DEFORMATION QUANTIZATION OF GERBES, I

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Abstract

This is the first in a series of articles devoted to deformation quantization of gerbes. Here we give basic definitions and interpret deformations of a given gerbe as Maurer-Cartan elements of a differential graded Lie algebra (DGLA). We classify all deformations of a given gerbe on a symplectic manifold, as well as provide a deformation-theoretic interpretation of the first Rozansky-Witten class.

1. Introduction

Deformation quantization of Poisson manifolds was first introduced in [BFFLS]. In the case when M is a symplectic manifold, deformation quantization of $C^\infty(M)$ was classified up to isomorphism in [DWL], [Fe], [D]. In the case of a complex manifold M with a holomorphic symplectic form, deformation quantizations of the sheaf of algebras $\mathcal{O}_M$ are rather difficult to study. They were classified, under additional cohomological assumptions, in [NT] (Theorem 4.1.6 of the present paper; cf. also [BK] for the algebraic case). All deformation quantizations of $\mathcal{O}_M$ were classified by Kontsevich in [K1].

In this paper we start a program of studying deformation quantization of stacks and gerbes. Stacks are a natural generalization of sheaves of algebras. They appear in geometry, microlocal analysis and mathematical physics, cf. [Gi], [Br], [DP], [Ka], [PS], and other works. We are going to discuss some of the motivations for the present work later in this introduction.

We start by defining stacks, gerbes and their deformations in the generality suited for our purposes. We then recall the language of differential graded Lie algebras (DGLAs) in deformation theory, along the lines of [D], [Ge], [S], [SS], [Dr], [HS]. After that, given a gerbe on a Poisson manifold, we define its deformation quantization. We first classify deformations of the trivial gerbe, i.e. deformations of the structure sheaf as a stack, on a symplectic manifold M, C^∞ or complex (Theorem 4.1.1; this result is very close to the main theorem of [P]). More precisely, we first reduce the classification problem to classifying certain *Q-algebras*, using the term of A. Schwarz (or *curved DGAs*, as they are called in [Bl]). (The link between these objects and gerbes was rather well understood for some time; for example, it is through such objects that gerbes appear in [Kapu]). We also give a new proof of the classification theorem for deformations of the sheaf of algebra of functions (Theorem 4.1.6). Then we show how the first Rozansky-Witten class [RW], [Kap], [K2]) is an obstruction for a certain canonical deformation of the trivial gerbe to be a sheaf, not just a stack. This canonical stack is very closely related to stacks of microdifferential operators defined in [Ka] and [PS].

Next, we show how to interpret deformation quantization of any gerbe in the language of DGLAs (Theorems 5.1.2 and 5.1.3). The proof is based on a DGLA interpretation of the deformation theory of any stack (within our generality); this is done in Theorem 5.3.5. We show that deformations of a stack are classified by the DGLA of De Rham-Sullivan forms with coefficients in *local Hochschild cochains of the twisted matrix algebra* associated to this stack. (This DGLA actually is a DGLA of special Hochschild cochains on an associative DGA; the cyclic homology of this DGA is the natural recipient of the Chern character of a twisted module over a stack. We will study this in the sequel).

Afterwards we prove a classification theorem for deformation quantizations of any gerbe on a symplectic manifold (Theorems 6.1.1 and 6.1.2). This can be viewed as an adaptation of Fedosov's methods [Fe], [Fe1] to the case of gerbes. Note that some ideas about deformation quantization of gerbes appeared already in Fedosov's work; cf. also [K], as well as [Ka] and [PS].

In subsequent papers we will extend the Kontsevich formality theorem to the gerbe context. In particular, let $\mathcal{A}$ be a gerbe on a smooth manifold. This gerbe defines a cohomology class in $H^3(M, \mathbb{C})$. Represent this class by a closed 3-form H. Recall that *a twisted Poisson structure* is a bivector P satisfying

$$[P, P] = < H, P \wedge P \wedge P >$$

(cf. [SW]). A formal weak Poisson structure is a formal series

$$P = \sum_{m \geq 0} (\sqrt{-1}\hbar)^{m+1} P_m$$

satisfying the equation above. (In particular, P_0 is a usual Poisson structure). We will prove that deformations of a gerbe $\mathcal{A}$ are in one-to-one correspondence with equivalence classes of formal week Poisson structures.

Note that Ševera constructed a stack starting from a weak Poisson structure ([Se]; cf. also [Se1]). So, in effect, we will show that any deformation quantization of a gerbe comes from Ševera's construction.

We will also extend the context of the present paper from manifolds to groupoids. We will also study characteristic classes of perfect complexes over a stack. Those classes will be defined by explicit formulas in the language of twisted cochains as in [OB], [OTT], [OTT1]. We will finish by a Riemann-Roch theorem for gerbes and their deformations.

This work is motivated by several goals. First, one can try to generalize the Atiyah-Singer index theorem from pseudo-differential to Fourier integral operators. More precisely, let A be a Fourier integral operator $L^2(X_1) \to L^2(X_2)$ whose wave front is a Lagrangian submanifold L in $T^*(X_1 \times X_2)$. Fix pseudo-differential projections e_i on $L^2(X_i)$. Consider the operator

$$e_2 A e_1 : e_1 L^2(X_1) \to e_2 L^2(X_2)$$

Under some assumptions it is possible to extend the usual index theoretical program to this case (symbols, ellipticity, Fredholmness), and to write an Atiyah-Singer type formula for the index of the resulting operator.

(The presence of the projectors e_1, e_2 is necessary. Indeed, in the applications X_1 and X_2 are of different dimensions. For example, X_1 is an affine space, X_2 the space of affine subspaces of given dimension, A the Radon transform, $e_1 = 1$ and e_2 the projection to the space of solutions of the John equations).

The assumptions one has to impose are as follows. First, one requires the projections of L to T^*X_i to be of constant rank. In this case the images of these projections are coisotropic submanifolds Σ_i. Second, we require the characteristic foliations on Σ_i to be fibrations.

If one does not impose this second condition, an index theorem becomes harder to formulate. To the characteristic foliations one can associate groupoids Γ_i with symplectic forms ω_i, as well as stacks deforming the trivial gerbes on the bases of the characteristic foliations. It seems that a higher index theorem for Fourier integral operators in this generality should rely on an algebraic index theorem for deformation quantizations of gerbes.

The second goal that motivates this paper is to understand deformation quantization of the moduli space of flat connections on a Riemann surface. This deformation quantization is related, on the one hand, to jets of differential operators in line bundles on the moduli space and, on the other hand, to quantum groups and their representations. The structures mentioned above (twisted Poisson structures and stacks) play a key role in the geometry of the moduli space, cf. for example [AKM], [AGS], [AMR], [CF], [RS].

There are other motivations for studying deformation quantization of gerbes, in particular the role of stacks and gerbes in quantum field theory. For example, Riemann-Roch and index theorems for deformation quantization of gerbes should lead to generalizations of index theorems such as in [MMS]. Among the applications other than the index theory, we would like to mention dualities between gerbes and noncommutative spaces, as in [Kapu], [Bl], [BBP]. The deformation-theoretical role of the first Rozansky-Witten class is also quite intriguing and worthy of further study.

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2. Stacks and cocycles

2.1. Let M be a smooth manifold (C^∞ or complex). In this paper, by a stack on M we will mean the following data:

1) An open cover $M = \cup U_i$;

2) a sheaf of rings $\mathcal{A}_i$ on every U_i;

3) an isomorphism of sheaves of rings $G_{ij} : \mathcal{A}_j|(U_i \cap U_j) \xrightarrow{\sim} \mathcal{A}_i|(U_i \cap U_j)$ for every i, j;

4) an invertible element $c_{ijk} \in \mathcal{A}_i(U_i \cap U_j \cap U_k)$ for every i, j, k satisfying

$$G_{ij}G_{jk} = \mathrm{Ad}(c_{ijk})G_{ik} \tag{2.1}$$

such that, for every i, j, k, l,

$$c_{ijk}c_{ikl} = G_{ij}(c_{jkl})c_{ijl} \tag{2.2}$$

If two such data $(U_i', \mathcal{A}_i', G_{ij}', c_{ijk}')$ and $(U_i'', \mathcal{A}_i'', G_{ij}'', c_{ijk}'')$ are given on M, an isomorphism between them is an open cover $M = \cup U_i$ refining both $\{U_i'\}$ and $\{U_i''\}$ together with isomorphisms $H_i : \mathcal{A}_i' \xrightarrow{\sim} \mathcal{A}_i''$ on U_i and invertible elements b_{ij} of $\mathcal{A}_i'(U_i \cap U_j)$ such that

$$G_{ij}'' = H_i \mathrm{Ad}(b_{ij})G_{ij}'H_j^{-1} \tag{2.3}$$

and

$$H_i^{-1}(c_{ijk}'') = b_{ij}G_{ij}'(b_{jk})c_{ijk}'b_{ik}^{-1} \tag{2.4}$$

A gerbe is a stack for which $\mathcal{A}_i = \mathcal{O}_{U_i}$ and $G_{ij} = \mathrm{id}$. In this case c_{ijk} form a two-cocycle in $Z^2(M, \mathcal{O}_M^*)$.

From a stack defined as above one passes to the following categorical data:

1) A sheaf of categories $\mathcal{C}_i$ on U_i for every i;

2) an invertible functor $G_{ij} : \mathcal{C}_j|(U_i \cap U_j) \xrightarrow{\sim} \mathcal{C}_i|(U_i \cap U_j)$ for every i, j;

3) an invertible natural transformation

$$c_{ijk} : G_{ij}G_{jk}|(U_i \cap U_j \cap U_k) \xrightarrow{\sim} G_{ik}|(U_i \cap U_j \cap U_k)$$

such that, for any i, j, k, l, the two natural transformations from $G_{ij}G_{jk}G_{kl}$ to G_{il} that one can obtain from the c_{ijk}'s are the same on $U_i \cap U_j \cap U_k \cap U_l$.

If two such categorical data $(U_i',\ \mathcal{C}_i',\ G_{ij}',\ c_{ijk}')$ and $(U_i'',\ \mathcal{C}_i'',\ G_{ij}'',\ c_{ijk}'')$ are given on M, an isomorphism between them is an open cover $M = \cup U_i$ refining both $\{U_i'\}$ and $\{U_i''\}$, together with invertible functors $H_i : \mathcal{C}_i' \xrightarrow{\sim} \mathcal{C}_i''$ on U_i and invertible natural transformations $b_{ij} : H_i G_{ij}'|(U_i \cap U_j) \xrightarrow{\sim} G_{ij}'' H_j|(U_i \cap U_j)$ such that, on any $U_i \cap U_j \cap U_k$, the two natural transformations $H_i G_{ij}' G_{jk}' \xrightarrow{\sim} G_{ij}'' G_{jk}'' H_k$ that can be obtained using H_i's, b_{ij}'s, and c_{ijk}'s are the same. More precisely:

$$((c_{ijk}'')^{-1} H_k)(b_{ik})(H_i c_{ijk}') = (G_{ij}'' b_{jk})(b_{ij} G_{jk}') \tag{2.5}$$

The above categorical data are defined from $(\mathcal{A}_i, G_{ij}, c_{ijk})$ as follows:

1) $\mathcal{C}_i$ is the sheaf of categories of $\mathcal{A}_i$-modules;

2) given an $\mathcal{A}_i$-module $\mathcal{M}$, the $\mathcal{A}_j$-module $G_{ij}(\mathcal{M})$ is the sheaf $\mathcal{M}$ on which $a \in \mathcal{A}_i$ acts via $G_{ij}^{-1}(a)$;

3) the natural transformation c_{ijk} between $G_{ij}G_{jk}(\mathcal{M})$ and $G_{jk}(\mathcal{M})$ is given by multiplication by $G_{ik}^{-1}(c_{ijk}^{-1})$.

From the categorical data defined above, one defines a sheaf of categories on M as follows. For an open V in M, an object of $\mathcal{C}(V)$ is a collection of objects X_i of $\mathcal{C}_i(U_i \cap V)$, together with isomorphisms $g_{ij} : G_{ij}(X_j) \xrightarrow{\sim} X_i$ on every $U_i \cap U_j \cap V$, such that

$$g_{ij} G_{ij}(g_{jk}) = g_{ik} c_{ijk}$$

on every $U_i \cap U_j \cap U_k \cap V$. A morphism between objects (X_i', g_{ij}') and (X_i'', g_{ij}'') is a collection of morphisms $f_i : X_i' \to X_i''$ (defined for some common refinement of the covers), such that $f_i g_{ij}' = g_{ij}'' G_{ij}(f_j)$.

Remark 2.1.1. What we call stacks is what is referred to in [DP] as descent data for a special kind of stacks of twisted modules (cf. Remark 1.9 in [DP]). Both gerbes and their deformations are stacks of this special kind. We hope that our terminology, which blurs the distinction between stacks and their descent data, will not cause any confusion.

Definition 2.1.2. *Consider a gerbe given by a two-cocycle* $c^{(0)}_{ijk}$. *Its* deformation quantization *is a stack such that:*

1) $\mathcal{A}_i = \mathcal{O}_{U_i}[[\hbar]]$ *as a sheaf, with the multiplication*

$$f * g = fg + \sum_{m=1}^{\infty} (\sqrt{-1}\hbar)^m P_m(f, g)$$

where $1 * f = f * 1 = f$ *and* P_m *are (holomorphic) bidifferential expressions;*

2) $G_{ij}(f) = f + \sum_{m=1}^{\infty} (\sqrt{-1}\hbar)^m T_m(f)$ *where* T_m *are (holomorphic) differential expressions;*

3) $c_{ijk} = \sum_{m=0}^{\infty} (\sqrt{-1}\hbar)^m c^{(m)}_{ijk}$.

An isomorphism between two deformation quantizations is an isomorphism (H_i, b_{ij}) *where*

$$H_i(f) = f + \sum_{m=1}^{\infty} (\sqrt{-1}\hbar)^m R_m(f)$$

where R_m *are (holomorphic) differential expressions, and*

$$b_{ij} = 1 + \sum_{m=1}^{\infty} (\sqrt{-1}\hbar)^m b^{(m)}_{ij}.$$

The aim of this paper is to classify up to isomorphism deformation quantizations of a given gerbe.

3. Differential graded Lie algebras and deformations

3.1. Here we give some definitions at the foundation of the deformation theory program along the lines of [D], [Ge], [S], [SS], [Dr], [HS]. Let

$$\mathcal{L} = \bigoplus_{m \geq -1} \mathcal{L}^m$$

be a differential graded Lie algebra (DGLA). We call *a Maurer-Cartan element* an element λ of $\hbar\mathcal{L}^1[[\hbar]]$ satisfying

$$d\lambda + \frac{1}{2}[\lambda, \lambda] = 0 \tag{3.1}$$

A gauge equivalence between two Maurer-Cartan elements λ and μ is an element $G = \exp X$ where $X \in \hbar\mathcal{L}^0[[\hbar]]$ such that

$$d + \mu = \exp \operatorname{ad} X \, (d + \lambda) \tag{3.2}$$

Given two gauge transformations $G = \exp X$, $H = \exp Y$ between λ and μ, *a two-morphism* from G to H is an element $c = \exp t$ of $\hbar\mathcal{L}^{-1}[[\hbar]]$ such that

$$\exp(X) = \exp(dt + [\mu, t])\exp Y \tag{3.3}$$

in the prounipotent group $\exp(\hbar\mathcal{L}^0[[\hbar]])$. The composition of gauge transformations G and H is the product GH in the prounipotent group $\exp\hbar(\mathcal{L}^0)[[\hbar]]$. The composition of two-morphisms c_1 and c_2 is the product c_1c_2 in the prounipotent group $\exp\hbar(\mathcal{L}^{-1}[[\hbar]])$. Here $\hbar\mathcal{L}^{-1}[[\hbar]]$ is viewed as a Lie algebra with the bracket

$$[a,\, b]_\mu = [a,\, \delta b + [\mu,\, b]] \tag{3.4}$$

We denote the above pronilpotent Lie algebra by $\hbar\mathcal{L}^{-1}[[\hbar]]_\mu$. The above definitions, together with the composition, provide the definition of *the Deligne two-groupoid* of $\mathcal{L}$.

Remark 3.1.1. Recently Getzler gave a definition of a Deligne n-groupoid of a DGLA concentrated in degrees above $-n$, cf. [G].

Now let $\mathcal{L}$ be a sheaf of DGLAs on M. *An $\mathcal{L}$-stack* on M is the following data:

1) A Maurer-Cartan element $\lambda_i \in \hbar\mathcal{L}^1[[\hbar]]$ on U_i for every i;

2) a gauge transformation $G_{ij} : \lambda_j|(U_i \cap U_j) \xrightarrow{\sim} \lambda_i|(U_i \cap U_j)$ for every i, j;

3) a two-morphism

$$c_{ijk} : G_{ij}G_{jk}|(U_i \cap U_j \cap U_k) \xrightarrow{\sim} G_{ik}|(U_i \cap U_j \cap U_k)$$

such that, for any i, j, k, l, the two two-morphisms from $G_{ij}G_{jk}G_{kl}$ to G_{il} that one can obtain from the c_{ijk}'s are the same on $U_i \cap U_j \cap U_k \cap U_l$.

If two such data $(U_i',\ \lambda_i',\ G_{ij}',\ c_{ijk}')$ and $(U_i'',\ \lambda_i'',\ G_{ij}'',\ c_{ijk}'')$ are given on M, an isomorphism between them is an open cover $M = \cup U_i$ refining both $\{U_i'\}$ and $\{U_i''\}$, together with gauge transformations $H_i : \mathcal{C}_i' \xrightarrow{\sim} \mathcal{C}_i''$ on U_i and two-morphisms $b_{ij} : H_iG_{ij}'|(U_i \cap U_j) \xrightarrow{\sim} G_{ij}''H_j|(U_i \cap U_j)$ such that, on any $U_i \cap U_j \cap U_k$, the two two-morphisms $H_iG_{ij}'G_{jk}' \xrightarrow{\sim} G_{ij}''G_{jk}''H_k$ that can be obtained using H_i's, b_{ij}'s, and c_{ijk}'s are the same.

Finally, given two isomorphisms (H_i', b_{ij}') and (H_i'', b_{ij}'') between the two data $(U_i,\ \lambda_i',\ G_{ij}',\ c_{ijk}')$ and $(U_i,\ \lambda_i'',\ G_{ij}'',\ c_{ijk}'')$, define a *two-isomorphism* between them to be a collection of two-morphisms $a_i : H_i' \to H_i''$ such that

$$b_{ij}'' \circ (a_i \circ G_{ij}') = (G_{ij}'' \circ a_i) \circ b_{ij}'$$

as two-morphisms from $H_i' \circ G_{ij}' \to G_{ij}'' \circ H_{ij}''$.

Proposition 3.1.2. *A morphism $f : \mathcal{L}_1 \to \mathcal{L}_2$ of sheaves of DGLAs induces a map from the set of isomorphism classes of $\mathcal{L}_1$-stacks on M to the set of isomorphism classes of $\mathcal{L}_2$-stacks on M. If f is a quasi-isomorphism, then the induced map is a bijection.*

The proof is standard.

Proposition 3.1.3. *Let $\mathcal{L}$ be a sheaf of DGLAs such that the sheaf cohomology $H^i(M, \mathcal{L}) = 0$ for $i > 0$. Then the set of isomorphism classes of $\mathcal{L}$-stacks on M is in one-to-one correspondence with the set of isomorphism classes of Maurer-Cartan elements of the DGLA $\Gamma(M, \mathcal{L})$.*

Proof. An $\mathcal{L}$-stack is a datum $(\lambda_i, G_{ij}, c_{ijk})$ satisfying (2.1), (2.2), where $\lambda_i \in \mathrm{MC}(\mathcal{L})$, G_{ij} are in $\exp(\hbar\mathcal{L}^0[[\hbar]])$ and $c_{ijk} \in \exp(\hbar\mathcal{L}^{-1}[[\hbar]]_{\lambda_i})$. An isomorphism of two such data is a collection (H_i, b_{ij}) satisfying (2.3), (2.4) where H_i are in $\exp(\hbar\mathcal{L}^0[[\hbar]])$ and $b_{ij} \in \exp(\hbar\mathcal{L}^{-1}[[\hbar]]_{\lambda_i})$. If $c_{ijk} = \exp(t_{ijk})$ and

$$t_{ijk} = \sum_{m=1}^{\infty} (\sqrt{-1}\hbar)^m t_{ijk}^{(m)}, \tag{3.5}$$

$$b_{ij} = \exp(\sum_{m=1}^{\infty} (\sqrt{-1}\hbar)^m u_{ij}^{(m)}), \tag{3.6}$$

then (2.2) implies

$$t_{ijk}^{(1)} - t_{ijl}^{(1)} + t_{ikl}^{(1)} - t_{jkl}^{(1)} = 0, \tag{3.7}$$

and (2.4) implies

$$t_{ijk}^{(1)} - t'^{(1)}_{ijk} = u_{ij}^{(1)} - u_{ik}^{(1)} + u_{jk}^{(1)}; \tag{3.8}$$

but the sheaf $\mathcal{L}$ is acyclic, so every stack datum is equivalent to another datum with $t_{ijk} = O(\hbar^2)$. Proceeding by induction, we can assume $t_{ijk} = O(\hbar^m)$. Now equations (3.7), (3.8) still hold if one replaces $t_{ijk}^{(1)}$ by $t_{ijk}^{(m)}$. By induction, we can replace our orginal datum by a datum for which $t_{ijk} = 0$.

Proceeding as above, we can find an isomorphism such that $b_{ij} = 1$ between our stack datum and a new datum with $c_{ijk} = 1$ and $G_{ij} = 1$. Now we have to check when such data are equivalent. Given an isomorphism (H_i, b_{ij}), we observe that by (2.4)

$$u_{ij}^{(1)} - u_{ik}^{(1)} + u_{jk}^{(1)} = 0;$$

because $\mathcal{L}$ is an acycllic sheaf, we can find $y_i^{(1)}$ such that

$$u_{ij}^{(1)} = y_i^{(1)} - y_j^{(1)};$$

then $\exp(\sqrt{-1}\hbar y_i^{(1)})$ defines a two-isomorphism between (H_i, b_{ij}) and some (H'_i, b'_{ij}) with $b'_{ij} = 1+O(\hbar^2)$. Proceeding by induction, we see that if two data with $c_{ijk} = 1$ and $G_{ij} = 1$ are isomorphic, an isomorphism can be chosen of the form (H_i, b_{ij}) with $b_{ij} = 1$. But such data are precisely Maurer-Cartan elements, and such isomorphisms are their gauge equivalences.

Definition 3.1.4. *For any associative algebra A, let $\mathcal{L}^H(A)$ be the Hochschild cochain complex equipped with the Gerstenhaber bracket* [Ge]. *The standard Hochschild differential is denoted by δ. For the sheaf of algebras C^∞_M on a smooth manifold, resp. $\mathcal{O}_M$ on a complex analytic manifold, let $\mathcal{L}^H_M$ be the sheaf of Hochschild cochains $D(f_1, \ldots, f_n)$ which are given by multi-differential, resp. holomorphic multi-differential, expressions in $f_1, \ldots, f_n$.*

One gets directly from the definitions the following

Lemma 3.1.5. *The set of isomorphism classes of deformation quantizations of the trivial gerbe on M is in one-to-one correspondence with the set of isomorphism classes of $\mathcal{L}^H_M$-stacks on M.*

3.2. Hochschild cochains at the jet level. For a manifold M, let J, or J_M, be the bundle of jets of smooth, resp. holomorphic, functions on M. By ∇_{can} we denote the canonical flat connection on the bundle J. Let $C^\bullet(J, J)$ be the bundle of Hochschild cochain complexes of J. More precisely, the fibre of this bundle is the complex of jets of multi-differential multi-linear expressions $D(f_1, \ldots, f_n)$. We denote by δ the standard Hochschild differential.

Proposition 3.2.1. *The set of isomorphism classes of deformation quantizations of the trivial gerbe on M is in one-to-one correspondence with the set of isomorphism classes of Maurer-Cartan elements of the DGLA*

$$\mathcal{L}^{H,J}(M) = A^\bullet(M, C^{\bullet+1}(J, J))$$

with the differential $\nabla_{\mathrm{can}} + \delta$. Here by $A^\bullet$ we mean C^∞ forms with coefficients in a bundle.

Proof. We have an embedding of sheaves of DGLA:

$$\mathcal{L}^H_M \to A^\bullet_M(C^{\bullet+1}(J, J))$$

which is a quasi-isomorphism, and the sheaf on the right hand side has zero cohomology in positive degrees. The proposition follows from Propositions 3.1.2 and 3.1.3.

4. Deformation quantization of the trivial gerbe on a symplectic manifold

4.1. Let (M, ω) be a symplectic manifold (C^∞ or complex analytic with a holomorphic symplectic form). In this section, we extend Fedosov's methods from [Fe] to deformations of the trivial gerbe. We say that a deformation quantization of the trivial gerbe on M corresponds to ω if, on every U_k, $f * g - g * f = \sqrt{-1}\hbar\{f, g\} + o(\hbar)$ where $\{\,,\,\}$ is the Poisson bracket corresponding to ω.

Let us observe that the group $H^2(M, \hbar\mathbb{C}[[\hbar]])$ acts on the set of equivalence classes of deformations of any stack: a class γ acts by multiplying c_{ijk} by $\exp \gamma_{ijk}$ where γ_{ijk} is a cocycle representing γ.

Theorem 4.1.1. *Denote by* $\mathrm{Def}(M, \omega)$ *the set of isomorphism classes of deformation quantizations of the trivial gerbe on* M *compatible with the symplectic structure* ω. *The action of* $H^2(M, \hbar\mathbb{C}[[\hbar]])$ *on* $\mathrm{Def}(M, \omega)$ *is free. The space of orbits of this action is in one-to-one correspondence with an affine space modelled on the vector space* $H^2(M, \mathbb{C})$ *(in the* C^∞ *case) or* $H^1(M, \mathcal{O}_M/\mathbb{C})$ *(in the complex case).*

Proof. As in [Fe], we will reduce the proof to a classification problem for certain connections in an infinite-dimensional bundle of algebras. First, note that in Proposition 3.2.1 we can replace the bundle of algebras J by the bundle of algebras

$$\mathrm{gr}J = \prod S^m(T^*_M).$$

Indeed, a standard argument shows that they are isomorphic as C^∞ bundles of algebras.

Under this isomorphism, the canonical connection ∇_{can} becomes a connection ∇_0 on $\mathrm{gr}J$. We are reduced to classifying up to isomorphism those Maurer-Cartan elements of $(A^\bullet(M, C^{\bullet+1}(\mathrm{gr}J, \mathrm{gr}J)), \nabla_0 + \delta)$ whose component in $A^0(M, C^2)$ is equal to $\frac{1}{2}\sqrt{-1}\hbar\{f, g\}$ modulo $\hbar$. In other words,these components must be, pointwise, deformation quantizations of $\prod S^m(T^*_M)$ corresponding to the symplectic structure. But all such deformations are isomorphic to the standard Weyl deformation from the definition below:

Definition 4.1.2. *The Weyl algebra of* T^*_M *is the bundle of algebras*

$$W = \mathrm{gr}J[[\hbar]] = \prod S^m(T^*_M)[[\hbar]]$$

with the standard Weyl product $*$.

Moreover, a smooth field of such deformations on M admits a smooth gauge transformation making it the standard Weyl deformation. Therefore, we have to classify up to isomorphism those Maurer-Cartan elements of $A^\bullet(M, C^{\bullet+1}(\mathrm{gr}J, \mathrm{gr}J))$ whose component in the subspace $A^0(M, C^2)$ is equal to $f * g - fg$. Here $*$ is the product in the standard Weyl deformation.

But such Maurer-Cartan elements are in one-to-one correspondence with pairs (A, c) where

$$A \in \hbar A^1(M, \mathrm{hom}(\mathrm{gr}J, \mathrm{gr}J))[[\hbar]]; \tag{4.1}$$

$$c \in \hbar A^2(M, \mathrm{gr}J)[[\hbar]], \tag{4.2}$$

such that, if

$$\nabla = \nabla_0 + A,$$

then

$$\nabla(f * g) = \nabla(f) * g + f * \nabla(g); \tag{4.3}$$

$$\nabla^2 = \mathrm{ad}(c);\ \nabla(c) = 0 \tag{4.4}$$

Two pairs (A, c) and (A', c') are equivalent if one is obtained from the other by a composition of transformations of the following two types.
a)

$$(A,\, c) \mapsto (\exp(\mathrm{ad}(X))(A),\, \exp(\mathrm{ad}(X))(c)) \tag{4.5}$$

where $X \in \hbar\mathrm{Der}(W)$;

b)

$$(A,\, c) \mapsto (A + B,\, c + \nabla B + \frac{1}{2}[B,\, B]) \tag{4.6}$$

where $B \in \hbar W$.

It is straightforward that the set of Maurer-Cartan elements discussed above, up to isomorphism, is in one-to-one correspondence with the set of pairs $(A,\, c)$ up to equivalence. It remains to show that the pairs $(A,\, c)$ are classified as in Theorem 4.1.1.

Let us start with notation. Let

$$\widetilde{\mathfrak{g}}^0 = \mathrm{gr}J$$

be the bundle of Lie algebras of formal power series with the standard Poisson bracket. Let $\mathfrak{g}^0 = \mathrm{gr}J/\mathbb{C}$ be the quotient bundle of Lie algebras. In other words, the fibre of $\mathfrak{g}^0$ is the Lie algebra of formal Hamiltonian vector fields on the tangent space. Also, put

$$\widetilde{\mathfrak{g}} = \frac{1}{\hbar}W$$

with the bracket $a * b - b * a$ where $*$ is the Weyl product, and

$$\mathfrak{g} = \widetilde{\mathfrak{g}} / \frac{1}{\hbar}\mathbb{C}[[\hbar]]$$

This is the Lie algebra of continuous derivations of the Weyl algebra. It maps surjectively to $\mathfrak{g}^0$ via $\frac{1}{\hbar}(f_0 + \hbar f_1 + \cdots) \mapsto f_0$. Put $|a| = m$ for $a \in S^m(T^*_M)$ and $|\hbar| = 2$. This defines the degree of any monomial in $S^m(T^*_M)[\hbar]$. By $\widetilde{\mathfrak{g}}^0_m$ we denote the subspace $S^{m+2}(T^*_M)$, and by $\widetilde{\mathfrak{g}}_m$ the set of $\frac{1}{\hbar}f$ where f is a polynomial from $S^{m+2}(T^*_M)[\hbar]$. Then

$$[\widetilde{\mathfrak{g}}^0_m, \widetilde{\mathfrak{g}}^0_r] \subset \widetilde{\mathfrak{g}}^0_{m+r}; \quad [\widetilde{\mathfrak{g}}_m, \widetilde{\mathfrak{g}}_r] \subset \widetilde{\mathfrak{g}}_{m+r};$$

$$\widetilde{\mathfrak{g}}^0 = \prod_{m \geq -2} \widetilde{\mathfrak{g}}^0_m; \ \widetilde{\mathfrak{g}} = \prod_{m \geq -2} \widetilde{\mathfrak{g}}_m$$

One defines $\mathfrak{g}^0_m$ and $\mathfrak{g}_m$ accordingly. We have

$$\mathfrak{g}^0 = \prod_{m \geq -1} \mathfrak{g}^0_m; \ \mathfrak{g} = \prod_{m \geq -1} \mathfrak{g}_m$$

In particular, the bundle $\widetilde{g}^0_{-1} = \mathfrak{g}^0_{-1} = \widetilde{g}_{-1} = \mathfrak{g}_{-1}$ is the cotangent bundle T^*_M. The symplectic form identifies this bundle with T_M.

Definition 4.1.3. *By A_{-1} we denote the canonical form* $\mathrm{id} \in A^1(M, T_M)$ *which we view as a form with values in $\widetilde{g}^0_{-1}$, etc. under the identifications above.*

The form A_{-1} is smooth in the C^∞ case and holomorphic in the complex case.

The connection ∇_0 can be expressed as

$$\nabla_0 = A_{-1} + \nabla_{0,0} + \sum_{k=1}^{\infty} A_k = \nabla_{0,0} + A^{(-1)} \tag{4.7}$$

where $\nabla_{0,0}$ is an $\mathfrak{sp}_n$-valued connection in the tangent bundle T_M and $A_k \in A^1(M, \mathfrak{g}^0_k)$. The form A_{-1} is in fact the canonical form from the above definition. In the case of a complex manifold, locally $\nabla_{0,0} = \partial + \overline{\partial} + A_{0,0}$ where $A_{0,0}$ is a $(1,0)$-form with values in $\mathfrak{sp}_n$. The form $A^{(-1)}$ can be viewed as a $\widetilde{\mathfrak{g}}^0$-valued one-form:

$$A^{(-1)} \in A^1(M, \widetilde{\mathfrak{g}}^0) \tag{4.8}$$

Let us look for ∇ of the form

$$\nabla = \nabla_0 + \sum_{m=0}^{\infty} (\sqrt{-1}\hbar)^m A^{(m)} \tag{4.9}$$

where $A^{(m)} \in A^1(M, \mathfrak{g}^0)$. The condition $\nabla^2 = o(\hbar)$ is equivalent to

$$\nabla_0 A^{(0)} + \frac{1}{2}[A^{(-1)}, A^{(-1)}]_2 = 0 \tag{4.10}$$

Here we use the notation

$$a * b - b * a = \sum_{m=1}^{\infty} (\sqrt{-1}\hbar)^m [a, b]_m$$

(in particular, $[\ ,\ ]_0$ is the Poisson bracket); we then extend the brackets $[a, b]_m$ to forms with values in the Weyl algebra. Since $[\nabla_{\mathrm{can}}, [\nabla_{\mathrm{can}}, \nabla_{\mathrm{can}}]] = 0$ and $[\nabla_0, \nabla_0] = 0$, we conclude that

$$\nabla_0 [A^{(-1)}, A^{(-1)}]_2 = 0$$

in $A^2(M, \widetilde{\mathfrak{g}}^0)$. Moreover, observe that the left hand side lies in fact in $A^2(M, \prod_{m \geq 0} \widetilde{\mathfrak{g}}^0_m)$.

Lemma 4.1.4. *If $c \in A^p(M, \widetilde{\mathfrak{g}}^0_m)$, $m \geq -1$, satisfies $[A_{-1}, c] = 0$, then $c = [A_{-1}, c']$ for $c' \in A^{p-1}(M, \widetilde{\mathfrak{g}}^0_{m+1})$.*

Proof. Indeed, the complex $A^\bullet(M, \widetilde{\mathfrak{g}}^0)$ with the differential $[A_{-1},\ ]$ is isomorphic to the complex of smooth sections of, resp, $A^{0,\bullet}$ forms with coefficients in, the bundle of complexes $S[[T^*_M]] \otimes \wedge(T^*_M)$ with the standard De Rham differential.

We now know that pairs (∇, c) exist. The theorem is implied by the following lemma (we use the notation of (4.1)-(4.6)).

Lemma 4.1.5. *1) For any two connections ∇ and ∇', $A^{(0)} - A'^{(0)}$ is a cocycle in $A^1(M, J/\mathbb{C})$; a pair (∇, c) is equivalent to a pair (∇', c') for some c' by some transformation (X, B) if and only if $A^{(0)} - A'^{(0)}$ is a coboundary;*

2) for any two pairs (∇, c) and (∇, c') with the same ∇, $c - c'$ is a closed form in $A^2(M, \hbar\mathbb{C}[[\hbar]])$; two such pairs are equivalent if and only if $c - c'$ is exact.

Proof. 1) The first statement of 1) follows from (4.10). To prove the second, note that

$$\nabla' = \exp \mathrm{ad}(X)(\nabla) + \mathrm{a}\, d(B),$$
$$B \in A^1(M, \hbar\widetilde{\mathfrak{g}})$$

with

$$X = \sum_{m=0}^{\infty} (\sqrt{-1}\hbar)^m X^{(m)}$$

and $X^{(m)} \in A^0(M, \mathfrak{g}^0)$, is possible if and only if

$$\nabla_0 X^{(0)} + A^{(0)} - A'^{(0)} = 0.$$

2) The first statement of 2) follows from (4.4). To prove the second, consider a lifting of ∇ to a $\widetilde{g}$-valued connection $\widetilde{\nabla}$. We have

$$c = \widetilde{\nabla}^2 + \theta$$

where $\theta \in A^2(M, \hbar\mathbb{C}[[\hbar]])$. One has

$$\nabla = \exp \mathrm{ad}(X)(\nabla) + B$$

if and only if the following two equalities hold:

$$\widetilde{\nabla} = \exp \mathrm{ad}(X)(\widetilde{\nabla}) + B + \alpha$$

for some $\alpha \in A^1(M, \mathbb{C}[[\hbar]])$;

$$c' = \exp \mathrm{ad}(X)(c) + \exp \mathrm{ad}(X)(B) + \frac{1}{2}[B, B].$$

But in this case

$$c' = \exp \mathrm{ad}(X)(\widetilde{\nabla}^2 + \theta) + [\exp \mathrm{ad}(X)(\widetilde{\nabla}), \widetilde{\nabla} - \exp \mathrm{ad}(X)(\widetilde{\nabla}) - \alpha] +$$

$$\frac{1}{2}[\widetilde{\nabla} - \exp \mathrm{ad}(X)(\widetilde{\nabla}), \widetilde{\nabla} - \exp \mathrm{ad}(X)(\widetilde{\nabla})] =$$

$$\frac{1}{2}[\exp \mathrm{ad}(X)(\widetilde{\nabla}), \exp \mathrm{ad}(X)(\widetilde{\nabla})] +$$

$$\theta + [\exp \mathrm{ad}(X)\widetilde{\nabla}, \widetilde{\nabla}] - \frac{1}{2}[\exp \mathrm{ad}(X)(\widetilde{\nabla}), \exp \mathrm{ad}(X)(\widetilde{\nabla})] - d\alpha + \frac{1}{2}[\widetilde{\nabla}, \widetilde{\nabla}] -$$

$$[\widetilde{\nabla}, \exp \mathrm{ad}(X)(\widetilde{\nabla})] + \frac{1}{2}[\exp \mathrm{ad}(X)(\widetilde{\nabla}), \exp \mathrm{ad}(X)(\widetilde{\nabla})] = \widetilde{\nabla}^2 + \theta - d\alpha$$

$$= c - d\alpha$$

This proves the theorem.

4.1.1. *Deformation quantization of the sheaf of functions.* Here we give another proof of a theorem from [NT] (cf. [BK] for the algebraic case).

Recall that (M, ω) is either a symplectic C^∞ manifold or a complex manifold with a holomorphic symplectic structure. By $\mathcal{O}_M$ we denote the sheaf of smooth, resp. holomorphic, functions.

Theorem 4.1.6. *Assume that the maps $H^i(M, \mathbb{C}) \to H^i(M, \mathcal{O}_M)$ are onto for $i = 1, 2$. Set*

$$H^2_F(M, \mathbb{C}) = \ker(H^2(M, \mathbb{C}) \to H^2(M, \mathcal{O}_M)).$$

Choose a splitting

$$H^2(M, \mathbb{C}) = H^2(M, \mathcal{O}_M) \oplus H^2_F(M, \mathbb{C}).$$

The set of isomorphism classes of deformation quantizations of the sheaf $\mathcal{O}_M$ compatible with ω is in one-to-one correspondence with a subset of the affine space

$$\frac{1}{\sqrt{-1}\hbar}\omega + H^2(M,\mathbb{C})[[\hbar]]$$

whose projection to

$$\frac{1}{\sqrt{-1}\hbar}\omega + H^2_F(M,\mathbb{C})[[\hbar]]$$

is a bijection.

Proof. First, observe that Lemma 3.1.5 and Proposition 3.2.1 have their analogs for deformations of the structure sheaf as a sheaf of algebras. The only difference is that the Hochschild complex $C^{\bullet+1}$ is replaced everywhere by $C^{\bullet+1}, \bullet \geq 0$. Similarly to (4.1)-(4.6), these deformations in the symplectic case are classified by forms $A \in A^1(M, \hbar\mathfrak{g})$ such that, if $\nabla = \nabla_0 + A$, $\nabla^2 = 0$. Two such forms are equivalent if, for $X \in A^0(M, \hbar\mathfrak{g})$,

$$\nabla' = \exp \operatorname{ad}(X) \nabla$$

To construct a flat connection ∇ , one has to solve recursively

$$R_n + \nabla_0 A^{(n+1)} = 0 \tag{4.11}$$

where

$$R_n = \frac{1}{2} \sum_{i,j \geq 0\ ; i+j+m=n+1} [A^{(i)}, A^{(j)}]_m$$

At every stage $\nabla_0 R_n = 0$; the class of R_n is in the image of the map

$$H^2(M, \mathcal{O}_M) \to H^2(M, \mathcal{O}_M/\mathbb{C})$$

which is zero under our assumptions.

Therefore flat connections ∇ exist. For any such connection we can consider its lifting to a $\widetilde{\mathfrak{g}}$-valued connection $\widetilde{\nabla}$. Put

$$\widetilde{\nabla}^2 = \theta = \sum_{m=-1}^{\infty} (\sqrt{-1}\hbar)^m \theta_m \in A^2(M, \frac{1}{\hbar}\mathbb{C}[[\hbar]]) \tag{4.12}$$

Let us try to determine all possible values of θ. First of all, $\theta_{-1} = \frac{1}{\sqrt{-1}\hbar}\omega$. There exists $\widetilde{\nabla}$ with $\theta_0 = 0$ (see (4.10) and the argument after it). To obtain other possible θ_0 we have to add to $\widetilde{\nabla}$ a form $A'^{(0)} - A^{(0)}$ whose image in $A^1(M, J/\mathbb{C})$ is $\widetilde{\nabla}$-closed. Therefore, the cohomology class of a possible θ_0 must be in the image of the map

$$H^1(M, \mathcal{O}_M/\mathbb{C}) \to H^2(M, \mathbb{C}),$$

which is precisely $H^2_F(M, \mathbb{C})$ under our assumptions.

Proceeding by induction, we see that, having constructed θ_i, $i \leq n$, and $\widetilde{\nabla}_{(n)}$ such that

$$\widetilde{\nabla}^2_{(n)} = \sum_{m=-1}^{n} (\sqrt{-1}\hbar)^m \theta_m + o(\hbar^n), \tag{4.13}$$

we can find θ_{n+1} and $\widetilde{\nabla}_{(n+1)} = \widetilde{\nabla}^{(n)} + o(\hbar^n)$ such that

$$\widetilde{\nabla}^2_{(n+1)} = \sum_{m=-1}^{n+1} (\sqrt{-1}\hbar)^m \theta_m + o(\hbar^{n+1}).$$

The cohomology class of such θ_{n+1} can be changed by adding any element of $H^2_F(M)$.

Proceeding by induction, we see that we can construct unique $\widetilde{\nabla}$ with any given projection of θ to $H^2_F(M)[[\hbar]]$. Now observe that, if $\nabla' = \exp \operatorname{ad}(X)\nabla$, then $\widetilde{\nabla}' = \exp \operatorname{ad}(X)\widetilde{\nabla} + \alpha$ for $\alpha \in A^1(M, \mathbb{C}[[\hbar]])$ and therefore $\theta' = \exp \operatorname{ad}(X)(\theta) + d\alpha$. Therefore two connections with non-cohomologous curvatures are not equivalent. An inductive argument, similar to the ones above, shows that two connections with cohomologous curvatures are equivalent. Indeed, by adding an α we can arrange for θ' and θ to be equal. Then we find $X = \sum(\sqrt{-1}\hbar)^m X_m$ by induction. At each stage we will have an obstruction in the image of the map

$$H^1(M, \mathcal{O}_M) \to H^1(M, \mathcal{O}_M/\mathbb{C}).$$

But this image is zero under our assumptions.

4.2. The first Rozansky-Witten class. We have seen in the previous section that, under the assumptions of Theorem 4.1.6, deformations of the sheaf of algebras $\mathcal{O}_M$ are classified by cohomology classes θ as in (4.13) where $\theta_{-1} = \frac{1}{\sqrt{-1}\hbar}\omega$; the (non-natural) projection of the set of all possible classes θ to $\frac{1}{\sqrt{-1}\hbar}\omega + H^2_F(M, \mathbb{C}[[\hbar]])$ is a bijection. More precisely, the (natural) projection of θ_{n+1} to $H^2(M, \mathcal{O}_M)$ is a nonlinear function in θ_i, $0 \leq i \leq n$. We are going to describe this function for the case $n = 0$.

Let M be a complex manifold with a holomorphic symplectic structure ω. We start by describing two ways of constructing cohomology classes in $H^2(M, \mathcal{O}_M)$. The first one is invented by Rozansky and Witten, cf. [RW], [Kap], [K2]. Let $\nabla_{0,0}$ be a torsion-free connection in the tangent bundle which is locally of the form $d + A_0$ for $A_0 \in A^{1,0}(M, \mathfrak{sp})$. Let $R = \overline{\partial} A_0$ be the $(1,1)$ component of the curvature of $\nabla_{0,0}$. We can view R as a $(1,1)$ form with coefficients in $S^2(T^*_M)$. Let z^i be holomorphic coordinates on M. By $\widehat{z}^i$ we denote the corresponding basis of T^*_M.

We write

$$R = \sum R_{abi\bar{j}} \hat{z}^a \hat{z}^b dz^i d\bar{z}^j \tag{4.14}$$

Put

$$\mathrm{RW}_{\Gamma_0}(M,\omega) = \sum R_{abi\bar{j}} R_{cdk\bar{l}} \omega^{ac} \omega^{bd} \omega^{ik} d\bar{z}^j d\bar{z}^l \tag{4.15}$$

Here Γ_0 refers to the graph with two vertices and three edges connecting them. In fact a similar form $\mathrm{RW}_\Gamma(M,\omega)$ can be defined for any finite graph Γ for which every vertex has three outgoing edges; the cohomology class of this form is independent of the connection [RW].

The other way of obtaining $(0,2)$ classes is as follows. For $\alpha = \sum \alpha_{i\bar{j}} dz^i d\bar{z}^j$ and $\beta = \sum \beta_{i\bar{j}} dz^i d\bar{z}^j$, put

$$\omega(\alpha,\beta) = \sum \alpha_{i\bar{j}} \beta_{k\bar{l}} \omega_{ik} d\bar{z}^j d\bar{z}^l \tag{4.16}$$

It is straightforward that the above operation defines a symmetric pairing

$$\omega : H^{1,1}(M) \otimes H^{1,1}(M) \to H^{0,2}(M).$$

Combined with the projection $H^2_F(M) \to H^{1,1}(M)$, this gives a symmetric pairing

$$\omega : H^2_F(M) \otimes H^2_F(M) \to H^2(M, \mathcal{O}_M).$$

Theorem 4.2.1. *Under the assumptions of Theorem 4.1.6, let a deformation of the sheaf of algebras $\mathcal{O}_M$ correspond to a cohomology class*

$$\theta = \sum (\sqrt{-1}\hbar)^m \theta_m, \ \theta_m \in H^2(M).$$

Then the projection of the class of θ_1 to $H^2(M, \mathcal{O}_M)$ is equal to

$$\mathrm{RW}_{\Gamma_0}(M,\omega) + \omega(\theta_0, \theta_0)$$

Proof. Let us start by observing that one can define the projection

$$\mathrm{Proj} : (A^{\bullet,\bullet}(M, \mathrm{gr}\, J), \nabla_0) \to (A^{0,\bullet}(M), \overline{\partial}) \tag{4.17}$$

as follows: if $\mathcal{I}$ is the DG ideal of the left hand side generated by dz^i and by the augmentation ideal of $\mathrm{gr}\, J$ then the right hand side is identified with the quotient of the left hand side by $\mathcal{I}$. It is straightforward that Proj is a quasi-isomorphism.

Using the notation introduced in and after Definition 4.1.3, we can write

$$\nabla_0 A^{(0)} + \frac{1}{2}[A^{(-1)}, A^{(-1)}]_2 = \theta_0 \tag{4.18}$$

and

$$\nabla_0 A^{(1)} + \frac{1}{2}[A^{(-1)}, A^{(-1)}]_3 + [A^{(-1)}, A^{(0)}]_2 + [A^{(-0)}, A^{(0)}]_1 = \theta_1. \tag{4.19}$$

Observe that:

a) $\mathrm{Proj}[A^{(-1)}, A^{(-1)}]_2 = \mathrm{Proj}[A^{(-1)}, A^{(-0)}]_2 = 0$;

b) $\mathrm{Proj}[A^{(-1)}, A^{(-1)}]_3$ depends only on the $(0,1)$ component of the form $A^{(-1)}_1$;

c) $\mathrm{Proj}[A^{(0)}, A^{(0)}]_1$ depends only on the $(0,1)$ component of the form $A^{(0)}_{-1}$.

The connection ∇_0 can be chosen in such a way that the form from b) is equal to

$$\sum R_{ijk\bar{l}} \widehat{z}^i \widehat{z}^j \widehat{z}^k d\overline{z}^l; \tag{4.20}$$

therefore for this connection

$$\frac{1}{2}\mathrm{Proj}[A^{(-1)}, A^{(-1)}]_3 = \mathrm{R}\, W_{\Gamma_0}(M, \omega).$$

Since $[A^{(-1)}, A^{(-1)}]_2 \in A^2(M, \widetilde{\mathfrak{g}}_{\geq 0})$, we can choose $A^{(0)} \in A^1(M, \widetilde{\mathfrak{g}}_{\geq 1})$; we conclude, because of b) and c), that there exists $\widetilde{\nabla}$ with $\theta_0 = 0$ such that the projection of θ_1 to $H^2(M, \mathcal{O}_M)$ is equal to $\mathrm{R}\, W_{\Gamma_0}(M, \omega)$.

Now we can produce a connection with a given θ_0 by adding to the above connection a form $A' - A$; for this new connection, the form from c) may be chosen as

$$\sum \alpha_{i\bar{j}} \widehat{z}^i d\overline{z}^j$$

where

$$\alpha = \sum \alpha_{i\bar{j}} dz^i d\overline{z}^j$$

is the $(1,1)$ component of a form representing the class θ. This implies

$$\mathrm{P}\, roj[A^{(0)}, A^{(0)}]_1 = \omega(\theta_0, \theta_0).$$

5. Deformations of a given gerbe

5.1. As above, let $\mathcal{A}$ be a gerbe on M; J_M is the bundle of algebras whose fiber at a point is the algebra of jets of C^∞, resp. holomorphic, functions on M at this point; this bundle has the canonical connection ∇_{can}. Horizontal sections of J_M correspond to smooth, resp. holomorphic, functions.

As above, by $\mathcal{O}_M$ we will denote the sheaf of smooth functions (in the C^∞ case) or the holomorphic functions (in the complex analytic case).

The two-cocycle c_{ijk} defining the gerbe belongs to the cohomology class in $H^2(M, \mathcal{O}_M/2\pi i\mathbb{Z})$. Project this class onto $H^2(M, \mathcal{O}_M/\mathbb{C})$.

Definition 5.1.1. *We denote the above class in $H^2(M, \mathcal{O}_M/\mathbb{C})$ by $R(\mathcal{A})$ or simply by R.*

The class R can be represented by a two-form R in $A^2(M, J_M/\mathbb{C})$.

Theorem 5.1.2. *Given a gerbe $\mathcal{A}$ on a manifold M. The set of deformations of $\mathcal{A}$ up to isomorphism is in one-to-one correspondence with the set of equivalence classes of Maurer-Cartan elements of the DGLA $A^\bullet(M, C^{\bullet+1}(J_M, J_M))$ with the differential $\nabla_{\mathrm{can}} + \delta + i_R$.*

Here $C^{\bullet+1}(J_M, J_M)$ is the complex of vector bundles of Hochschild cochains of the jet algebra; $R \in A^2(M, J_M/\mathbb{C})$ is a form representing the class from Definition 5.1.1; i_R is the Gerstenhaber bracket with the Hochschild zero-cochain R. Explicitly, if r is an element of an algebra A,

$$i_r D(a_1, \ldots, a_n) = \sum_{i=0}^{n} (-1)^i D(a_1, \ldots, a_i, r, \ldots, a_n).$$

In Theorem 5.1.2 this operation is combined with the wedge multiplication on forms.

If the manifold M is complex, we can formulate the theorem in terms of Dolbeault complexes.

Theorem 5.1.3. *Given a holomorphic gerbe $\mathcal{A}$ on a complex manifold M. The set of deformations of $\mathcal{A}$ up to isomorphism is in one-to-one correspondence with the set of equivalence classes of Maurer-Cartan elements of the DGLA $A^{0,\bullet}(M, C^{\bullet+1}(\mathcal{O}_M, \mathcal{O}_M))$ with the differential $\overline{\partial} + \delta + i_R$.*

Here $R \in A^{0,2}(M, \mathcal{O}_M/\mathbb{C})$ is a form representing the class from Definition 5.1.1; i_R is the Gerstenhaber bracket with the Hochschild zero-cochain R.

The rest of this section is devoted to the proof of the theorems above. First, we will construct a DGLA whose Maurer-Cartan elements classify deformations of any stack (Theorem 5.3.5). In order to that, we will start by noticing that a stack datum can be defined in terms of the simplicial nerve of a cover; if we replace the nerve by its first barycentric subdivision, we arrive at a notion of an $\mathcal{L}$-stack where $\mathcal{L}$ is a simplicial sheaf of DGLAs (Definitions 5.3.2, 5.3.3). We reduce the problem to classifying such $\mathcal{L}$-stacks in Lemma 5.3.4. Then we replace our simplicial sheaf of DGLAs by a quasi-isomorphic acyclic simplicial sheaf of DGLAs. For the latter, classifying $\mathcal{L}$-stacks is the same as classifying Maurer-Cartan elements of the DGLA of global sections, whence

Theorem 5.3.5. It states that deformations of a stack are classified by Maurer-Cartan elements of *local Hochschild cochains of the twisted matrix algebra*.

Then we return to the generality of a gerbe. We start with a coordinate change that replaces twisted matrices by usual matrices, at a price if making the differential and the transition isomorphisms more complicated (Lemma 5.3.8). The second coordinate change ((5.13) and up) allows to get rid of matrices alltogether.

5.2. Twisted matrix algebras. For any simplex σ of the nerve of an open cover $M = \cup U_i$ corresponding to $U_{i_0} \cap \ldots \cap U_{i_p}$, put $I_\sigma = \{i_o, \ldots, i_p\}$ and $U_\sigma = \cap_{i \in I} U_i$. Define the algebra $\mathrm{Matr}^\sigma_{\mathrm{tw}}(\mathcal{A})$ whose elements are finite matrices

$$\sum_{i,j \in I_\sigma} a_{ij} E_{ij}$$

such that $a_{ij} \in \mathcal{A}_i(U_\sigma)$. The product is defined by

$$a_{ij} E_{ij} \cdot a_{lk} E_{lk} = \delta_{jl} a_{ij} G_{ij}(a_{jk}) c_{ijk} E_{ik}$$

We call a Hochschild cochain D of $\mathrm{Matr}^\sigma_{\mathrm{tw}}(\mathcal{A})$ *local* if:

a) $D(E_{i_1 j_1}, \ldots, E_{i_k j_k}) = 0$ whenever $j_p \neq i_{p+1}$ for some p between 1 and $k-1$;

b) $D(E_{i_1 j_1}, \ldots, E_{i_k j_k})$ is a factor of $E_{i_1 j_k}$ by an element of $\mathcal{A}$.

Local cochains form a DGL subalgebra of all Hochschild cochains $C^{\bullet+1}(\mathrm{Matr}^\sigma_{\mathrm{tw}}(\mathcal{A}), \mathrm{Matr}^\sigma_{\mathrm{tw}}(\mathcal{A}))$. Denote it by $\mathcal{L}^{H,\mathrm{local}}(\mathrm{Matr}^\sigma_{\mathrm{tw}}(\mathcal{A}))$.

Remark 5.2.1. It is easy to define a sheaf of categories on U_σ whose complex of Hoschild cochains is exactly the complex of local Hochschild cochains above.

5.3. De Rham-Sullivan forms. For any p-simplex σ of the nerve of an open cover $M = \cup U_i$ corresponding to $U_{i_0} \cap \ldots \cap U_{i_p}$, let

$$\mathbb{Q}[\Delta_\sigma] = \mathbb{Q}[t_{i_0}, \ldots, t_{i_p}]/(t_{i_0} + \ldots + t_{i_p} - 1)$$

and

$$\Omega^\bullet[\Delta_\sigma] = \mathbb{Q}[t_{i_0}, \ldots, t_{i_p}]\{dt_{i_0}, \ldots, dt_{i_p}\}/(t_{i_0} + \ldots + t_{i_p} - 1, dt_{i_0} + \ldots + dt_{i_p})$$

As usual, define *De Rham-Sullivan forms* as collections $\omega_\sigma \in \Omega^\bullet[\Delta_\sigma]$ where σ runs through all simplices, subject to $\omega_\tau|\Delta_\sigma = \omega_\sigma$ on U_τ whenever $\sigma \subset \tau$. De Rham-Sullivan forms form a complex with the differential $(\omega_\sigma) \mapsto (d_{\mathrm{DR}}\omega_\sigma)$. We denote the space of all k-forms by $\Omega^k_{\mathrm{DRS}}(M)$.

We need to say a few words about the functoriality of Hochschild cochains. Usually, given a morphism of algebras $A \to B$, there is no natural morphism between $C^\bullet(A, A)$ and $C^\bullet(B, B)$ (both map to $C^\bullet(A, B)$. Nevertheless, in our special case, there are maps $\mathrm{Matr}^\sigma_{\mathrm{tw}} \to \mathrm{Matr}^\tau_{\mathrm{tw}}$ on U_τ if $\sigma \subset \tau$. These maps do induce morphisms of sheaves of *local* cochains on the open subset U_τ in the opposite direction; we call these morphisms *the restriction maps.* And, as before, we consider Hochschild cochain complexes already as sheaves of complexes. For example, in all the cases we are interested in, Hochschild cochains are given by multidifferential maps.

Definition 5.3.1. *Let $\Omega^\bullet_{\mathrm{DRS}}(M, \mathcal{L}^{H,\mathrm{local}}(\mathrm{Matr}_{\mathrm{tw}}(\mathcal{A})))$ be the space of all collections*

$$D_\sigma \in \mathcal{L}^{H,\mathrm{local}}(\mathrm{Matr}^\sigma_{\mathrm{tw}}(\mathcal{A})) \otimes \Omega^k(\Delta_\sigma)$$

such that for $\sigma \subset \tau$ the restriction of the cochain $D_\tau|\Delta_\sigma$ to $\mathrm{Matr}^\sigma_{\mathrm{tw}}(\mathcal{A})$ is equal to D_σ on U_τ. These spaces form a DGLA with the bracket $[(D_\sigma), (E_\sigma)] = ([D_\sigma, E_\sigma])$ and the differential $(D_\sigma) \mapsto ((d_{\mathrm{DR}} + \delta)D_\sigma)$.

The DGLAs above are examples of a structure that we call *a simplicial sheaf of DGLAs.*

Definition 5.3.2. *A simplicial sheaf $\mathcal{L}$ is a collection of sheaves $\mathcal{L}_\sigma$ on U_σ, together with morphisms of sheaves $r_{\sigma\tau} : \mathcal{L}_\tau \to \mathcal{L}_\sigma$ on U_τ for all $\sigma \subset \tau$, such that $r_{\sigma\tau} r_{\tau\theta} = r_{\sigma\theta}$ for any $\sigma \subset \tau \subset \theta$. A simplicial sheaf of DGLAs $\mathcal{L}$ is a simplicial sheaf such that all $\mathcal{L}_\sigma$ are DGLAs and all $r_{\sigma\tau}$ are morphisms of DGLAs.*

Definition 5.3.3. *For a simplicial sheaf of DGLAs $\mathcal{L}$,* an $\mathcal{L}$-stack *is a collection of Maurer-Cartan elements $\lambda_\sigma \in \hbar\mathcal{L}^1(U_\sigma[[\hbar]])$, together with gauge transformations $G_{\sigma\tau} : r_{\sigma\tau}\lambda_\tau \to \lambda_\sigma$ on U_τ and two-morphisms $c_{\sigma\tau\theta} : G_{\sigma\tau} r_{\sigma\tau}(G_{\tau\theta}) \to G_{\sigma\theta}$ on U_θ for any $\sigma \subset \tau \subset \theta$, subject to*

$$c_{\sigma\tau\omega} G_{\sigma\tau}(r_{\sigma\tau}(c_{\tau\theta\omega})) = c_{\sigma\theta\omega} c_{\sigma\tau\theta}$$

for any $\sigma \subset \tau \subset \theta \subset \omega$.

We leave to the reader the definition of isomorphisms (and two-isomorphisms) of $\mathcal{L}$-stacks. Given a simplicial sheaf $\mathcal{L}$, one defines the cochain complex

$$C^p(\mathcal{L}) = \prod_{\sigma_0 \subset \ldots \subset \sigma_p} \mathcal{L}(U_{\sigma_p})$$

Put

$$(Ds)_{\sigma_0 \ldots \sigma_{p+1}} = s_{\sigma_1 \ldots \sigma_{p+1}} + \sum_{i=1}^{p} (-1)^i s_{\sigma_0 \ldots \widehat{\sigma_i} \ldots \sigma_{p+1}} + r_{\sigma_p, \sigma_{p+1}} (-1)^{p+1} s_{\sigma_0 \ldots \sigma_p}$$

We say that $\mathcal{L}$ is *acyclic* if for every q the cohomology of this complex is zero for $p > 0$. We say that a morphism of simplicial sheaves of

DGLAs is a quasi-isomorphism if the induced morphism $\mathcal{L}_\sigma^{(1)} \to \mathcal{L}_\sigma^{(2)}$ is a quasi-isomorphism of complexes of sheaves for any simplex σ. The analogs of Propositions 3.1.2 and 3.1.3 are true for simplicial sheaves of DGLAs, with proofs virtually identical.

The collection of sheaves $\mathcal{L}^{H,\mathrm{local}}(\mathrm{Matr}_{\mathrm{tw}}^\sigma(\mathcal{A}))$ forms a simplicial sheaf of DGLAs if one sets $r_{\sigma\tau}(\omega)$ to be the restriction of the ω to the algebra $\mathrm{Matr}_{\mathrm{tw}}^\sigma(\mathcal{A})$. We denote this simplicial sheaf of DGLAs by $\mathcal{L}^{H,\mathrm{local}}(\mathrm{Matr}_{\mathrm{tw}}(\mathcal{A}))$.

Lemma 5.3.4. *Isomorphism classes of deformations of any stack $\mathcal{A}$ are in one-to-one correspondence with isomorphism classes of $\mathcal{L}^{H,\mathrm{local}}(\mathrm{Matr}_{\mathrm{tw}}(\mathcal{A}))$-stacks.*

Proof. Given a deformation, it defines a Maurer-Cartan element of $\mathcal{L}^{H,\mathrm{local}}(\mathrm{Matr}_{\mathrm{tw}}^\sigma(\mathcal{A}))$ for every σ, namely the Hochschild cochain corresponding to the deformed product on $\mathrm{Matr}_{\mathrm{tw}}(\mathcal{A})$. It is immediate that this cochain is local. The restriction $r_{\sigma\tau}$ sends these cochains to each other, so a deformation of $\mathcal{A}$ does define an $\mathcal{L}^{H,\mathrm{local}}$-stack. Conversely, to have an $\mathcal{L}^{H,\mathrm{local}}$-stack is the same as to have a deformed stack datum $\tilde{\mathcal{A}}_\sigma$ on every U_σ (with respect to the cover by $U_i \cap U_\sigma = U_\sigma$, $i \in I_\sigma$), together with an isomorphism $\tilde{\mathcal{A}}_\tau \to \tilde{\mathcal{A}}_\sigma$ on U_τ for $\sigma \subset \tau$ and a two-isomorphism on U_θ for every $\sigma \subset \tau \subset \theta$. Trivializing the stacks $\tilde{\mathcal{A}}_\sigma$ on U_σ, we see that isomorphism classes of such data are in one-to-one correspondence with isomorphism classes of the following:

1) a deformation $\mathbb{A}_\sigma$ of the sheaf of algebras $\mathcal{A}_{i_0}$ on U_σ where $I_\sigma = \{i_0, \ldots, i_p\}$;

2) an isomorphism of deformations $\mathbb{A}_\tau \to \mathbb{A}_\sigma | U_\tau$ for every $\sigma \subset \tau$;

3) an invertible element of $\mathbb{A}_\sigma(U_\theta)$ for every $\sigma \subset \tau \subset \theta$,

satisfying the equations that we leave to the reader. Finally, one can establish a one-to-one correspondence between isomorphism classes of the above data and isomorphism classes of deformations of $\mathcal{A}$. This is done using an explicit formula utilizing the fact that sequences $\sigma_0 \subset \ldots \subset \sigma_p$ are numbered by simplices of the barycentric subdivision of σ_p (cf. [Seg]).

Theorem 5.3.5. *Isomorphism classes of deformations of any stack $\mathcal{A}$ are in one-to-one correspondence with isomorphism classes of Maurer-Cartan elements of the DGLA $\Omega^\bullet_{\mathrm{DRS}}(M, \mathcal{L}^{H,\mathrm{local}}(\mathrm{Matr}_{\mathrm{tw}}(\mathcal{A})))$.*

(cf. Definition 5.3.1).

Proof. Define the simplicial sheaf of DGLAs as follows. Put

$$\mathcal{L}_\sigma = \Omega^\bullet_{\mathrm{DRS}}(\Delta_\sigma, \mathcal{L}^{H,\mathrm{local}}(\mathrm{Matr}_{\mathrm{tw}}^\sigma(\mathcal{A}))),$$

with the differential $d_{\rm DR}+\delta$ and transition homomorphisms

$$r_{\sigma\tau}(D_\tau) = D_\tau|\Delta_\sigma \text{ restricted to } \mathrm{Matr}^{\sigma}_{\mathrm{tw}}(\mathcal{A}).$$

We denote this simplicial sheaf of DGLAs by

$$\underline{\Omega}^{\bullet}_{\mathrm{DRS}}(M, \mathcal{L}^{H,\mathrm{local}}(\mathrm{Matr}_{\mathrm{tw}}(\mathcal{A}))).$$

It is acyclic as a simplicial sheaf. Therefore, by a simplicial analog of Proposition 3.1.3, isomorphism classes of stacks over it are in one-to-one correspondence with isomorphism classes of Maurer-Cartan elements of the DGLA $\Omega^{\bullet}_{\mathrm{DRS}}(M, \mathcal{L}^{H,\mathrm{local}}(\mathrm{Matr}_{\mathrm{tw}}(\mathcal{A})))$, because the latter is its zero degree Čech cohomology. Now, the embedding

$$\mathcal{L}^{H,\mathrm{local}}(\mathrm{Matr}_{\mathrm{tw}}(\mathcal{A}))) \to \underline{\Omega}^{\bullet}_{\mathrm{DRS}}(M, \mathcal{L}^{H,\mathrm{local}}(\mathrm{Matr}_{\mathrm{tw}}(\mathcal{A})))$$

is a quasi-isomorphism of simplicial sheaves of DGLAs (the left hand side is the zero degree De Rham cohomology, and the higher De Rham cohomology vanishes locally). By a simplicial analog of Proposition 3.1.2, isomorphism classes of $\mathcal{L}$-stacks are in one-to-one correspondence for the two simplicial sheaves of DGLAs above.

Now that we reduced the problem of classifying deformations of a gerbe to the problem of classifying Maurer-Cartan elements of a DGLA, our next aim is to simplify this DGLA.

5.3.1. *First coordinate change: untwisting the matrices.* Recall that we are working on a manifold M with a coordinate cover $\{U_i\}_{i\in I}$ and a Čech two-cocycle c_{ijk} with coefficients in $\mathcal{O}^*_M$.

In what follows, we will denote by $\Omega^k(\Delta_\sigma, \mathcal{O}(U_\sigma))$, etc. the space of smooth forms on the simplex Δ_σ with values in $\mathcal{O}(U_\sigma)$, etc.

Locally, c can be trivialized. We assume that the cover is good and write

$$c_{ijk} = h_{ij}(\sigma)h_{ik}(\sigma)^{-1}h_{jk}(\sigma) \tag{5.1}$$

on U_σ for a simplex σ, where h_{ij} are elements of $\Omega^0(\Delta_\sigma, \mathcal{O}(U_\sigma))$. As a consequence,

$$d_{\mathrm{DR}}\mathrm{log}h_{ij}(\sigma) - d_{\mathrm{DR}}\mathrm{log}h_{ik}(\sigma) + d_{\mathrm{DR}}\mathrm{log}h_{jk}(\sigma) = 0 \tag{5.2}$$

Remark 5.3.6. At this stage the cochains $h_{ij}(\sigma)$, $a_i(\sigma,\tau)$ can be chosen to be constant as functions on simplices. But later they will be required to satisfy Lemma 5.3.10, and for that they have to be dependent on the variables t_i.

Note that two local trivializations of the two-cocycle c differ by a one-cocycle which is itself locally trivial. Therefore

$$h_{ij}(\sigma) = a_i(\sigma,\tau)h_{ij}(\tau)a_j(\sigma,\tau)^{-1} \tag{5.3}$$

on U_τ where a_i are some invertible elements of $\Omega^0(\Delta_\sigma, \mathcal{O}(U_\tau))$. We have another local trivialization:

$$d_{\mathrm{DR}}\mathrm{log}h_{ij}(\sigma) = \beta_i(\sigma) - \beta_j(\sigma) \tag{5.4}$$

on U_σ, where $\beta_i(\sigma)$ are elements of $\Omega^1(\Delta_\sigma, \mathcal{O}(U_\sigma))$. Now introduce the coordinate change

$$a_{ij}E_{ij} \mapsto a_{ij}h_{ij}(\sigma)E_{ij} \tag{5.5}$$

Definition 5.3.7. *By* $\mathrm{Matr}_\sigma(\mathcal{A})$ *we denote the sheaf on* U_σ *whose elements are finite sums* $\sum a_{ij}E_{ij}$ *where* $a_{ij} \in \mathcal{A}_i$. *The multiplication is the usual matrix product.*

One gets immediately

Lemma 5.3.8. *Put*

$$a(\sigma, \tau) = \mathrm{diag}\, a_i(\sigma, \tau)$$

and

$$\beta(\sigma) = \mathrm{diag}\, \beta_i(\sigma)$$

Consider the spaces of all collections

$$D_\sigma \in \Omega^k(\Delta_\sigma, \mathcal{L}^{H,\mathrm{local}}(\mathrm{Matr}^\sigma(\mathcal{O})))$$

such that for $\sigma \subset \tau$ *the restriction of the cochain* $D_\tau|\sigma$ *to* $\mathrm{Matr}^\sigma(\mathcal{A})$ *is equal to* $\mathrm{Ad}(a(\sigma,\tau))(D_\sigma)$ *on* U_τ. *These spaces form a DGLA with the bracket* $[(D_\sigma), (E_\sigma)] = ([D_\sigma, E_\sigma])$ *and the differential* $(D_\sigma) \mapsto ((d_{\mathrm{DR}} + \delta + \mathrm{ad}(\beta(\sigma)))D_\sigma)$. *The coordinate change* (5.5) *provides an isomorphism of this DGLA and the DGLA* $\Omega^\bullet_{\mathrm{DRS}}(M, \mathcal{L}(\mathrm{Matr}_{\mathrm{tw}}(\mathcal{A})))$ *from Definition 5.3.1.*

5.3.2. *Second coordinate change.* We have succeeded in replacing the sheaf of DGLAs of Hochschild complexes of twisted matrices by the sheaf of DGLAs of Hochschild complexes of usual matrices, at a price of having more complicated differential and transition functions. Both involve conjugation (or commutator) with a diagonal matrix. Our next aim is to make these diagonal matrices have all the entries to be the same. This will allow us eventually to get rid of matrices altogether.

We already have one such diagonal matrix. Indeed, from (5.4) one concludes that

$$d_{\mathrm{DR}}\beta_i(\sigma) = d_{\mathrm{DR}}\beta_j(\sigma) \tag{5.6}$$

and therefore

$$d_{\mathrm{DR}}\beta(\sigma) \in \Omega^2(\Delta_\sigma, \mathcal{O}(U_\sigma))$$

is well-defined. The other one is

$$\gamma(\sigma, \tau) = d_{\mathrm{DR}}\mathrm{log}a_i(\sigma, \tau) - \beta_i(\sigma) + \beta_i(\tau) \tag{5.7}$$

To see that this expression does not depend on i, apply $d_{\mathrm{DR}}\log$ to (5.3) and compare the result with (5.4). Thus, we have a well-defined element

$$\gamma(\sigma,\tau) \in \Omega^1(\Delta_\sigma, \mathcal{O}(U_\tau)).$$

Also, from (5.3) we observe that

$$s(\sigma,\tau,\theta) = a_i(\sigma,\tau)a_i(\sigma,\theta)^{-1}a_i(\tau,\theta) \tag{5.8}$$

does not depend on i and therefore defines an invertible element

$$s(\sigma,\tau,\theta) \in \Omega^0(\Delta_\sigma, \mathcal{O}(U_\theta)).$$

The above cochains form a cocycle in the following sense:

$$d_{\mathrm{DR}}(d_{\mathrm{DR}}\beta) = 0; \tag{5.9}$$

$$d_{\mathrm{DR}}\beta(\sigma) - d_{\mathrm{DR}}\beta(\tau) = -d_{\mathrm{DR}}\gamma(\sigma,\tau); \tag{5.10}$$

$$\gamma(\sigma,\tau) - \gamma(\sigma,\theta) + \gamma(\tau,\theta) = d_{\mathrm{DR}}\log s(\sigma,\tau,\theta); \tag{5.11}$$

$$s(\sigma,\tau,\theta)s(\rho,\tau,\theta)^{-1}s(\rho,\sigma,\theta)s(\rho,\sigma,\tau)^{-1} = 1 \tag{5.12}$$

Lemma 5.3.9. *The cohomology of the Čech bicomplex of the complex of simplicial sheaves* $\mathcal{O}_M^* \xrightarrow{d_{\mathrm{DR}}\log} \Omega_M^1 \ldots \xrightarrow{d_{\mathrm{DR}}} \Omega_M^2 \xrightarrow{d_{\mathrm{DR}}} \ldots$ *is isomorphic to* $H^\bullet(M, \mathcal{O}_M^*)$*. Under this isomorphism, the cohomology class of the cocycle* $(d_{\mathrm{DR}}\beta, \gamma, s)$ *of this complex becomes the cohomology class of the cocycle* c_{ijk}*.*

The proof is straightforward, using the fact that sequences $\sigma_0 \subset \ldots \subset \sigma_p$ are numbered by simplices of the barycentric subdivision of σ_p (cf. [Seg]; compare with the proof of Lemma 5.3.4).

We need another lemma to prooceed.

Lemma 5.3.10. *The cochains* $a_i(\sigma,\tau)$ *can be chosen as follows:*

$$a_i(\sigma,\tau) = a_0(\sigma,\tau)\widetilde{a}_i(\sigma,\tau)$$

where $a_0(\sigma,\tau)$ *does not depend on* i *and* $\widetilde{a}_i(\sigma,\tau)$ *take values in the subgroup* $\Omega^0(\Delta_\sigma, \mathbb{C}\cdot 1)^*$*.*

Proof. Choose local branches of the logarithm. We have from (5.8)

$$\log a_i(\alpha,\sigma) - \log a_i(\alpha,\tau) + \log a_i(\sigma,\tau) - \log s(\alpha,\sigma,\tau) = 2\pi\sqrt{-1}N_i(\alpha,\sigma,\tau)$$

where $N_i(\alpha,\sigma,\tau)$ are constant integers. The Čech complex of the simplicial sheaf $\sigma \mapsto \Omega^0(\Delta_\sigma, \mathcal{O}_{U_\sigma})$ is zero in positive degrees. Let S be a contracting homotopy from this complex to its zero cohomology. Put

$$b_i(\sigma) = \exp(S(\log a_i(\alpha,\sigma)));$$

then

$$b_i(\sigma)b_i(\tau)^{-1} = a_i(\sigma,\tau)^{-1}\widetilde{a}_i(\sigma,\tau)a(\sigma,\tau)$$

where

$$\widetilde{a}_i(\sigma,\tau) = \exp(2\pi\sqrt{-1}S(N_i(\alpha,\sigma,\tau)))$$

and

$$a(\sigma,\tau) = \exp(S(s(\alpha,\sigma,\tau)))$$

Therefore we can, from the start, replace $h_{ij}(\sigma)$ by $b_i(\sigma)h_{ij}(\sigma)b_j(\sigma)^{-1}$ in (5.1), and $a_i(\sigma,\tau)$ by $\widetilde{a}_i(\sigma,\tau)a(\sigma,\tau)$ in (5.3). This proves the lemma.

Now consider the operator

$$i_{\beta(\sigma)} : \Omega^\bullet(\Delta_\sigma, C^{\bullet+1}(\mathrm{Matr}(\mathcal{O}))) \to \Omega^{\bullet+1}(\Delta_\sigma, C^\bullet(\mathrm{Matr}(\mathcal{O})))$$

This operator acts by the Gerstenhaber bracket (at the level of $C^\bullet$), combined with the wedge product at the level of $\Omega^\bullet$, with the cochain $\beta(\sigma) \in \Omega^1(\Delta_\sigma, C^0(\mathrm{Matr}(\mathcal{O})))$. One has

$$[\delta, i_{\beta(\sigma)}] = \mathrm{ad}_{\beta(\sigma)} : \Omega^\bullet(\Delta_\sigma, C^\bullet(\mathrm{Matr}(\mathcal{O}))) \to \Omega^{\bullet+1}(\Delta_\sigma, C^\bullet(\mathrm{Matr}(\mathcal{O})))$$

and

$$[d_{\mathrm{DR}}, i_{\beta(\sigma)}] = i_{d_{\mathrm{DR}}\beta(\sigma)} : \Omega^\bullet(\Delta_\sigma, C^{\bullet+1}(\mathrm{Matr}(\mathcal{O}))) \to \Omega^{\bullet+2}(\Delta_\sigma, C^\bullet(\mathrm{Matr}(\mathcal{O})))$$

Now define the second coordinate change as

$$\exp(i_{\beta(\sigma)}) \tag{5.13}$$

on $\Omega^\bullet(\Delta_\sigma, C^\bullet(\mathrm{Matr}(\mathcal{O})))$. This coordinate change turns the DGLA from Lemma 5.3.8 into the following DGLA. Its elements are collections of elements

$$\omega_\sigma \in \Omega^\bullet(\Delta_\sigma, C^\bullet(\mathrm{Matr}^\sigma(\mathcal{O}(U_\sigma)))) \tag{5.14}$$

such that the restriction of $D_\tau|\Delta_\sigma$ to the subalgebra $\mathrm{Matr}^\sigma(\mathcal{O}(U_\sigma))$ is equal to

$$\exp(i_{\beta(\sigma)} - i_{\beta(\tau)})\mathrm{Ad}(a(\sigma,\tau))D_\sigma; \tag{5.15}$$

the differential is

$$d_{\mathrm{DR}} + \delta + i_{d_{\mathrm{DR}}\beta(\sigma)} \tag{5.16}$$

We can replace (5.15) by

$$\exp(i_{\gamma(\sigma,\tau)} - i_{d_{\mathrm{DR}}\log a_0(\sigma,\tau)} - i_{d_{\mathrm{DR}}\log\widetilde{a}(\sigma,\tau)}))\mathrm{Ad}(a_0(\sigma,\tau))D_\sigma \tag{5.17}$$

where $\widetilde{a}(\sigma,\tau) = \mathrm{diag}\,\widetilde{a}_i(\sigma,\tau)$ (cf. Lemma 5.3.10).

5.4. Getting rid of matrices. Consider the morphism

$$C^\bullet(\mathcal{O}_{U_\sigma}) \to C^\bullet(\mathrm{Matr}^\sigma(\mathcal{O}_{U_\sigma})$$

defined as follows. Put $\overline{\mathcal{O}} = \mathcal{O}/\mathbb{C}$. Then for $D \in C^p(\mathcal{O}, \mathcal{O})$, $D : \overline{\mathcal{O}}^{\otimes p} \to \mathcal{O}$, define

$$\widetilde{D}(m_1 a_1, \dots, m_p a_p) = m_1 \dots m_p D(a_1, \dots, a_p)$$

where $a_i \in \mathcal{O}$ and $m_i \in M(\mathbb{C})$. The following is true:

a) the cochains $\widetilde{D}$ are invariant under isomorphisms $\mathrm{Ad}(m)$ for $m \in GL(\mathbb{C})$;

b) the cochains $\widetilde{D}$ become zero after substituting and argument from $M(\mathbb{C})$.

It is well known that the map $D \mapsto \widetilde{D}$ is a quasi-isomorphism with respect to the Hochschild differential δ. Therefore this map establishes a quasi-isomorphism of the DGLA from (5.14), (5.16), (5.15), (5.17) with the following DGLA: its elements are collections $D_\sigma \in \Omega^\bullet(\Delta_\sigma, C^{\bullet+1}(\mathcal{O}(U_\sigma)))$ such that

$$D_\tau | \Delta_\sigma = \exp(i_{\gamma(\sigma,\tau)} - i_{d\log a_0(\sigma,\tau)}) D_\sigma \tag{5.18}$$

on U_τ, with the differential

$$d_{\mathrm{DR}} + \delta + i_{d_{\mathrm{DR}}\beta(\sigma)}. \tag{5.19}$$

Now consider any cocycle $r(\sigma) \in \Omega^2(U_\sigma, \mathcal{O}/\mathbb{C})$, $t(\sigma,\tau) \in \Omega^1(U_\tau, \mathcal{O}/\mathbb{C})$;

$$r(\sigma) - r(\tau) + t(\sigma, \tau) = 0;$$

$$t(\sigma,\tau) - t(\sigma,\theta) + t(\tau,\theta) = 0$$

Such a cocycle defines a of DGLA of collections D_σ as above, where (5.18) gets replaced by

$$D_\tau | \Delta_\sigma = \exp(i_{t(\sigma,\tau)}) D_\sigma \tag{5.20}$$

and the differential is $d_{\mathrm{DR}} + \delta + i_{r(\sigma)}$ If two cocycles differ by the differential of $u(\sigma) \in \Omega^1(\Delta^\sigma, \mathcal{O}(U_\sigma)/\mathbb{C})$, then operators $\exp(i_{u(\sigma)})$ define an isomorphism of DGLAs. Finally, put $r(\sigma) = \beta(\sigma)$ and $t(\sigma,\tau) = \gamma(\sigma,\tau) - d\log a_0(\sigma,\tau)$. This is a cocycle of $\check{\mathrm{C}}^\bullet(M, \mathcal{A}_M(\mathcal{O}/\mathbb{C}))$. It lies in the cohomology class of the cocycle $(\log s,\ \gamma,\ d_{\mathrm{DR}}\beta)$ from Lemma 5.3.9. Now replace this cocycle by a cohomologous cocycle which has $t = 0$.

This proves that isomorphism classes of deformations of a gerbe $\mathcal{A}$ are in one-to-one correspondence with isomorphism classes of Maurer-Cartan elements of the DGLA of collections $D_\sigma \in \Omega^\bullet(\Delta_\sigma, C^{\bullet+1}(\mathcal{O}_{U_\sigma}, \mathcal{O}_{U_\sigma})$ such that $D_\sigma | U_\tau = D_\tau$; the differential is $d_{\mathrm{DR}} + \delta + i_R$ where $R \in \Omega^2_{\mathrm{DRS}}(M, \mathcal{O}/\mathbb{C})$ represents the class R as defined in the beginning of this section. To pass to the DGLA of jets (Theorem 5.1.2) or of Dolbeault forms (Theorem 5.1.3), we apply Proposition 3.1.2.

6. Deformations of gerbes on symplectic manifolds

6.1. For a gerbe on M defined by a cocycle c, we denote by c the class of this cocycle in $H^2(M, \mathcal{O}_M/2\pi i\mathbb{Z})$ and by ∂c its boundary in $H^3(M, 2\pi i\mathbb{Z})$.

Theorem 6.1.1. *Let $\mathcal{A}$ be a gerbe on a symplectic manifold (M, ω). The set of isomorphism classes of deformations of $\mathcal{A}$ compatible to ω:*

a) is empty if the image of the class ∂c under the map $H^3(M, 2\pi i\mathbb{Z}) \to H^3(M, \mathbb{C})$ is non-zero;

b) is in one-to-one correspondence with the space $Def(M, \omega)$ (Theorem 4.1.1) if the image of the class ∂c under the map $H^3(M, 2\pi i\mathbb{Z}) \to H^3(M, \mathbb{C})$ is zero.

Let R be the projection of c to $H^2(M, \mathcal{O}_M/\mathbb{C})$, as in Definition 5.1.1.

Theorem 6.1.2. *Let $\mathcal{A}$ be a gerbe on a complex symplectic manifold (M, ω). The set of isomorphism classes of deformations of $\mathcal{A}$ compatible to ω is:*

a) is empty if $R \neq 0$;

b) is in one-to-one correspondence with the space $Def(M, \omega)$ if $R = 0$.

Proof. The arguments from the proof of Theorem 4.1.1 show that deformations of a gerbe are classified exactly as in (4.1)-(4.4), with one exception: equation (4.2) should be replaced by the requirement that the class of c modulo $A^2(M, \mathbb{C} + \hbar \mathrm{gr} J)[[\hbar]]$ should coincide with R where R is a form defined before Theorem 5.1.2. Therefore, if $R = 0$, the classification goes unchanged; if $R \neq 0$ in $H^2(M, \mathcal{O}_M/\mathbb{C})$, then

$$\nabla_0 A^{(0)} + \frac{1}{2}[A^{(-1)}, A^{(-1)}]_2 = R \tag{6.1}$$

shows that no connection ∇ exists.

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