-\sum_{i=1}^n t_i$. Consider the operator

$$a(t_1,\ldots,t_n) = \sum_{i=0}^n t_i e(x_i) = e(x_0) + \delta$$

where

$$\delta = \sum_{i=1}^n t_i(e(x_i) - e(x_0)) \tag{18}$$

Since $\|a(t_1,\ldots,t_n) - e(x_0)\| \leq \rho$, the spectrum of $a(t_0,\ldots,t_n)$ is contained in the union of 2 discs of radius $\rho < 1/2$ and centers at 0 and 1. Let $e(t_1,\ldots,t_n)$ be the spectral projector on the part of the spectrum inside the disc $|\lambda-1| < 1/2$:

$$e(t_1,\ldots,t_n) = \frac{1}{2\pi i}\int_{|\lambda-1|=1/2} (\lambda - a(t_1,\ldots,t_n))^{-1} d\lambda \tag{19}$$

Notice that $e(t_1,\ldots,t_n)$ depends smoothly on $t_1,\ldots,t_n$, since $a(t_1,\ldots,t_n)$ depends smoothly on them and its spectrum does not intersect the contour of integration.

Now we can rewrite $e(t_1,\ldots,t_n)$ in yet another form.

$$\begin{aligned}
e(t_1,\ldots,t_n) &= \frac{1}{2\pi i}\int_{|\lambda-1|=1/2} (\lambda - a(t_1,\ldots,t_n))^{-1} d\lambda \\
&= \frac{1}{2\pi i}\int_{|\lambda-1|=1/2} (\lambda - (e(x_0)+\delta))^{-1} d\lambda \\
&= \frac{1}{2\pi i}\int_{|\lambda-1|=1/2} (1-(\lambda - e(x_0))^{-1}\delta)^{-1}(\lambda - e(x_0))^{-1} d\lambda \\
&= \frac{1}{2\pi i}\sum_{k=0}^{\infty}\int_{|\lambda-1|=1/2} ((\lambda - e(x_0))^{-1}\delta)^k(\lambda - e(x_0))^{-1} d\lambda
\end{aligned} \tag{20}$$

We now use the identity

$$(\lambda - e(x_0))^{-1} = \frac{e(x_0)}{\lambda-1} + \frac{1-e(x_0)}{\lambda}$$

to rewrite the k-th term:

$$\frac{1}{2\pi i}\int_{|\lambda-1|=1/2}((\lambda-e(x_0))^{-1}\delta)^k(\lambda-e(x_0))^{-1}d\lambda$$
$$=\frac{1}{2\pi i}\int_{|\lambda-1|=1/2}\left(\left(\frac{e(x_0)}{\lambda-1}+\frac{1-e(x_0)}{\lambda}\right)\delta\right)^k\left(\frac{e(x_0)}{\lambda-1}+\frac{1-e(x_0)}{\lambda}\right)d\lambda$$
$$=\frac{1}{2\pi i}\int_{|\lambda-1|=1/2}\sum_{m+l=k+1}\frac{1}{(\lambda-1)^m\lambda^l}(b_0\delta b_1\delta\dots\delta b_k)d\lambda \quad (21)$$

Here each b_j equals either $e(x_0)$ or $(1-e(x_0))$; each such monomial contains m factors of the first type and l factors of the second. We see that the expression under the integral has the only pole inside the contour of integration at the point 1, and the residue at this point can be explicitly computed. Indeed, by the binomial formula, $(\lambda-1)^{-m}\lambda^{-l}=\sum_{p=0}^{\infty}\binom{-l}{p}(\lambda-1)^{-m}(\lambda-1)^{-p}$, and hence $res|_{\lambda=1}(\lambda-1)^{-m}\lambda^{-l}=\binom{-l}{m-1}=(-1)^{m-1}\binom{l+m-2}{m-1}=(-1)^{m-1}\binom{k-1}{m-1}$ (here we suppose, as usual, that $\binom{k-1}{-1}=0$) , and

$$\frac{1}{2\pi i}\int_{|\lambda-1|=1/2}((\lambda-e(x_0))^{-1}\delta)^k(\lambda-e(x_0))^{-1}d\lambda$$
$$=\sum(-1)^{m-1}\binom{k-1}{m-1}b_0\delta b_1\delta\dots\delta b_k \quad (22)$$

Here in the term with the coefficient $(-1)^{m-1}\binom{k-1}{m-1}$ there are m factors b_j equal to $e(x_0)$.

Hence

$$e(t_1,t_2,\dots,t_n)=\sum_k\sum(-1)^{m-1}\binom{k-1}{m-1}b_0\delta b_1\delta\dots\delta b_k \quad (23)$$

For n even we define a function

$$Ch^n(x_0,x_1,\dots,x_n)=b_n\int\limits_{\sum_{j=1}^n t_j\le 1,t_j\ge 0} Tr\ e(t_1,t_2,\dots,t_n)(de(t_1,t_2,\dots,t_n))^n \quad (24)$$

with $b_n=\frac{(-1)^{n/2}}{(2\pi i)^{n/2}(n/2)!}$. Here $de(t_1,t_2,\dots,t_n)=\sum_{j=1}^n\frac{\partial e(t_1,t_2,\dots,t_n)}{\partial t_j}dt_j$ — a matrix-valued 1-form on the n-simplex $\Delta^n=\{t_j,j=1,\dots,n\mid\sum_{j=1}^n t_j\le 1,t_j\ge 0\}=\{t_j,j=0,\dots,n\mid\sum_{j=0}^n t_j=1,t_j\ge 0\}$

We are now going to prove that Ch^n represents the n-th component of the Chern character of the bundle E .

Theorem 6. *Let E be a complex vector bundle over a compact topological space X, represented by the idempotent $e(x)$, and let Ch^n be the Alexander-Spanier cocycle defined by (24). Then it represents the n-th component of the Chern character of the bundle E.*

Remark 7. One can show that the second component of the Chern character – the first Chern class – can be described also by the following cocycle ϕ, cohomologous to $Ch^2(x_0, x_1, x_2)$. Let the points x_0, x_1 and x_2 be sufficiently close. $e(x_k) \in M_N(\mathbb{C})$ is an orthogonal projection and let $E_k \subset \mathbb{C}^N$ be its image, $k = 0,\ 1\ ,\ 2$. Let P_{kl} be the orthogonal projection from E_k to E_l. Then $P_{12}P_{23}P_{31}$ is a linear transformation in E_1, sufficiently close to 1, and put

$$\phi(x_0, x_1, x_2) = -\frac{1}{\pi}\Im \log\det(P_{12}P_{23}P_{31})$$

where $\Im$ denotes the imaginary part. (Here log can be unambiguously defined by requiring that $\log(1) = 0$, since $\det(P_{12}P_{23}P_{31})$ is close to 1).

To prove the theorem 6 we need several lemmas.

Lemma 8. *Ch^n is an Alexander-Spanier cocycle.*

Proof. First, notice that Ch^n is clearly a continuous function (in the neighborhood of the diagonal). Now suppose that we are given $n+2$ points $x_0, \ldots, x_{n+1}$. Similarly to (19) we can construct an idempotent over $\Delta^{n+1} = \{t_j, j = 0, \ldots, n+1 \mid \sum_{j=0}^{n+1} t_j = 1, t_j \geq 0\}$

$$F(t_0, \ldots, t_{n+1}) = \frac{1}{2\pi i}\int_{|\lambda-1|=1/2}\left(\lambda - \sum_{j=0}^{n+1} t_j e(x_j)\right)^{-1} d\lambda \tag{25}$$

We can consider for any n a form $TrF(dF)^n$ on Δ^{n+1}. This form is 0 for odd n and is closed for even n. Indeed, since $F^2 = F$, we have $FdF + dFF = dF$. Multiplying by F we get $FdFF = 0$. Also $FdFdF = dFdF - dFFdF = dFdF - dFdF + dFdFF = dFdFF$. Using this we get for n odd:

$$TrF(dF)^n = TrF(dF)^nF = TrFdFF(dF)^{n-1} = 0$$

For n even we get:

$$dTrF(dF)^n = dTrF(dF)^nF = Tr(dF)^{n+1}F + TrF(dF)^{n+1} = 0,$$

where we have used the calculation for odd n. Now, the restriction of F to the face given by equation $t_j = 0$ coincides (after an obvious renumeration of variables) with the idempotents constructed by the formula (19) from the points $x_0, \ldots, \hat{x}_j, \ldots, x_{n+1}$. This implies that

$$\begin{aligned}\partial Ch^n(x_0, \ldots, x_{n+1}) &= \sum_{j=0}^{n+1}(-1)^j Ch^n(x_0, \ldots, \hat{x}_j, \ldots, x_{n+1}) \\ &= b_n \int_{\partial\Delta^{n+1}} TrF(dF)^n = \int_{\Delta^{n+1}} dTrF(dF)^n = 0 \qquad (26)\end{aligned}$$

by Stokes' theorem,$b_n = \frac{(-1)^{n/2}}{(2\pi i)^{n/2}(n/2)!}$. □

Lemma 9. *The cohomology class of Ch^n depends only on the bundle E, and not on the particular idempotent representing the bundle.*

Proof. Suppose that we have two different embeddings of our bundle into trivial bundles, and let $e_0(x)$, $e_1(x)$ be the two corresponding idempotents. We can suppose that e_0 and e_1 are homotopic (this can be achieved by "enlarging" original trivial bundles). This homotopy can be chosen to be at least piecewise-smooth. Now we will show that the cocycles corresponding to the two smoothly homotopic idempotents differ by a coboundary. Let $e_\tau(x)$ be this homotopy. Define by (19) the idempotent over Δ^n $e_\tau(t_1, \ldots .t_n)$. In the proof of this lemma we will write it just as e_τ. Notice that for each τ the restriction of $e_\tau(t_1, \ldots, t_n)$ to the face $t_j = 0$ depends only on $e(x_k), k \neq j$. Let Ch^n_τ be defined similarly by (24).

$$\frac{d}{d\tau}Ch^n \ = \ b_n \int\limits_{\sum_{j=1}^n t_j \leq 1, t_j \geq 0} \frac{d}{d\tau}Tr \ \ e_\tau(t_1, t_2, \ldots, t_n)(de_\tau(t_1, t_2, \ldots, t_n))^n \qquad (27)$$

$b_n = \frac{(-1)^{n/2}}{(2\pi i)^{n/2}(n/2)!}$. But

$$\frac{d}{d\tau} Tr\ e_\tau (de_\tau)^n = Tr\ \frac{d}{d\tau}(e_\tau)(de_\tau)^n + \sum_{j=1}^{n} Tr\ e_\tau (de_\tau)^{j-1}(\frac{d}{d\tau} de_\tau)(de_\tau)^{n-j}$$
$$= d\left(\sum_{j=1}^{n}(-1)^j Tr\ e_\tau (de_\tau)^{j-1}(\frac{d}{d\tau}e_\tau)(de_\tau)^{n-j}\right) - \sum_{j=1}^{n} Tr\ (de_\tau)^j (\frac{d}{d\tau}e_\tau)(de_\tau)^{n-j}$$
$$= d\left(\sum_{j=1}^{n}(-1)^j Tr\ e_\tau (de_\tau)^{j-1}(\frac{d}{d\tau}e_\tau)(de_\tau)^{n-j}\right) \quad (28)$$

since $\sum_{j=1}^{n} Tr\ (de_\tau)^j (\frac{d}{d\tau}e_\tau)(de_\tau)^{n-j} = 0$.

Indeed, we have as before

$$\frac{d}{d\tau}e_\tau = \frac{d}{d\tau}e_\tau^2 = e_\tau \frac{d}{d\tau}e_\tau + (\frac{d}{d\tau}e_\tau)e_\tau$$

and from this $e_\tau(\frac{d}{d\tau}e_\tau)e_\tau = 0$. Hence

$$\sum_{j=1}^{n} Tr\ (de_\tau)^j (\frac{d}{d\tau}e_\tau)(de_\tau)^{n-j}$$
$$= \sum_{j=1}^{n} Tr\ (de_\tau)^j (\frac{d}{d\tau}e_\tau)e_\tau^2 (de_\tau)^{n-j} + \sum_{j=1}^{n} Tr\ (de_\tau)^j e_\tau^2 (\frac{d}{d\tau}e_\tau)(de_\tau)^{n-j}$$
$$= 2\sum_{j=1}^{n} Tr\ (de_\tau)^j \left(e_\tau(\frac{d}{d\tau}e_\tau)e_\tau\right)(de_\tau)^{n-j} = 0 \quad (29)$$

Notice that the form $\sum_{j=1}^{n}(-1)^j Tr\ e_\tau (de_\tau)^{j-1}(\frac{d}{d\tau}e_\tau)(de_\tau)^{n-j}$ restricted to the face of Δ^n depends only on the values of e_τ at the vertices of this face. This allows us to define an Alexander-Spanier $n-1$ cochain $T_\tau(x_0, \ldots, x_{n-1})$. Construct, by (19), $e_\tau = e_\tau(t_1, \ldots, t_{n-1})$. Then

$$T_\tau(x_0, \ldots, x_{n-1})$$
$$= \frac{(-1)^{n/2}}{(2\pi i)^{n/2}(n/2)!} \int_{\sum_{j=1}^{n-1} t_j \le 1, t_j \ge 0} \sum_{j=1}^{n}(-1)^j Tr\ e_\tau (de_\tau)^{j-1}(\frac{d}{d\tau}e_\tau)(de_\tau)^{n-j} \quad (30)$$

Now, with $b_n = \frac{(-1)^{n/2}}{(2\pi i)^{n/2}(n/2)!}$

$$
\begin{aligned}
\frac{d}{d\tau} Ch^n_\tau = b_n \int\limits_{\sum_{j=1}^n t_j \le 1, t_j \ge 0} & \frac{d}{d\tau} Tr\ e_\tau(t_1, t_2, \ldots, t_n)(de_\tau(t_1, t_2, \ldots, t_n))^n \\
= b_n \int\limits_{\partial\Delta^n} & \sum_{j=1}^n (-1)^j Tr\ e_\tau (de_\tau)^{j-1} (\frac{d}{d\tau} e_\tau)(de_\tau)^{n-j} \\
= & \sum_{j=0}^n (-1)^j T_\tau(x_0, \ldots, \hat{x}_j, \ldots, x_n) \\
& = (\partial T_\tau)(x_0, \ldots, x_n) \quad (31)
\end{aligned}
$$

by Stokes' theorem, and

$$
Ch^n_1 - Ch^n_0 = \partial \int_0^1 T_\tau d\tau \tag{32}
$$

This proves the lemma. ☐

Lemma 10. *If E is a smooth vector bundle over a smooth manifold, Ch^n represents the n-th component of the Chern character.*

Proof. We will show that under the canonical map from the Alexander-Spanier complex to de the Rham cohomological complex Ch^n is mapped to the differential form representing the n-th component of Chern character constructed from the curvature of the natural connection on E induced by the embedding in the trivial bundle. Namely, $e(x)$ defines connection by $\nabla\xi = ed\xi, \xi \in \Gamma(E)$. The curvature of this connection is $\nabla^2 = edede$. By the Chern-Weil theory differential form $\frac{(-1)^{n/2}}{(2\pi i)^{n/2}(n/2)!} Tr\ e(de)^n$ represents the component of the Chern character of the bundle E in the n-th cohomology. Now we will compute the image of Ch^n under the isomorphism with de Rham cohomology. We consider an arbitrary point x_0 and n curves $x_j(\epsilon_j), j = 1, \ldots, n$ such that $x_j(0) = x_0$ and let the tangent vector of x_j at x_0 be v_j. We compute (δ is from (18)):

$$
\delta(x_0, x_1(\epsilon_1), \ldots, x_n(\epsilon_n))|_{\epsilon_j=0} = 0
$$

$$
\frac{\partial}{\partial \epsilon_k} \delta|_{\epsilon_j=0} = t_k(\mathcal{L}_{v_k} e)(x_0)
$$

Here $\mathcal{L}_{v_k}$ denotes directional derivative. Using this and differentiating (23) we get:

$$\frac{\partial}{\partial \epsilon_k} e(t_1, \ldots, t_n)|_{\epsilon_j=0}$$
$$= (1 - e(x_0))(\mathcal{L}_{v_k} e)(x_0) e(x_0) + e(x_0)(\mathcal{L}_{v_k} e)(x_0)(1 - e(x_0)) = \mathcal{L}_{v_k} e(x_0) \quad (33)$$

Here we used the identities $(\mathcal{L}_{v_k} e)(x_0) = e(x_0)(\mathcal{L}_{v_k} e)(x_0) + (\mathcal{L}_{v_k} e)(x_0) e(x_0)$ and $e(x_0)(\mathcal{L}_{v_k} e)(x_0) e(x_0) = 0$ obtained, as before, by differentiating the relation $e(x)^2 = e(x)$, and then multiplying it by $e(x)$. From this and (23) we obtain:

$$de(t_1, \ldots, t_n)|_{\epsilon_j=0} = 0 \quad (34)$$
$$\frac{\partial}{\partial \epsilon_k} de(t_1, \ldots, t_n)|_{\epsilon_j=0} = (\mathcal{L}_{v_k} e)(x_0) dt_k \quad (35)$$

We can now compute (with $b_n = \frac{(-1)^{n/2}}{(2\pi i)^{n/2} (n/2)!}$) :

$$\frac{\partial}{\partial \epsilon_1} \ldots \frac{\partial}{\partial \epsilon_n} Ch^n \left(x_0, x_1(\epsilon_1), \ldots, x_n(\epsilon_n)\right)|_{\epsilon_j=0}$$
$$= b_n \int\limits_{\sum_{j=1}^n t_j \leq 1, t_j \geq 0} Tr\, \frac{\partial}{\partial \epsilon_1} \ldots \frac{\partial}{\partial \epsilon_n} e(t_1, t_2, \ldots, t_n)(de(t_1, t_2, \ldots, t_n))^n|_{\epsilon_j=0}$$
$$= b_n \int\limits_{\sum_{j=1}^n t_j \leq 1, t_j \geq 0} \sum_{\sigma \in S_n} sgn(\sigma) e(x_o)(\mathcal{L}_{v_{\sigma 1}} e)(x_0) \ldots (\mathcal{L}_{v_{\sigma n}} e)(x_0) dt_1 \ldots dt_n$$
$$= b_n \frac{1}{n!} \sum_{\sigma \in S_n} sgn(\sigma) e(x_o)(\mathcal{L}_{v_{\sigma 1}} e)(x_0) \ldots (\mathcal{L}_{v_{\sigma n}} e)(x_0)$$
$$= b_n e(de)^n_{x_0}(v_1, \ldots, v_n) \quad (36)$$

and the result is already antisymmetric in $v_0, \ldots, v_n$. □

We can now prove the main result of this section.

Proof of Theorem 6. In the smooth situation this result is proved in Lemma 10. For the general case notice that E is isomorphic to a pull-back of some smooth vector bundle over a smooth manifold, given by the classifying map into some Grassmanian. As our considerations are functorial under continuous mappings and, in the smooth case, our cocycles represent the components of the Chern character the statement of the theorem is true for some embedding of E into a trivial bundle. But then by Lemma 9 it is true for any such embedding. $\square$

References

[CM] A. Connes, H. Moscovici. Cyclic cohomology, the Novikov conjecture, and hyperbolic groups, *Topology* 29:3 (1990), 345-388.

[CST] A. Connes, D. Sullivan, N.Teleman. Quasiconformal mappings, operators on Hilbert space and local formulae for characteristic classes, *Topology* 33 (1994),663-681.

[Ge] E. Getzler. The odd Chern character in cyclic homology and spectral flow. *Topology* 25 (1993), 489-507

[MW] H. Moscovici, F.-B. Wu, Localization of topological Pontrjagin classes via finite propagation speed, *Geom. and Func. Analysis* 4 (1994), 52-92.