Relative Moduli of Vector Bundles and the Log-Minimal Model Program on $\overline{M}_g$

by

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Thesis directed by Associate Professor Sebastian Casalaina-Martin

Recent work on the log-minimal model program for the moduli space of curves, as well as past results of Caporaso, Pandharipande, and Simpson motivate an investigation of compactifications of the universal moduli space of slope semi-stable vector bundles over moduli spaces of curves arising in the Hassett–Keel program. Our main result is the construction of a compactification of the universal moduli space of vector bundles over several of these moduli spaces, along with a complete description in the case of pseudo-stable curves.

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Chapter 1

Introduction

Moduli spaces of vector bundles over smooth curves have long been a subject of interest in algebraic geometry. Recall that due to the work of Mumford, Newstead, and Seshadri there is a projective moduli space $U_{e,r}(C)$ of slope semi-stable vector bundles of degree e and rank r over any fixed complex projective curve C. The result has since been generalized to other settings by many others. In particular, in [21], Simpson constructed a relative moduli space of slope semi-stable sheaves on families of polarized projective schemes.

As a consequence of Simpson's result, there is a relative moduli space of slope semi-stable vector bundles for the universal curve $C_g^\circ \to M_g^\circ$ over the moduli space of smooth, automorphism-free curves, polarized by the relative canonical bundle (for $g \geq 2$). Though the coarse moduli space of Deligne–Mumford stable curves, $\overline{M}_g$, does not admit a universal curve, it is natural to ask whether there is a moduli space of slope semi-stable bundles over $\overline{M}_g$ that compactifies the universal moduli space over the space of automorphism-free curves. Caporaso and Pandharipande affirmatively answered this question in the case $r = 1$ and for general rank respectively in [5, 17].

Precisely, in [17], Pandharipande constructed a relative moduli space for $g \geq 2$

$$\overline{U}_{e,r,g} \to \overline{M}_g$$

parametrizing slope semi-stable torsion-free sheaves of uniform rank r and degree e, with a dense open subset that can be identified with the uniform rank locus in Simpson's moduli space.

More recently, with the aim of providing a modular interpretation for the canonical model of the moduli space of curves, there has been interest in understanding alternate modular compactifications of the moduli space of curves. The Hassett–Keel program outlines a principle for applying the log-minimal model program to the moduli space of genus g curves to obtain a modular interpretation of the canonical model by studying spaces of the form

$$\overline{M_g}(\alpha) := \mathrm{Proj}\left(\bigoplus_n H^0(\overline{M_g}, n(K_{\overline{M}_g} + \alpha\Delta))\right),$$

where Δ is the boundary divisor in $\overline{M}_g$ and $\alpha \in [0,1] \cap \mathbb{Q}$. One has $\overline{M}_g(1) = \overline{M}_g$ and $\overline{M}_g(0)$ equal to the canonical model of M_g for $g \gg 0$ ([11, 7, 9]). We direct the reader as well to [10, 15] for more details.

The first steps in the program have been worked out in [13, 14, 2]. In particular, Hassett and Hyeon showed ([13]) that the first birational modification occurs at $\alpha = 9/11$ and is a divisorial contraction to the space

$$\overline{M}_g(9/11) \cong \overline{M}_g^{ps},$$

where $\overline{M}_g^{ps}$ is Schubert's moduli space of pseudo-stable curves (see [18]). Recall that $\overline{M}_g^{ps}$ is a coarse moduli space for the moduli functor of pseudo-stable curves (see Section 3.1 for a precise definition). The contraction essentially replaces elliptic tails with cusps. Recall that, as seen in [12], the moduli space $\overline{M}_2^{ps}$ contains a GIT semi-stable point.

Given α and a modular interpretation of $\overline{M}_g(\alpha)$, it is natural to ask if there exists a compactified universal moduli space of slope semi-stable sheaves

$$\overline{U}_{e,r,g}(\alpha) \to \overline{M}_g(\alpha).$$

This does not follow immediately from Simpson's construction. For instance, though $M_g^\circ \subset \overline{M}_g^{ps}$, there is no universal curve over $\overline{M}_g^{ps}$. The answer for $7/10 < \alpha \leq 9/11$ is a corollary to our main theorem, proved in §3.

Theorem 1.1. *For all $\alpha \in \mathbb{Q} \cap (2/3, 1]$, $e, r \geq 1$ and $g \geq 3$, there exists a projective variety $\overline{U}_{e,r,g}(\alpha)$*

with a canonical projection $\pi : \overline{U}_{e,r,g}(\alpha) \to \overline{M}_g(\alpha)$ *such that the diagram*

$$\begin{array}{ccc} U_{e,r}(C_g^\circ/M_g^\circ) & \hookrightarrow & \overline{U}_{e,r,g}(\alpha) \\ \downarrow & & \downarrow \pi \\ M_g^\circ & \hookrightarrow & \overline{M}_g(\alpha) \end{array}$$

is cartesian and the top row is a compactification of $U_{e,r}(C_g^\circ/M_g^\circ)$*, the moduli space obtained by applying Simpson's construction to the universal curve* $C_g^\circ \to M_g^\circ$*. Over the GIT stable points of* $\overline{M}_g(\alpha)$*, the points of* $\overline{U}_{e,r,g}(\alpha)$ *correspond to aut-equivalence classes of slope semi-stable torsion-free sheaves of rank* r *and degree* e*. Moreover, the fiber of* π *over any GIT stable* $[C] \in \overline{M}_g(\alpha)$ *is isomorphic to* $U_{e,r}(C)/\operatorname{Aut}(C)$*.*

Corollary 1.2. *For all* $e, r \geq 1$*,* $g \geq 3$*, and* $\alpha > 7/10$*, the points of* $\overline{U}_{e,r,g}(\alpha)$ *correspond to aut-equivalence classes of slope semi-stable torsion-free sheaves of rank* r *and degree* e*. Moreover, the fiber of* π *over* $[C] \in \overline{M}_g(\alpha)$ *is isomorphic to* $U_{e,r}(C)/\operatorname{Aut}(C)$*. Finally,* $\overline{U}_{e,r,g}(\alpha)$ *co-represents* $\overline{\mathcal{U}}_{e,r,g}(\alpha)$*.*

We note that the case $r = 1$ was established in [4] using a different approach more in line with that of [5].

Remark 1.1. Over any moduli stack of curves, $\mathcal{M}$, there is always an Artin stack $\mathcal{U}$ of slope semi-stable torsion-free sheaves. The above theorem can be framed as demonstrating that for certain $\mathcal{M}$, $\mathcal{U}$ admits a good moduli space U with an ample line bundle.

Alternatively, one can also form an intermediate stack by applying Simpson's construction: to any family of curves $\mathcal{C} \to S$, $\mathcal{U}^{Simp}(\mathcal{C}/S)$ is defined to be $U(\mathcal{C}/S)$.

Because Simpson's construction is canonical, $\mathcal{U}^{Simp}$ forms a stack, relatively projective over $\mathcal{M}$. In particular, if $\mathcal{M}$ is Deligne–Mumford, so is $\mathcal{U}^{Simp}$, and therefore it admits a good moduli space which will also be a good moduli space for $\mathcal{U}$.

It is not immediately obvious that the good moduli space is projective, but in a discussion with the author, Alexeev observed that again beccause Simpson's construction is canonical and the good moduli space is relatively projective locally, the relative polarizations glue and the good moduli space is projective.

The assumption that $\mathcal{M}$ is Deligne–Mumford is critical for this construction, and the GIT construction presented here applies in greater generality.

We will consider weakening the Deligne–Mumford condition in the base in future work.

We now outline the paper and our strategy for proving the result. For concreteness, our outline is for the case of pseudo-stable curves. Because $\overline{M}_g(\alpha)$ is a GIT quotient of a Hilbert or Chow scheme for $\alpha > 2/3$ ([14]), the same argument goes through more or less immediately.

First, we observe that for a given degree e, twisting the sheaves under consideration by an ample line bundle forms an isomorphism with the same moduli problem for some higher degree; in other words, it suffices to assume e is large (see Remark 3.4 for details). For $g \geq 3$, let H_g denote the appropriate locus in the Hilbert scheme corresponding to 4-canonically embedded pseudo-stable curves, with universal curve $U_H \subset H_g \times \mathbb{P}^N$. Let $\nu : U_H \to \mathbb{P}^n$ denote the projection map. A given rank r and degree e uniquely determine a Hilbert polynomial, $\Phi(t)$. To streamline notation, let $n = \Phi(0)$. We define $\pi : Q^r \to H_g$ to be the locus in a relative Quot scheme parametrizing sheaves of uniform rank r. A point $\xi \in Q^r$ corresponds to an equivalence class of the following data: a point of H_g corresponding to a curve C; a presentation of sheaves $\mathbb{C}^n \otimes \mathcal{O}_C \to E \to 0$ such that the Hilbert polynomial of E with respect to $\omega_C^{\otimes 4}$ is Φ and E is of uniform rank. The groups SL_{N+1} and SL_n act naturally on Q^r by changing the coordinates of the curve's embedding and the presentation of the sheaf, respectively. For any k, we have an $SL_{N+1} \times SL_n$-linearized ample line bundle $\mathcal{L}_k := \pi^*\mathcal{O}_{H_g}(k) \otimes \mathcal{O}_{\mathfrak{Quot}}(1)$ on Q^r. We may therefore define

$$\overline{U}^{ps}_{e,r,g,k} := Q^r /\!\!/_{\mathcal{L}_k} SL_{N+1} \times SL_n.$$

A standard variation of GIT argument (see Propositions 2.1 and 2.2 in Section 2.3) tells us that when $k \gg 0$, the GIT (semi-)stability of a point of Q^r is entirely determined by the (semi-)stability with respect to the action of SL_n. In fact, this is the same as the GIT (semi-)stability of the point with respect to the action of SL_n on the fiber $\mathfrak{Quot}^{\Phi,\omega_C^{\otimes 4}}_{\mathbb{C}^n \otimes \mathcal{O}_C, C}$, linearized by the restriction of $\mathcal{O}_{\mathfrak{Quot}}(1)$. It is well-known (e.g., [21]) that when the linearization on the fiber has sufficiently high degree, GIT (semi-)stability is equivalent to slope (semi-)stability. However, we require a bound on the linearization which holds independent of the curve under consideration, or in other words, for every fiber of $U_H \to H_g$ simultaneously.

Theorem 5.1 provides such a bound by establishing the existence of an $SL_{N+1} \times SL_n$-linearized ample line bundle $\mathcal{O}_{\mathfrak{Quot}}(1)$ for which fiberwise GIT (semi-)stability is equivalent to slope (semi-)stability. This completes the proof of the main result (Theorem 1.1).

This fiberwise result (Theorem 5.1) is similar to Simpson's result, but differs in two important ways. In [21], Simpson constructed a compactification of the moduli space of slope semi-stable vector bundles for any projective (possibly singular) curve C. His construction utilizes an asymptotic description of the GIT stability of sheaves. Specifically, he studies the component Q of the Quot scheme containing slope semi-stable, rank r, torsion-free sheaves of degree d on C. Studying a very ample linearization $\mathcal{O}_Q(1)$, he proves that for $N \gg 0$, GIT (semi-)stability with respect to $\mathcal{O}_Q(N)$ is equivalent to slope (semi-)stability. Our results differ in that we restrict attention to the component Q^r of Q containing sheaves of **uniform** rank r. We moreover assume that C is Gorenstein and provide specific bounds for N, depending only on d, g, r, and the singularities of C.

Our results are also closely related to work of Pandharipande, who established the result for nodal curves in [17]. Our argument closely follows his, but differs in two ways. With an eye towards future generalization, we construct the moduli of vector bundles over a fixed Gorenstein curve C using an arbitrary polarization of the curve. In the construction of the compactified universal moduli space, the curves we consider are pseudo-stable or more generally α-stable. This complicates certain bounds and is dealt with mostly in Section 2.2.

Moving forward, we would like to construct analogous moduli spaces over each of the Hassett–Keel moduli spaces and complete the point classification for $\alpha \leq 7/10$. At the time of writing, the latest results due to [2] include a classification up to $\alpha = 2/3 - \epsilon$. One of the first obstructions to applying the same techniques to classify the points is the existence of strictly semi-stable curves in $\overline{M}_g(\alpha)$ for $\alpha \leq 7/10$. Additionally, most of the techniques in this paper require the assumption that the curves are all reduced. Recent work by Chen and Kass ([6]) has hinted at a way forward which we are actively pursuing. Finally, the above result demonstrates that $\overline{U}^{ps}_{e,r,g}$ is a "good moduli space," in the sense of [1], for the Artin stack, $[Q^{SS}/\!\!/G]$. This approach to the problem is considered in greater detail in forthcoming work of the author.

Now we briefly outline the structure of the paper. We establish notation and recall various standard results in section §2. In section §3, we precisely state the moduli problem of interest, we perform the construction of the compactified universal moduli space in detail, and demonstrate that the constructed space co-represents the moduli functor. Section §5 contains the details of the fiberwise problem, i.e., the construction of $\overline{U}_{e,r}(C)$ for a fixed curve C. This section contains the specifics of the construction of the uniform bounds so essential in section §3.2.

Chapter 2

Preliminaries

2.1 Notations and Conventions

Here we fix notation and recall standard useful results which we will use later.

Notation 2.1 (Curve)**.** A **curve** is a proper connected one-dimensional scheme over the complex numbers. The **genus** of a curve C will refer to the arithmetic genus of C, $h^1(C, \mathcal{O}_C)$.

Definition 2.2. For a reduced curve C, the **class of singularity types** of C is defined to be

$$T = T(C) := \{[\widehat{\mathcal{O}}_{C,x}] : x \in C\},$$

where $[\widehat{\mathcal{O}}_{C,x}]$ denotes the isomorphism class of the completed local ring $\widehat{\mathcal{O}}_{C,x}$. We say two curves C and C' **have the same class of singularity types** if $T(C) = T(C')$. Given a set $\mathscr{T}$ of isomorphism classes of complete local rings, we say that a curve **has at worst singularities of type** $\mathscr{T}$ if $T(C) \subset \mathscr{T}$.

Remark 2.3. As there are only a finite number of singular points in a given reduced curve, and the complete local ring on a smooth point is the completion of a polynomial ring, there are only a finite number of isomorphism classes of rings in $T(C)$. This definition is related to the common definition of singularity type of a curve, but does not keep track of the count of each singularity type. For example, all singular nodal curves have the same class of singularity types.

Recall the following version of asymptotic Riemann–Roch.

Lemma 2.1 ([20, Corollary 8, p. 152])**.** *Let (C, L) be a polarized curve with $\deg L = d$. For the irreducible components of C, $\{C_i\}$, denote by L_i the restriction $L|_{C_i}$. Let $d_i = \deg L_i$. Then for any coherent sheaf F, we have*

$$\chi(F \otimes L^t) = \chi(F) + t \sum_i r_i d_i,$$

where $r_i := \dim_{k(\eta_i)} F|_{C_i} \otimes k(\eta_i)$ and η_i is the generic point of C_i.

Motivated by this lemma, one makes the following definition of rank and degree of a sheaf on a curve.

Definition 2.4 (Rank and Degree)**.** Let (C, L) be a polarized curve of genus g with $\deg L = d$ and let F be a coherent sheaf on C. If $\Phi(t) = \chi(F \otimes L^t)$, the **rank** and **degree** of F **with respect to** L are defined so that

$$\Phi(t) = \deg_L F + \operatorname{rank}_L F \chi(\mathcal{O}_C) + t \operatorname{rank}_L F \deg L$$

holds.

It follows that if C is irreducible the generic rank agrees with rank_L. In this case, neither rank_L nor $\deg_L$ depend on L.

Definition 2.5. A sheaf F on C is said to be of uniform rank if there exists a number r such that for every component C_i of C, $\operatorname{rank} F|_{C_i} = r$.

Remark 2.6. If F is of uniform rank, then $\operatorname{rank}_L F$ and $\deg_L F$ are both integers and are independent of L. Indeed, this follows from Lemma 2.1 because $\sum r_i d_i = rd$.

Remark 2.7. In particular, because $d_i > 0$ for every i, $r_i \leq rd$. We will make use of this fact later in the paper.

We will make extensive use of the fact that, for a coherent sheaf F of uniform rank and a line bundle M, we have

$$\operatorname{rank}_L(F \otimes M) = \operatorname{rank}_L F, \quad \deg_L(F \otimes M) = \deg_L(F) + \operatorname{rank}_L(F) \deg(M).$$

A coherent sheaf F on C is said to be **pure** if for every non-zero subsheaf $F' \subset F$, the dimension of the support of F' is equal to the dimension of the support of F. A coherent sheaf F on C is said to be **torsion-free** if it is pure and the support of F is equal to C.

Definition 2.8. Let (C, L) be a polarized curve and F a torsion-free sheaf on C. F is said to be slope stable (slope semi-stable) with respect to L if for every nonzero, proper subsheaf $0 \to E \to F$,

$$\frac{\chi(E)}{\sum s_i d_i} < (\leq) \frac{\chi(F)}{\sum r_i d_i},$$

where s_i and r_i denote the ranks of E and F on each irreducible component of C, and d_i is the degree of L restricted to each irreducible component.

Remark 2.9. If (C, L) is a polarized nonsingular curve and F is a vector bundle on C, then F is slope-stable (slope-semistable) with respect to L if and only if for each nonzero subsheaf $0 \to E \to F$,

$$\frac{\deg(E)}{\operatorname{rank}(E)} < (\leq) \frac{\deg(F)}{\operatorname{rank}(F)}.$$

We caution the reader that if C is reducible, then even if we restrict to sheaves of uniform rank so that rank_L and $\deg_L$ do not depend on L, the slope stability condition of Definition 2.8 does in general depend on L, because E is not required to be of uniform rank.

2.2 Sheaves on Singular Curves

We will make use of several results from [20], which we include here for convenience. For the following results, let $r > 0$ be an integer, and Y an irreducible projective curve over $\mathbb{C}$. Let $x \in Y$ be a point and $\nu : Y^\nu \to Y$ the normalization of Y. Let $x_1, \ldots, x_p$ be the points of Y^ν over x. Let $\overline{\mathcal{O}}_{Y,x}$ be the integral closure of $\mathcal{O}_{Y,x}$.

We are interested in the interaction of singularities on a curve with sheaves on the curve. The following lemma allows us to bound the dimension of certain quotients in terms of analytic invariants of the curve in question.

Lemma 2.2 ([20, Lemma 7, p. 150]). *Let x be a point of an irreducible curve Y. Let M be an $\mathcal{O}_{Y,x}$-module, torsion-free of rank r. Then*

$$\dim_{\mathbb{C}}(M/\mathfrak{m}_x M) \leq (1 + \dim_{\mathbb{C}} \overline{\mathcal{O}}_{Y,x}/\mathcal{O}_{Y,x}) \cdot r.$$

The statement and proof of Lemma 2.2 assume Y is irreducible. We have the following bound for reducible Y.

Corollary 2.3. *Suppose (Y, L) is a polarized projective curve with $\deg L = d$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_Y$-module of multirank $(r_1, \ldots, r_p)$. Then for any $y \in Y$,*

$$\dim_{\mathbb{C}} (\mathcal{F}_y/\mathfrak{m}_y\mathcal{F}_y) \leq d\delta \max_i(r_i), \tag{2.1}$$

where

$$\delta = \max_{A \in T(Y)} (1 + \dim_{\mathbb{C}} \overline{A}/A).$$

Proof. This follows by bounding the restriction of $\mathcal{F}$ to each component of Y ([20, p. 152]) and observing that d is greater than the number of irreducible components of Y. Indeed, from [20, p. 152], we have the claim

$$\dim_{\mathbb{C}} (\mathcal{F}_y/\mathfrak{m}_y\mathcal{F}_y) \leq \sum r_l\delta_l,$$

where the summation is over the components of the normalization containing y. By r_l, we denote the rank of the stalk of $\mathcal{F}|_{Y_l,y}$, and δ_l is the bound of Lemma 2.2. Thus we arrive at the claimed bound by taking the maximum over l of r_l and δ_l and observing that the number of irreducible components must be at most d. □

2.3 Variation of GIT for quotients of products

In this section, we study the properties of quotients of products. Let G and H be reductive groups. Let (X, L_X) and (Y, L_Y) be polarized schemes with linearized actions of G on both X and Y, and of H on Y. Assume that the actions of G and H commute on Y. Then we have an induced action of $G \times H$ on the product $X \times Y$ given by

$$(g, h) \cdot (x, y) = (g \cdot x, g \cdot (h \cdot y)).$$

Moreover, we have many linearizations on $X \times Y$ corresponding to

$$L_X^{\otimes a} \boxtimes L_Y^{\otimes b} := \pi_X^* L_X^{\otimes a} \otimes \pi_Y^* L_Y^{\otimes b}$$

for all $(a, b) \in \mathbb{Z}^2_{>0}$.

With several group actions under consideration, we fix some notation: the superscripts S and SS will indicate stability and semi-stability with respect to the product action of $G \times H$. Stability and semi-stability for G alone will be indicated by superscripts S_G and SS_G, and similarly for H alone. We will also refer to the (semi-)stable locus in X with respect to G as $X^{(S)S_G}$.

Our plan, following [17], is to shift the weight of the polarization almost entirely to X. This, we will show, reduces the stability condition for $G \times H$ on $X \times Y$ to the stability condition for H on Y. The following key propositions, understood in the context of variation of GIT, makes this precise:

Proposition 2.1. *Let $\pi_X : X \times Y \to X$ be the natural projection map. Then for $a/b \gg 0$, we have, with respect to the linearization $L_X^{\otimes a} \boxtimes L_Y^{\otimes b}$ on $X \times Y$,*

$$\pi_X^{-1}(X^{S_G}) \subset (X \times Y)^{S_G}_{[a,b]} \subset (X \times Y)^{SS_G}_{[a,b]} \subset \pi_X^{-1}(X^{SS_G}).$$

Proof. This is a standard result in variation of GIT; see e.g. [22, Lemma 4.1]. This particular formulation is equivalent to Propositions 7.1.1 and 7.1.2 in [17]. □

Proposition 2.2 ([17, Prop. 8.2.1])**.** *Let $Q \subset \pi_X^{-1}(X^{S_G})$ be a closed subscheme. Then for a, b as in Proposition 2.1, we have*

$$Q^{S_H}_{[a,b]} = Q^{S}_{[a,b]} \text{ and } Q^{SS_H}_{[a,b]} = Q^{SS}_{[a,b]}.$$

Proof. We sketch a variation of GIT argument here. An explicit proof in coordinates can also be found in [17, Prop. 8.2.1]. We only prove the statement for stable loci; the semi-stable case is identical.

To begin, certainly, $Q^{S}_{[a,b]} \subset Q^{S_H}_{[a,b]}$, so it suffices to demonstrate the opposite inclusion. For this, let λ be a one-parameter subgroup of $G \times H$, with components λ_G and λ_H. Let μ denote the

Hilbert–Mumford index and fix $(x, y) \in Q^{S_H}_{[a,b]}$. From local arguments, we have

$$\mu^{L^a_X \boxtimes L^b_Y}((x, y), \lambda) = a\mu^{L_X}(x, \lambda_G) + b\mu^{L_Y}(y, \lambda_G) + b\mu^{L_Y}(y, \lambda_H). \tag{2.2}$$

From the first inclusion of Proposition 2.1, and the assumption that $Q \subset \pi_X^{-1}(X^{S_G})$, it follows that for $\lambda_G \neq 1$ the sum of the first two terms on the right-hand side of (2.2) is negative. If $\lambda_G = 1$, the first two terms sum to 0. From the assumption that $(x, y) \in Q^{S_H}_{[a,b]}$, it follows that the last term is also negative. Therefore, $(x, y) \in Q^{S}_{[a,b]}$. □

Chapter 3

Construction of $\overline{U}_{e,r,g}(\alpha)$

In this chapter, we construct the compactified universal moduli space as a GIT quotient. First, in §3.1 we state the moduli problem. The GIT construction takes place in §3.2, and then we prove that the GIT quotient co-represents the moduli functor over the locus of stable curves in §3.3.

3.1 The moduli problem

Here, we describe the moduli functor of sheaves we wish to study. Throughout, we assume that $g \geq 2$. The notion of α-stability is developed in [2] to describe the various stability conditions that arise in the Hassett–Keel program. Defined for $2/3 - \epsilon < \alpha \leq 1$, α-stable curves are, in particular, reduced curves with at worst type A_4 singularities. We refer the reader to [2, Def. 2.5] for the complete definition of α-stability. For $\alpha \in (7/10, 9/11]$, the reader may also refer to Definition 4.1.

Definition 3.1. Let e, r, and g be integers such that $r \geq 1$ and $g \geq 2$. Let $\alpha \in (2/3, 1] \cap \mathbb{Q}$. The functor $\overline{\mathcal{U}}_{e,r,g}(\alpha)$ associates to each scheme S the following set of equivalence classes of data:

- A family of genus g α-stable curves $\mu : \mathcal{C} \to S$; i.e., a flat proper morphism such that every geometric fiber is an α-stable curve of genus g
- A coherent sheaf $\mathcal{F}$ on $\mathcal{C}$, flat over S, such that on geometric fibers $\mathcal{F}$ is slope-semistable, torsion-free of uniform rank r and degree e.

Two pairs $(\mu : \mathcal{C} \to S, \mathcal{F})$ and $(\mu' : \mathcal{C}' \to S, \mathcal{F}')$, are equivalent if there exists an S-isomorphism $\phi : \mathcal{C} \to \mathcal{C}'$ and a line bundle L on S such that $\mathcal{F} \cong \phi^* \mathcal{F}' \otimes \mu^* L$.

Remark 3.2. Recall that for $\alpha > 2/3 - \epsilon$, $\bar{\mathscr{M}}_g(\alpha)$ admits a good moduli space, $\overline{M}_g(\alpha)$. For $\alpha > 7/10$, $\overline{M}_g(\alpha)$ is in fact a coarse moduli space. For $7/10 < \alpha \leq 9/11$, $\overline{M}_g(\alpha)$ is isomorphic to $\overline{M}_g^{ps}$, Schubert's moduli space of pseudo-stable curves. There is a divisorial contraction of coarse moduli spaces $\overline{M}_g \to \overline{M}_g^{ps}$ sending Deligne–Mumford stable curves with an elliptic tail to cuspidal curves ([13]).

We will also require the so-called "fiberwise" moduli functor.

Definition 3.3. Let (C, L) be any polarized curve. The functor $\mathcal{U}_{e,r}(C)$ associates to each scheme S the set of equivalence classes (in the sense of Def. 3.1) of sheaves $\mathcal{F}$ on $S \times C$, flat over S, such that for each $s \in S$, $\mathcal{F}_s$ is slope semi-stable and torsion-free of uniform rank r and degree e.

Remark 3.4. There is an isomorphism of functors

$$\overline{\mathcal{U}}_{e,r,g}(\alpha) \to \overline{\mathcal{U}}_{e \pm (2g-2),r,g}(\alpha), \tag{3.1}$$

$$(\mu : \mathcal{C} \to S, \mathcal{F}) \mapsto (\mu : \mathcal{C} \to S, \mathcal{F} \otimes \omega_{\mathcal{C}/S}^{\otimes \pm 1}),$$

and similarly we have

$$\mathcal{U}_{e,r}(C) \xrightarrow{\sim} \mathcal{U}_{e \pm \deg L, r}(C).$$

As a result, it suffices to study the moduli functors for large e.

3.2 GIT construction of the compactified universal moduli space

We now construct a GIT quotient which we will see in §3.3 co-represents the restriction of the moduli functor $\overline{\mathcal{U}}_{e,r,g}(\alpha)$ to GIT-stable curves in $\overline{\mathcal{M}}_g(\alpha)$.

Proof of Theorem 1.1. We have broken the proof into several parts. The construction and point classification are carried out in Proposition 3.1. The classification of the fibers over stable curves is carried out in Remark 3.6. The inclusions in the diagram are a consequence of the description of the fibers. □

Proposition 3.1. *Using the notation of Theorem 1.1, there exists a projective variety* $\overline{U}_{e,r,g}(\alpha)$ *with a canonical projection* $\pi : \overline{U}_{e,r,g}(\alpha) \to \overline{M}_g(\alpha)$*. The points of* $\overline{U}_{e,r,g}(\alpha)$ *lying over stable curves correspond to aut-equivalence classes of slope semi-stable torsion-free sheaves of rank* r *and degree* e*.*

Proof. We proceed with the proof in three parts: first we set up the GIT problem, then we proceed with the construction of the moduli space, and last we classify the orbit closures over GIT-stable curves. For the sake of concision, the statements and proofs of various independent supporting arguments will be found after this proof.

Part 1 - Setup From [14] and [2], for $\alpha > 2/3$ there is a scheme H_g, either a Hilbert scheme or Chow scheme depending on α, equipped with the action of a reductive group G, along with a G-linearized line bundle $\mathcal{O}_{H_g}(1)$ such that

$$\overline{M}_g(\alpha) \cong H_g /\!\!/_{\mathcal{O}_{H_g}(1)} G.$$

Let $U_H \hookrightarrow H_g \times \mathbb{P}^N$ be the universal curve over H_g. Let $\nu : U_H \to \mathbb{P}^N$ be the projection map. Define $d = \deg \nu^* \mathcal{O}_{\mathbb{P}^N}(1)|_C$ for any curve $C \in H_g$.

We will construct our moduli space using a relative Quot scheme. Specifically, let

$$Q \subset \mathfrak{Quot}^{\nu^*\mathcal{O}_{\mathbb{P}^N}(1),\Phi}_{\mathbb{C}^n \otimes \mathcal{O}_{U_H}, U_H, H_g},$$

be the locus of quotients of uniform rank, where Φ is a Hilbert polynomial with respect to $\nu^*\mathcal{O}_{\mathbb{P}^N}(1)$ ensuring that all parametrized sheaves have rank r and degree e. A point $\xi \in Q$ corresponds to an equivalence class of the following data:

- a point of H_g corresponding to an α-stable curve C embedded in projective space by $\nu^*\mathcal{O}_{\mathbb{P}^N}(1)|_C$;
- a presentation of sheaves $\mathbb{C}^n \otimes \mathcal{O}_C \to E \to 0$ such that the Hilbert polynomial of E with respect to $\nu^*\mathcal{O}_{\mathbb{P}^N}(1)|_C$ is Φ (i.e., $\deg E = e$ and $\operatorname{rank} E = r$) and E is of uniform rank.

It is well-known that Q is a union of connected components, but for lack of a reference for our specific case, we establish this in Lemma 3.1 below.

The action of G on H_g lifts naturally to an action on Q. Moreover, Q is equipped with an action of SL_n by changing coordinates in $\mathbb{C}^n \otimes \mathcal{O}_{U_H}$. These two actions commute and therefore induce an action of $G \times SL_n$ on Q. In order to apply the results from variation of GIT above (specifically, Prop. 2.2), we express the problem in terms of a quotient of a product: there is a closed immersion respecting the group actions

$$Q \subset H_g \times \mathfrak{Quot}^{\nu^*\mathcal{O}_{\mathbb{P}^N}(1),\Phi}_{\mathbb{C}^n\otimes\mathcal{O}_{U_H},U_H,H_g}.$$

We now recall the very ample line bundle $\mathcal{O}_{\mathfrak{Quot}}(1)$ on the Quot scheme. Recall from the construction of the Quot scheme that tensoring a quotient by powers of an ample line bundle, e.g. the relative dualizing sheaf,

$$(\mathcal{O}_C^n \to E \to 0) \mapsto (\mathcal{O}_C^n \otimes \omega^t \to E \otimes \omega^t \to 0)$$

and applying global sections defines a rational map into a Grassmannian. For $t \gg 0$, this becomes an embedding. There is a number $t(d, g, r, e)$, depending only on d, g, r, and e defined in Theorem 5.1, and we take $t = t(d, g, r, e)$. Take k from Prop. 2.2 to be the least integer such that the conclusion of the proposition holds. Define $\mathcal{O}_{\mathfrak{Quot}}(t)$ to be the pullback of the very ample line bundle on the Grassmannian from the Plücker embedding. Let

$$\mathscr{L}_{k,t} = \mathcal{O}_{H_g}(k) \boxtimes \mathcal{O}_{\mathfrak{Quot}}(t),$$

where $\boxtimes$ denotes the tensor product of the respective pullbacks. The line bundle $\mathscr{L}_{k,t}$ admits an $G \times SL_n$-linearization.

Part 2 - Construction of $\overline{U}_{e,r,g}(\alpha)$ We now have everything required to define our GIT quotient:

$$\overline{U}_{e,r,g}(\alpha) := Q /\!\!/_{\mathscr{L}_{k,t}} (G \times SL_n).$$

The reader will recall that a description for large e suffices because of the isomorphism between moduli functors for sheaves of different degrees described in (3.1).

By specifying the degree, d, of our ample line bundle, and the class of singularity types of α-stable curves T, Theorem 5.1, produces a number $E(d,r,g)$ (see (5.2) of §5.5 for an explicit description), depending only on T, d, r and g. We now assume that $e > E(d,r,g,T)$.

First, we need to establish that for $e > E(d,r,g,T)$, all slope semi-stable sheaves (with respect to the canonical polarization) appear in Q. This is an essentially well-known boundedness statement, but for want of a specific reference we present a proof in Lemma 5.8 (by definition $E(d,r,g) > e_4$).

Let Q_G denote the pre-image of H_g^S. Because we selected k/t so that the conclusion of the variation of GIT result from Proposition 2.2 holds, the $G \times SL_n$-stable (semi-stable) locus of Q_G with respect to the linearization $\mathscr{L}_{k,t}$ is equal to the SL_n-stable (semi-stable) locus. In other words, this implies that if C is GIT-stable, then a pair (C,F) is GIT (semi-)stable if and only if F is GIT (semi-)stable as a point of the fiber of Q_G over $[C] \in H_g$ with respect to the linearization induced by the restriction of $\mathscr{L}_{k,t}$. Note that the fiber of Q_G over $[C] \in H_g$ is naturally embedded in the Quot scheme $\mathfrak{Quot}^{\nu^*\mathcal{O}_{\mathbb{P}^N}(1)|_C,\Phi}_{\mathbb{C}^n \otimes \mathcal{O}_C, C}$.

If we can establish that GIT (semi-)stability with respect to the restriction of $\mathscr{L}_{k,t}$ is equivalent to slope (semi-)stability (with respect to the specified polarization) in each fiber simultaneously, then we will have demonstrated that (C,F) is GIT (semi-)stable if and only if F is slope (semi-)stable. If we were working with a fixed curve, we could use Simpson's result to argue that this is true for large e and large t. The proof of Simpson's result, however, only constructs bounds for a fixed family of curves. For our argument, we must construct bounds for e and t which work for all α-stable curves simultaneously, i.e. a bound which depends only on d, the allowable singularities, r and g.

Theorem 5.1 provides such an explicit bound for e, namely $E(d,r,g,T(\alpha))$, where $T(\alpha)$ denotes the class of singularity types prescribed by α. Specifically, because we selected t to be greater than the $t(d,g,r,e)$ described in Theorem 5.1, we may apply the theorem and conclude that when $e > E(d,r,g,T)$ the GIT (semi-)stability of F with respect to the linearization on the fiber of Q over $[C]$ induced by $\mathscr{L}_k$ is equivalent to the slope (semi-)stability of F. We have thus established that for $k \gg 0$ and a GIT-stable curve C, the GIT (semi-)stability of (C, F) with respect to $G \times SL_n$ is equivalent to slope (semi-)stability of F.

Now we construct the projection map π. The morphism

$$Q^{SS} \to H_g^{SS} \to \overline{M}_g(\alpha)$$

is equivariant with respect to the group action, and so by the universal property of the GIT quotient, induces a morphism

$$\overline{U}_{e,r,g}(\alpha) \xrightarrow{\pi} \overline{M}_g(\alpha),$$

sending a curve and a sheaf to the underlying curve.

Part 3 - Orbit closures.

First, observe that because Q is a union of connected components (Lemma 3.1), the space $\overline{U}_{e,r,g}(\alpha)$ is the GIT quotient of a closed subset of a projective scheme, and is therefore projective. By our construction above, Proposition 2.2 guarantees that the stable and semi-stable loci over GIT-stable curves are completely described by the fiberwise stable and semi-stable loci, described in Theorem 5.1. The locus of slope semi-stable vector bundles on a smooth curve is open because it is the preimage under π of an open subset of $\overline{M}_g(\alpha)$.

Next, we classify the orbit closures over the GIT-stable curves. Let $\xi \in Q_G^{SS} = Q^{SS} \cap \pi^{-1}(H_g^S)$ and suppose that $\bar{\xi} \in Q_G^{SS}$ lies in the orbit closure of ξ. It is immediate that $\pi(\bar{\xi})$ is in the orbit closure of $\pi(\xi)$. Thus, if ξ corresponds to (C, F) and $\bar{\xi}$ corresponds to $(\overline{C}, \overline{F})$, we see that C and $\overline{C}$ are projectively equivalent. The G-orbit closure of $\bar{\xi}$ consists of the images of $\overline{F}$ under projective automorphisms of $\overline{C}$. On the other hand, the SL_n-orbit closure of ξ is known (e.g., [21, Thm. 1.21]) to consist of sheaves E which are aut-equivalent to F.

We will demonstrate that these two orbit closures intersect, which will prove that ξ and $\bar{\xi}$ are aut-equivalent. Consider a path

$$\gamma = (\gamma_1, \gamma_2) : \Delta \setminus \{p\} \to G \times SL_n,$$

such that

$$\lim_{z\to p}\gamma(z)\cdot\xi=\bar{\xi}.$$

Composing the path with the group action induces

$$\mu:\Delta\setminus\{p\}\to Q_G,\quad \mu(z):=\gamma_2(z)\cdot\xi.$$

As Q is projective, μ extends to Δ. Notice that $\mu(p)$ is in the SL_n-orbit closure of ξ. If we can demonstrate that $\mu(p)$ is also in the G-orbit closure of $\bar{\xi}$, then we are done. We have

$$\lim_{z\to p}\gamma_1(z)\cdot\mu(p)=\lim_{z\to p}\gamma_1(z)\cdot\lim_{z\to p}(\gamma_2(z)\cdot\xi)=\lim_{z\to p}(\gamma_1(z)\cdot\gamma_2(z)\cdot\xi)=\bar{\xi}.$$

This completes the proof of the theorem. □

Remark 3.5. The classification of the orbit closures fails for GIT strictly semi-stable curves because the argument relies on a description of the GIT semi-stable points in the fiber over the curve.

For lack of a better reference, we include the following lemma to establish that the locus of sheaves of uniform rank in the Quot scheme above is a union of connected components. It is similar to [17, Lemma 8.1.1], with the difference that we work with arbitrary families of curves instead of Deligne–Mumford stable curves. We note in passing that the result holds in greater generality. In particular, with an eye towards future work, the result applies to other loci in the Hilbert scheme arising in the Hassett–Keel program (e.g., [14]).

Lemma 3.1. *Let $g\geq 2$ and r be integers. Define $\Phi(t)=e+r(1-g)+drt$ and let $n=\Phi(0)$. Let $\kappa:\mathcal{C}\to\mathcal{B}$ be a projective, flat family of genus g curves parametrized by an irreducible curve such that the relative ample $\mathcal{L}$ has relative degree d. Define*

$$Q\subset\mathfrak{Quot}^{\mathcal{L},\Phi}_{\mathbb{C}^n\otimes\mathcal{O}_{\mathcal{C}},\mathcal{C},\mathcal{B}}$$

to be the subset corresponding to quotients

$$\mathbb{C}^n\otimes\mathcal{O}_C\to E\to 0,$$

where E has uniform rank r on C. Then the subscheme Q is open and closed in $\mathfrak{Quot}^{\mathcal{L},\Phi}_{\mathbb{C}^n\otimes\mathcal{O}_{\mathcal{C}},\mathcal{C},\mathcal{B}}$.

Proof. Let $\mathcal{E}$ be a κ-flat coherent sheaf.

Suppose there exists a $b^* \in \mathcal{B}$ such that $\mathcal{E}_{b^*}$ has uniform rank r on $\mathcal{C}_{b^*} = C$. Let $\{\mathcal{C}_i\}$ be the irreducible components of $\mathcal{C}$. The morphism κ is flat and surjective of relative dimension 1, and so each $\mathcal{C}_i$ contains a component of C. By the semi-continuity of

$$r(z) := \dim_{k(z)}(\mathcal{E} \otimes k(z)),$$

there is an open set $U_i \subset \mathcal{C}_i$ where $r(z) \leq r$.

The set $U = \cap_i \kappa(U_i) \subset \mathcal{B}$ is open, and has the property that for every $b \in U$ the rank of $\mathcal{E}_b$ at the generic point of each irreducible component of $\mathcal{C}_b$ is at most r. We will show that $U = \mathcal{B}$ and conclude that $\mathcal{E}_b$ is of uniform rank for every $b \in U$.

By way of contradiction, suppose that there exists a $b' \in \mathcal{B}$ such that $\mathcal{E}_{b'}$ is not of uniform rank r. Then again by semi-continuity, there is an i so that $r(z) < r$ on an open $W \subset \mathcal{C}_i$: As $\mathcal{E}$ is flat over $\mathcal{B}$, the Hilbert polynomial of $\mathcal{E}_b$ is constant. In particular, the coefficient rd of t is constant. By Remark 2.7, $\sum_j r_j d_j = rd$, where r_j is the generic rank on the j-th component of $\mathcal{C}_{b'}$. If $\mathcal{E}_{b'}$ is not of uniform rank, then some r_j is greater than r and some r_i is less than r. Fixing the component $\mathcal{C}_i$ containing that component, we may appeal to upper semi-continuity and see that there is an open subset with rank bounded by $r_i < r$.

But for any $b \in U \cap \kappa(W)$, the multiranks of $\mathcal{E}_b$ is at most r on each component and strictly less than r on at least one component. By Lemma 2.1, $\mathcal{E}_b$ cannot have Hilbert polynomial $\Phi(t)$, a contradiction.

Thus, there was no such b', and so for every $b \in \mathcal{B}$, $\mathcal{E}_b$ has uniform rank r, proving the lemma. □

3.3 The quotient $\overline{U}_{e,r,g}(\alpha)$ co-represents $\overline{\mathcal{U}}_{e,r,g}(\alpha)$ over stable curves

We introduce a piece of notation for the following. The functor $\overline{\mathcal{U}}_{S,e,r,g}(\alpha)$ is the restriction of $\overline{\mathcal{U}}_{e,r,g}(\alpha)$ to GIT-stable curves. The same notation indicates the restriction of $\overline{U}_{e,r,g}(\alpha)$.

Theorem 3.2. *For any e, r, g with $r \geq 1$ and $g \geq 2$, the scheme $\overline{U}_{S,e,r,g}(\alpha)$ co-represents the functor $\overline{\mathcal{U}}_{S,e,r,g}(\alpha)$.*

Recall that to say $\overline{U}_{S,e,r,g}(\alpha)$ co-represents the functor $\overline{\mathcal{U}}_{S,e,r,g}(\alpha)$ is to say that $\overline{U}_{S,e,r,g}(\alpha)$ is initial with respect to morphisms from $\overline{\mathcal{U}}_{S,e,r,g}(\alpha)$ to schemes:

$$\begin{array}{ccc} \overline{\mathcal{U}}_{S,e,r,g}(\alpha) & \longrightarrow & \overline{U}_{S,e,r,g}(\alpha) \\ & \searrow & \downarrow \\ & & Z. \end{array}$$

Proof. First, note that by our definitions and the isomorphism

$$\overline{\mathcal{U}}_{S,e,r,g}(\alpha) \cong \overline{\mathcal{U}}_{S,e\pm(2g-2),r,g}(\alpha),$$

it suffices to prove the claim for $e > E(d, g, r, T)$. Now, we construct a natural transformation

$$\phi : \overline{\mathcal{U}}_{e,r,g}(\alpha) \to \mathrm{Hom}(-, \overline{U}_{e,r,g}(\alpha)).$$

Let $e > E(d, g, r, T)$. For a scheme S, let $(\mu, \mathcal{C}, \mathcal{F}) \in \overline{\mathcal{U}}_{e,r,g}(\alpha)(S)$. The sheaf $\mu_* \nu^* \mathcal{O}_{\mathbb{P}^N}(1)$ is locally free of rank $N+1$. Additionally, as we have taken e sufficiently large, for all $s \in S$ Lemma 5.8 states (taking $F = E = \mathcal{F}_s$) that $h^1(\mathcal{C}_s, \mathcal{F}_s) = 0$ and $\mathcal{F}_s$ is generated by global sections. In particular, $H^0(\mathcal{C}_s, \mathcal{F}_s) = \chi(\mathcal{F}_s) =: n$. Thus $\mu_* \mathcal{F}$ is locally free of rank n. Let $\{W_i\}$ be an open cover of S trivializing both $\mu_* \nu^* \mathcal{O}_{\mathbb{P}^N}(1)$ and $\mu_* \mathcal{F}$:

$$\alpha_i : \mathbb{C}^{N+1} \otimes \mathcal{O}_{W_i} \xrightarrow{\cong} \mu_* \nu^* \mathcal{O}_{\mathbb{P}^N}(1)|_{W_i},$$

$$\beta_i : \mathbb{C}^n \otimes \mathcal{O}_{W_i} \xrightarrow{\cong} \mu_* \mathcal{F}|_{W_i}.$$

If $V_i = \mu^{-1}(W_i)$, then pulling back we obtain compositions

$$\mathbb{C}^{N+1} \otimes \mathcal{O}_{V_i} \xrightarrow{\cong} \mu^*(\mu_* \nu^* \mathcal{O}_{\mathbb{P}^N}(1)|_{V_i}) \to \nu^* \mathcal{O}_{\mathbb{P}^N}(1)|_{V_i},$$

$$\mathbb{C}^n \otimes \mathcal{O}_{V_i} \xrightarrow{\cong} \mu^*(\mu_* \mathcal{F}|_{V_i}) \to \mathcal{F}|_{V_i}.$$

The second morphisms, and hence the compositions, are surjective because both $\nu^* \mathcal{O}_{\mathbb{P}^N}(1)$ and $\mathcal{F}_s$ are globally generated; the former because it is very ample and the latter by Lemma 5.8 as mentioned above. Moreover, by construction, the induced maps on global sections are surjective as well. A dimension count shows that they are isomorphisms. By the universal property of Q, we obtain morphisms $\phi_i : W_i \to Q$.

We now pause to restate what we have established about the fiberwise behavior of $\mathcal{F}$:

- $\mathcal{F}$ is fiberwise slope-semistable and torsion-free of uniform rank
- the fiberwise presentation of $\mathcal{F}$ induces an isomorphism on global sections.

Theorem 5.1 tells us that we may **uniformly** select a lower bound on t depending only on d, g, r, e such that for larger t, such families of sheaves are in the semi-stable locus of Q. In other words, $\phi_i(W_i) \subset Q^{SS}$.

As $\phi_i|_{W_i \cap W_j}$ differs from $\phi_j|_{W_i \cap W_j}$ precisely by the trivializations defined above, we obtain a well-defined morphism

$$S \to \overline{U}_{S,e,r,g}(\alpha).$$

The naturality of the universal property of Q implies that the defined ϕ is also natural.

The proof is complete pending the universality of ϕ. This is, however, a straightforward diagram chase and is left to the reader. □

Remark 3.6. Now we study the fibers of $\pi : \overline{U}_{S,e,r,g}(\alpha) \to \overline{M}_{S,g}(\alpha)$. Because $\overline{U}_{e,r,g}(\alpha)$ is a universal categorical quotient (see [16]), for a GIT-stable curve C the fiber $\overline{U}_{e,r,g}(\alpha) \times_{\overline{M}_g(\alpha)} [C]$ co-represents the fiber $\overline{\mathcal{U}}_{e,r,g}(\alpha) \times_{\overline{\mathcal{M}}_g(\alpha)} [C]$. By definition the fiber of the functor parametrizes families of sheaves on isotrivial families of curves isomorphic to C. The functor $\mathcal{U}_{e,r}(C)$ parametrizes families of sheaves on trivial families of curves isomorphic to C. Co-representing a functor is equivalent to co-representing its sheafification. From the description of the functors above, we see that in the étale topology we have the following identification of sheafifications

$$\left(\overline{\mathcal{U}}_{e,r,g}(\alpha) \times_{\overline{\mathcal{M}}_g(\alpha)} [C]\right)^+ \cong \left(\mathcal{U}_{e,r}(C)/\operatorname{Aut}(C)\right)^+.$$

Thus, we need to show that $U_{e,r}(C)/\operatorname{Aut}(C)$ co-represents $(\mathcal{U}_{e,r}(C)/\operatorname{Aut}(C))^+$. Now, $U_{e,r}(C)$ co-represents $\mathcal{U}_{e,r}(C)$, and so $U_{e,r}(C)/\operatorname{Aut}(C)$ co-represents the functor $\mathcal{U}_{e,r}(C)/\operatorname{Aut}(C)$. Also, it is certainly the case that $U_{e,r}(C)/\operatorname{Aut}(C)$ co-represents the sheaf $(\mathcal{U}_{e,r}(C)/\operatorname{Aut}(C))^+$. Thus, $U_{e,r}(C)/\operatorname{Aut}(C)$ co-represents the fiber $\overline{\mathcal{U}}_{e,r,g}(\alpha) \times_{\overline{\mathcal{M}}_g(\alpha)} [C]$.

Chapter 4

Properties of $\overline{U}^{ps}_{e,r,g}$

4.1 The irreducibility of $\overline{U}^{ps}_{e,r,g}$

Before defining the universal moduli functor, let us recall the definition of a pseudo-stable curve.

Definition 4.1. A projective curve is pseudo-stable if

- it is connected, reduced, and has only nodes and cusps as singularities;
- every subcurve of genus one meets the rest of the curve in at least two points;
- the canonical sheaf of the curve is ample.

Given a scheme S, a **family of genus** g **pseudo-stable curves parametrized by** S is a morphism $f : C \to S$, where f is a flat and proper morphism such that every geometric fiber is a pseudo-stable curve of genus g. Two families $f : C \to S$ and $g : D \to S$ are isomorphic if they are isomorphic over S. Recall the moduli functor $\bar{\mathscr{M}}^{ps}_g$ which associates to a scheme S the set of all families of genus g pseudo-stable curves parametrized by S modulo isomorphism.

Now, we establish the irreducibility of $\overline{U}^{ps}_{e,r,g}$. The following lemmas extend Lemmas 9.1.1 and 9.2.3 of [17] and lay the groundwork for a deformation-theoretic argument.

Lemma 4.1. *Let $\mu : \mathcal{C} \to S$ be a family of pseudo-stable, genus $g \geq 2$ curves. Let $\mathcal{E}$ be a μ-flat coherent sheaf on $\mathcal{C}$. The condition that $\mathcal{E}_s$ is a slope-semistable torsion-free sheaf of uniform rank on $\mathcal{C}_s$ is open on S*

Proof. Suppose $\mathcal{E}_{s_0}$ is a slope-semistable sheaf of uniform rank n on $\mathcal{C}_{s_0}$ for some $s_0 \in S$. There exists an integer m such that

(1) $h^1(\mathcal{E}_s \otimes \omega_{\mathcal{C}_s}^m, \mathcal{C}_s) = 0$ for all $s \in S$.

(2) $\mathcal{E}_s \otimes \omega_{\mathcal{C}_s}^m$ is generated by global sections for all $s \in S$.

(3) $\deg(\mathcal{E}_{s_0} \otimes \omega_{\mathcal{C}_{s_0}}^m) > E(g, r)$.

It is enough to prove the lemma for $\mathcal{F} := \mathcal{E} \otimes \omega_{\mathcal{C}/S}^m$. Let f be the Hilbert polynomial of $\mathcal{F}$.

We claim that there is an open $W \subset S$ containing s_0 and a morphism

$$\phi : W \to Q$$

such that $\mathcal{F}$ is isomorphic to the pullback of the universal quotient. The sheaf $\mu_* \omega_{\mathcal{C}/S}^4$ is locally free of rank $N + 1 := 4(2g - 2) - g + 1$. By the above, $\mu_* \mathcal{F}$ is locally free of rank n. Let W be an open subset of S containing s_0 such that both $\mu_* \omega_{\mathcal{C}/S}^4$ and $\mu_* \mathcal{F}$ are trivialized. On $V := \mu^{-1}(W)$, we obtain

$$\mathbb{C}^{N+1} \otimes \mathcal{O}_V \xrightarrow{\cong} \mu^*(\mu_* \omega_{\mathcal{C}/S}^4|_V) \to \omega_{\mathcal{C}/S}^4|_V,$$

$$\mathbb{C}^n \otimes \mathcal{O}_V \xrightarrow{\cong} \mu^*(\mu_* \mathcal{F}|_V) \to \mathcal{F}|_V.$$

The second morphisms, and hence the compositions, are surjective because both $\omega_{\mathcal{C}/S}^4$ and $\mathcal{F}_s$ are globally generated; the former because $g \geq 3$ and the latter by the above. Moreover, by construction, the induced maps on global sections are surjective as well. A dimension count shows that they are isomorphisms. Hence, W has the desired property by the universal property of Q.

Because $\phi(s_0) \in Q_{[k,1]}^{SS}$, which is open in Q, the lemma is proven. □

Lemma 4.2. *Let C be a pseudo-stable curve of genus g. Let E be a slope-semistable torsion-free sheaf of uniform rank r on C. Then there exists a family $\mu : \mathcal{C} \to \Delta_0$ and a μ-flat coherent sheaf $\mathcal{E}$ on $\mathcal{C}$ such that:*

(1) Δ_0 is a pointed curve

(2) $\mathcal{C}_0 \cong C$*, and for every* $t \neq 0$*,* $\mathcal{C}_t$ *is a complete, nonsingular, irreducible genus* g *curve.*

(3) $\mathcal{E}_0 \cong E$*, and for every* $t \neq 0$*,* $\mathcal{E}_t$ *is a slope-semistable torsion-free sheaf of rank* r*.*

Proof. Let $z \in C$ be a singular point. Because E is torsion-free of uniform rank r, we have

$$E_z \cong \mathcal{O}_z^{\oplus a_z} \oplus \mathfrak{m}_z^{\oplus r - a_z},$$

where a_z is an integer determined by E called the local semirank of E (see [3]). This follows when C has a node at z from Propositions (2) and (3) of chapter (8) of [20]. When C has a cusp at z, the statement follows from the main theorem of [3].

Because of its structure, a deformation of E_z may be given by merely deforming $\mathfrak{m}_z$. We will exploit this feature of E_z to produce a local deformation $(\mathcal{C}_z, \mathcal{E}_z)$ of (C_z, E_z) over the disc which smooths C at z and deforms E_z into a vector bundle. If z is a node of C, then [17, Lemma 9.2.2] gives an explicit deformation of $\mathfrak{m}_z$, which we have seen is adequate. If z is a cusp, then let S be a neighborhood of z isomorphic to $\mathrm{Spec}(\mathbb{C}[x,y]/(y^2 - x^3))$. Let $\mu : \mathrm{Spec}(\mathbb{C}[x,y,t]/(y^2 - x^3 - 2t^6) \to \mathrm{Spec}(\mathbb{C}[t])$ be the projection map. We claim that the ideal $\mathcal{I} := (x - t^2, y - t^3)$ is the desired deformation of $\mathfrak{m}_z$. To see this, observe that $\mathcal{I}$ defines a section of μ, whose image we will call L, satisfying an exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_S \to \mathcal{O}_L \to 0. \tag{4.1}$$

Because $\mathcal{O}_S$ is torsion-free over $\mathbb{C}[t]$, so is $\mathcal{I}$. Thus, $\mathcal{I}$ is μ-flat because $\mathbb{C}[t]$ is a Dedekind domain. Moreover the section of μ is an isomorphism, and so $\mathcal{O}_L$ is μ-flat. Hence (4.1) is exact after restriction to z. Thus, $\mathcal{I}_0 \cong m_z$ and so $\mathcal{I}$ is the desired deformation.

At this point, we have produced a local deformation $(\mathcal{C}_z, \mathcal{E}_z)$ of the germ (C_z, E_z) over the disc which smooths C_z and deforms E_z to a vector bundle. Let $(\hat{\mathcal{C}}_z, \hat{\mathcal{E}}_z)$ be the associated formal deformation. The collection of these formal local deformations at each singular point defines a formal deformation for the deformation functor $\mathrm{Def}^{loc}(C,E) := \prod_z \mathrm{Def}(C_z, E_z)$, where $\mathrm{Def}(C_z, E_z)$ is the local deformation functor for the pair (C_z, E_z). As established in [8, Section A.], the morphism

$$\mathrm{Def}(C,E) \xrightarrow{loc} \mathrm{Def}^{loc}(C,E)$$

is smooth, where $\mathrm{Def}(C, E)$ is the deformation functor of the pair (C, E) and *loc* is the natural restriction map. Because *loc* is smooth, we may lift the formal local deformation to a global formal deformation $(\hat{\mathcal{C}}, \hat{\mathcal{E}})$ of (C, E). This global formal deformation is effective; the proof is identical to the standard proof for deformations of schemes, e.g., [19, Thm. 2.5.13]. This effective deformation is then algebraizable by a special case of Artin's algebraization theorem (e.g., [19, Thm. 2.5.14]). Let $(\mathcal{C}, \mathcal{E})$ be an algebrized deformation. Restriction to a disc gives the theorem.

Alternatively, a direct gluing argument may be carried out to explicitly construct a global deformation from the local deformation, as in [17, Lemma 9.2.3]. □

Proposition 4.1. *$\overline{U}^{ps}_{e,r,g}$ is an irreducible variety.*

Proof. Consider $\pi_{SS} : Q^{SS} \to H_g$. By [20, Prop. 24], the scheme

$$\pi_{SS}^{-1}([C])$$

is irreducible for each nonsingular curve C, $[C] \in H_g$. Because the locus $H_g^0 \subset H_g$ of nonsingular curves is irreducible, $\pi_{SS}^{-1}(H_g^0)$ is irreducible. By Lemma 4.2, $\pi_{SS}^{-1}(H_g^0)$ is dense in Q^{SS}. There is a surjection

$$Q^{SS} \to \overline{U}^{ps}_{e,r,g},$$

whence we conclude $\overline{U}^{ps}_{e,r,g}$ is irreducible. □

Chapter 5

Moduli of sheaves on a fixed curve

Our main result for this section is the calculation of the fiberwise GIT quotient. The goal is to extract the uniform bounds we used to solve the global version of this GIT problem in Theorem 1.1. We compare GIT stability for sheaves on a fixed curve with slope stability:

Theorem 5.1. *Let g, r, and d be integers. Define $\Phi(t) = e + r(1-g) + rdt$, and let $n = \Phi(0)$. Let (C, L) be a polarized reduced Gorenstein curve such that $\deg L = d$ and let $\hat{t}(d, g, r, e)$ have the property that for all $t > \hat{t}(d, g, r, e)$, the morphism*

$$i_t : \mathfrak{Quot}^{\Phi,L}_{\mathbb{C}^n \otimes \mathcal{O}_C/C} \to \mathbf{G}(\Phi(t), (\mathbb{C}^n \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t)))^*)$$

is a closed embedding.

Let T be a class of singularity types. There exists integers $E(d, g, r, T)$ and $t(d, g, r, e)$ (see Definition 5.2 for an explicit description) such that for all $e > E(d, g, r, T)$, $t > t(d, g, r, e)$ and any genus g curve C with degree d polarization L and class of singularity types T, we have the following: if $\xi \in \mathfrak{Quot}^{\Phi,L}_{\mathbb{C}^n \otimes \mathcal{O}_C/C}$ corresponds to a quotient

$$\mathbb{C}^n \otimes \mathcal{O}_C \to E \to 0,$$

where E is of rank r, then ξ is GIT stable (semi-stable) with respect to the SL_n-linearization determined by i_t if and only if E is a slope stable (semi-stable) torsion-free sheaf on C and the induced function

$$\mathbb{C}^{\chi(E)} \otimes H^0(C, \mathcal{O}_C) \to H^0(C, E)$$

is an isomorphism.

A few remarks are in order.

Remark 5.1. Theorem 5.1 is a direct generalization of [17, Thm. 2.1.1]. Here, we remove the hypotheses that the curve be Deligne–Mumford stable, and that the polarization be given by the dualizing bundle.

Simpson establishes the same stability criterion in [21], but the the bounds for which the criterion holds depend *a priori* on the curve under consideration. Consequently, uniform bounds must be established if we are to solve the moduli problem in a global setting without a universal curve. Our uniform bound allows us to compare GIT stability with slope stability for all curves in consideration at once, and therefore apply the result to families of sheaves of uniform rank over families of curves.

We now outline the strategy of the proof. We begin with GIT destabilization arguments. Specifically, we demonstrate that any quotient which is not slope-semistable, torsion-free, with an induced isomorphism on global sections is GIT unstable. Then, we study GIT stabilization. We establish that the remaining sheaves are GIT semi-stable, and show that GIT stability is equivalent to slope-stability. The bulk of the argument is a straightfoward extension of the argument presented in [17]. The inclusion of α-stable curves primarily affects the statement and proof of Proposition 5.6, where a more subtle bound of the numerical properties of singularities is required. Many of the following results are straightforward extensions of or identical to previous results, and for the sake of concision we provide a reference rather than a proof.

5.1 Numerical Criterion for Grassmannians

The following lemma, adapted from [17], provides a useful reformulation of the Hilbert-Mumford numerical criterion for Grassmannians. The lemma constitutes the principal technical tool we use for destabilizing sheaves on curves.

Lemma 5.2 ([17, Lemma 2.3.1])**.** *Let g, r, and d be integers. Define $\Phi(t) = e + r(1-g) + rdt$, and let $n = \Phi(0)$. Let (C, L) be a polarized reduced Gorenstein curve such that $\deg L = d$ and let*

$\hat{t}(d, g, r, e)$ have the property that for all $t > \hat{t}(d, g, r, e)$, the morphism

$$i_t : \mathfrak{Quot}^{\Phi,L}_{\mathbb{C}^n \otimes \mathcal{O}_C/C} \to \mathbf{G}(\Phi(t), (\mathbb{C}^n \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t)))^*)$$

is a closed embedding.

Let $\xi \in \mathfrak{Quot}^{\Phi,L}_{\mathbb{C}^n \otimes \mathcal{O}_C/C}$ correspond to a quotient

$$\mathbb{C}^n \otimes \mathcal{O}_C \to E \to 0.$$

Let $U \subset \mathbb{C}^n$ be a subspace, and define $W := \operatorname{im}(U \otimes H^0(C, \mathcal{O}_C)) \subset H^0(C, E)$. Let G be the subsheaf of E generated by W. Then if

$$\frac{\dim(U)}{n} > \frac{h^0(C, G \otimes L^t)}{\Phi(t)},$$

ξ is GIT unstable.

5.2 Destabilization of sheaves

We recall our setup. Let g, r, and d be integers. Define $\Phi(t) = e + r(1 - g) + rdt$, and let $n = \Phi(0)$. Let (C, L) be a polarized reduced Gorenstein curve such that $\deg L = d$ and let $\hat{t}(d, g, r, e)$ have the property that for all $t > \hat{t}(d, g, r, e)$, the morphism

$$i_t : \mathfrak{Quot}^{\Phi,L}_{\mathbb{C}^n \otimes \mathcal{O}_C/C} \to \mathbf{G}(\Phi(t), (\mathbb{C}^n \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t)))^*)$$

is a closed embedding.

Our strategy for understanding the points of the GIT quotient is to begin with the destabilization of certain points. First, we demonstrate that semi-stability requires the presentation of a sheaf to induce an injection on global sections. This allows us to then place bounds on the size of torsion. Next, we destabilize slope-unstable sheaves, and finally demonstrate that the presentation must be surjective on global sections. Combined with the bound on torsion, we will see that GIT semi-stable sheaves must be torsion-free. In summary, by the end of the section we will have established that for large e and t (with explicit bounds), GIT semi-stable sheaves must be torsion-free, slope semi-stable, and have a presentation which induces an isomorphism on global sections.

5.2.1 Injectivity for the global section map

We immediately destabilize a large class of points. The following proposition is a straightforward generalization of [17, Prop 2.3.1].

Proposition 5.1. *Let C be a genus g curve with degree d polarization L. Let $\xi \in \mathfrak{Quot}^{\Phi,L}_{\mathbb{C}^n\otimes\mathcal{O}_C/C}$ correspond to a quotient*

$$\mathbb{C}^n \otimes \mathcal{O}_C \to E \to 0.$$

Then if $\mathbb{C}^n \otimes H^0(C,\mathcal{O}_C) \to H^0(C,E)$ is not injective, ξ is GIT unstable.

5.2.2 Restricting the size of Torsion

The main result of this section places restrictions on the size of torsion in GIT semi-stable sheaves. Once we establish that semi-stability of a presentation of a sheaf requires an isomorphism on global sections, we will conclude that sheaves with torsion are GIT unstable. The proposition is a straightforward extention of [17, Prop. 3.2.1].

Proposition 5.2. *There exists a bound $t_0(d,g,r,e) > \hat{t}(d,g,r,e)$ such that for each $t > t_0(d,g,r,e)$ and genus g curve C with degree d polarization L, the following holds: if $\xi \in \mathfrak{Quot}^{\Phi,L}_{\mathbb{C}^n\otimes\mathcal{O}_C/C}$ corresponds to*

$$\mathbb{C}^n \otimes \mathcal{O}_C \to E \to 0,$$

and $\operatorname{im}(\mathbb{C}^n \otimes H^0(C,\mathcal{O}_C)) \cap H^0(C,\tau_E) \neq 0$, where τ_E is the torsion subsheaf of E, then ξ is GIT-unstable.

5.2.3 Destabilizing slope unstable sheaves

In this section, we destabilize certain slope-unstable sheaves using straightforward extensions of results in [17]. The primary result is

Proposition 5.3 ([17, Prop. 4.1.1])**.** *There exist bounds $e_1(d,g,r) > r(g-1)$ and $t_1(d,g,r,e) > \hat{t}(d,g,r,e)$ such that for any pair $e > e_1(d,g,r)$ and $t > t_1(d,g,r,e)$ and genus g curve C with* *degree d polarization L, the following holds: if $\xi \in \mathfrak{Quot}^{\Phi,L}_{\mathbb{C}^n \otimes \mathcal{O}_C/C}$ corresponds to*

$$\mathbb{C}^n \otimes \mathcal{O}_C \to E \to 0,$$

where the induced morphism on global sections is an isomorphism and E is a slope-unstable torsion-free sheaf, then ξ is GIT unstable with respect to the linearization induced by i_t.

Lemma 5.3 ([17, Lemma 4.2.1])**.** *Assume the hypotheses of Proposition 5.3.*

There exists an integer $e_1(d,g,r) > r(g-1)$ such that for each $e > e_1(d,g,r)$, the following holds:

if $\xi \in \mathfrak{Quot}^{\Phi,L}_{\mathbb{C}^n \otimes \mathcal{O}_C/C}$ corresponds to

$$\mathbb{C}^n \otimes \mathcal{O}_C \to E \to 0,$$

where E is slope-unstable and torsion-free, then there exists a nonzero, proper, destabilizing subsheaf $0 \to F \to E$ and an exact sequence

$$0 \to \overline{F} \to F \to \tau \to 0,$$

where $\overline{F}$ is generated by global sections and τ is torsion.

5.2.4 Surjectivity of the global section map

In this section, we establish the surjectivity of the global sections morphism on GIT semistable sheaves. If $\xi \in \mathfrak{Quot}^{\Phi,L}_{\mathbb{C}^n \otimes \mathcal{O}_C/C}$ corresponds to

$$\mathbb{C}^n \otimes \mathcal{O}_C \to E \to 0,$$

because $n = \chi(E)$, we always have $n \leq h^0(C,E)$. In particular, if $H^0(C, \mathbb{C}^n \otimes \mathcal{O}_C) \to H^0(C,E)$ is a surjection, we have $n \geq h^0(C,E)$, which implies that $h^1(C,E) = 0$. On the other hand, if $h^1(C,E) = 0$, then $n = \chi(E)$. Excluding GIT unstable sheaves, we may assume by Proposition 5.1 that the global sections morphism is an injective morphism of finite-dimensional vector spaces of the same dimension, and therefore an isomorphism. Our main result for this section thus concerns $h^1(C,E)$:

Proposition 5.4 ([17, Prop. 5.2.1]). *There exist bounds* $e_2(d,g,r) > r(g-1)$ *and* $t_2(d,g,r,e) > \hat{t}(d,g,r,e)$ *such that for each pair* $e > e_2(d,g,r)$, $t > t_2(d,g,r,e)$ *and genus* g *curve* C *with degree* d *polarization* L, *the following holds: if* $\xi \in \mathfrak{Quot}^{\Phi,L}_{\mathbb{C}^n\otimes\mathcal{O}_C/C}$ *corresponds to*

$$\mathbb{C}^n \otimes \mathcal{O}_C \to E \to 0,$$

where $h^1(C,E) \neq 0$, *then* ξ *is GIT unstable.*

Proof. We sketch the proof. We will postpone the proofs of Lemmas 5.4 and Lemma 5.5 below in order to streamline the argument. The lemmas compute the bounds e_2 and t_2, and we now assume $e > e_2$ and $t > t_2$. By Proposition 5.1, we may assume that the global sections map ψ is injective. By Proposition 5.2, we may assume that the codimension of the image of ψ is bounded by the dimension of the space of torsion sections. We see that the hypotheses of Lemma 5.5 hold. Thus we may apply Lemma 5.4, which produces a subspace which destabilizes ξ by Lemma 5.2. □

The following two lemmas are straightforward generalizations of results in [17]. Together, they provide the bounds needed for the proof of Proposition 5.4.

Lemma 5.4 ([17, Lemma 5.1.1]). *There exists an integer* $e_2(d,g,r) > r(g-1)$ *such that for each* $e > e_2(d,g,r)$ *the following holds:*

Suppose that E *is a coherent sheaf on* C *having Hilbert polynomial* $\Phi(t)$ *with respect to* L, *and* τ *is the maximal torsion sub-sheaf of* E. *If*

(1) $h^1(C,E) \neq 0$,

(2) $\chi(\tau) < gd(rd+1)+1$,

then there exists a nonzero, proper subsheaf F *of* E *with multirank* (s_i) *not identically zero such that*

(1) F *is generated by global sections,*

(2)

$$\frac{\chi(F) - (gd(rd+1)+1)}{\sum s_i d_i} > \frac{\chi(E)}{rd} + 1.$$

Lemma 5.5 ([17, Lemma 5.1.2]). *Let $e > e_2(d, g, r)$ as in Lemma 5.4.*

There exists an integer $t_2(d, g, r, e) > t_0(d, g, r, e)$ such that for each $t > t_2(d, g, r, e)$ the following holds:

If $\xi \in \mathfrak{Quot}^{\Phi,L}_{\mathbb{C}^n \otimes \mathcal{O}_C/C}$ corresponds to

$$\mathbb{C}^n \otimes \mathcal{O}_C \to E \to 0,$$

where E is a coherent sheaf satisfying

(i) $\psi : \mathbb{C}^n \otimes H^0(C, \mathcal{O}_C) \to H^0(C, E)$ is injective,

(ii) $h^1(C, E) \neq 0$,

(iii) the torsion subsheaf τ of E satisfies $\chi(\tau) < gd(rd + 1) + 1$,

then there exists a nonzero subspace $W \subset \psi(\mathbb{C}^n \otimes H^0(C, \mathcal{O}_C))$ generating a nonzero, proper subsheaf $0 \to G \to E$ such that

$$\frac{\dim W}{n} > \frac{h^0(C, G \otimes L^t)}{\Phi(t)}.$$

5.3 GIT-semistable Sheaves

In this section, we demonstrate that the classes of quotients destabilized above are in fact the only unstable quotients. The results in this section are straightforward generalizations of results from [17], and so for brevity we omit proofs but provide the reader with references. The central result is the following

Proposition 5.5 ([17, Prop. 6.1.1]). *There exist bounds $e_3(d, g, r) > r(g-1)$ and $t_3(d, g, r, e) > \hat{t}(d, g, r, e)$ such that for each pair $e > e_3(d, g, r)$, $t > t_3(d, g, r, e)$ and genus g curve C with degree d polarization L, the following holds: if $\xi \in \mathfrak{Quot}^{\Phi,L}_{\mathbb{C}^n \otimes \mathcal{O}_C/C}$ corresponds to a quotient*

$$\mathbb{C}^n \otimes \mathcal{O}_C \to E \to 0,$$

where

$$\psi : \mathbb{C}^n \otimes H^0(C, \mathcal{O}_C) \to H^0(C, E)$$

is an isomorphism and E is a slope-stable (slope-semistable), torsion-free sheaf, then ξ is a GIT stable (semi-stable) point.

Proof. The proof follows by explicitly constructing a basis satisfying the Numerical Criterion for Grassmannians. The degree and tensor bounds are established in Lemma 5.6 and Lemma 5.7. □

The following lemmas are used in the proof of the above proposition. We include them for completeness, as they are essentially the same as results from [17].

Lemma 5.6 ([17, Lemma 6.2.1])**.** *Let q be an integer. Then there exists an integer $e_3(d, g, r, q)$ such that for each $e > e_3(d, g, r, q)$, the following holds:*

If E is a slope-semistable, torsion-free sheaf on C with Hilbert polynomial $\Phi(t)$ and

$$0 \to F \to E$$

is a nonzero subsheaf with multirank (s_i) satisfying $h^1(C, F) \neq 0$, then

$$\frac{\chi(F) + q}{\sum s_i d_i} < \frac{\chi(E)}{rd} - 1.$$

Lemma 5.7 ([17, Lemma 6.2.2])**.** *Let $e > e_3(d, g, r, b) > r(g - 1)$ be as described in Lemma 5.6. Then there exists an integer $t_3(d, g, r, e) > \hat{t}(d, g, r, e)$ such that for each $t > t_3(d, g, r, e)$, the following holds:*

If $\xi \in \mathfrak{Quot}^{\Phi, L}_{\mathbb{C}^n \otimes \mathcal{O}_C / C}$ corresponds to a quotient

$$\mathbb{C}^n \otimes \mathcal{O}_C \to E \to 0,$$

where E is a torsion-free, slope-semistable sheaf on C and $0 \to F \to E$ is a nonzero, proper subsheaf generated by global sections, then

$$\frac{h^0(C, F)}{n} \leq \frac{h^0(C, F \otimes L^t)}{\Phi(t)}.$$

Moreover, if E is slope-stable, then

$$\frac{h^0(C, F)}{n} < \frac{h^0(C, F \otimes L^t)}{\Phi(t)}.$$

5.4 Strict Slope-semistability

In this section, we demonstrate that strict slope-semistability implies strict GIT semistability. The following proposition is similar to [17, Prop. 6.4.1], and the proofs are similar. Here the weakened hypotheses on the singularities of the curve require greater care, and the arguments begin to diverge more substantially from [17] than in the previous sections. In light of this, we include greater detail.

Proposition 5.6. *There exist bounds $e_4(d,g,r,T)$ and $t_4(d,g,r,e)$ such that for each pair $e > e_4(d,g,r,T)$ and $t > t_4(d,g,r,e)$ and every genus g curve C with degree d polarization L and class of singularity types T, the following holds:*
If $\xi \in \mathfrak{Quot}^{\Phi,L}_{\mathbb{C}^n\otimes\mathcal{O}_C/C}$ corresponds to a quotient

$$\mathbb{C}^n \otimes \mathcal{O}_C \to E \to 0$$

where

$$\mathbb{C}^n \otimes H^0(C,\mathcal{O}_C) \to H^0(C,E)$$

is an isomorphism, and E is torsion-free and strictly slope-semistable, then ξ is GIT strictly semistable.

Proof. Let $0 \to F \to E$ be a nonzero proper semistabilizing subsheaf. Suppose F is generated by global sections and $h^1(C,F) = 0$. Then we may directly apply the Numerical Criterion for Grassmannians to the linearized SL_n action on

$$\mathbf{G}(\Phi(t), (\mathbb{C}^n \otimes \mathrm{Sym}^t(H^0(C,L)))^*).$$

The point ξ corresponds to the quotient

$$\psi^t : \mathbb{C}^n \otimes \mathrm{Sym}^t(H^0(C,L)) \to H^0(C, E\otimes L^t) \to 0.$$

Let $U \otimes H^0(C,\mathcal{O}_C)$ be the pre-image of $H^0(C,F)$. Let $\bar{v} = (v_1,\ldots,v_n)$ be a basis of $\mathbb{C}^n$ such that $v_1,\ldots,v_{h^0(C,F)}$ is a basis for U. Define $w(v_i) = 0$ for $1 \le i \le h^0(C,F)$, and 1 otherwise. We must show that for any $\Phi(t)$-tuple $(a_1, \ldots, a_{\Phi(t)})$ of $\bar{v}$-pure elements of $\mathbb{C}^n \otimes \mathrm{Sym}^t(H^0(C, L))$ which projects to a basis of $H^0(C, E \otimes L^t)$,

$$\sum_{i=1}^{n} \frac{w(v_i)}{n} = \sum_{j=1}^{\Phi(t)} \frac{w(a_j)}{\Phi(t)}.$$

Because ξ is semi-stable, we have

$$\sum_{i=1}^{n} \frac{w(v_i)}{n} \geq \sum_{j=1}^{\Phi(t)} \frac{w(a_j)}{\Phi(t)}.$$

Thus, it suffices to demonstrate the reverse inequality. By a few algebraic manipulations, we reduce this to

$$h^0(C, F)rdt \geq h^0(C, E)(\sum s_i d_i)t,$$

but in fact we have equality because F is strictly slope semistable.

The proof is now finished, pending a demonstration that there is a nonzero, proper semistabilizing subsheaf of E generated by global sections and whose first cohomology group vanishes. This is the content of Lemma 5.8. □

The following lemma is similar to [17, Lemma 6.4.1]. The key difference in our result is the bound δ, which has been modified to hold for α-stable curves.

Lemma 5.8. *Let $T = T(C)$ be the class of singularity types of C. There exists an integer $e_4(d, g, r, T)$ such that for any $e > e_4(d, g, r, T)$ the following holds: if E is any slope-semistable, torsion free sheaf on C with Hilbert polynomial $\Phi(t)$, and $0 \to F \to E$ is a nonzero subsheaf with multirank (s_i) satisfying*

$$\frac{\chi(F)}{\sum s_i d_i} = \frac{\chi(E)}{rd},$$

then

(i) $h^1(C, F) = 0$.

(ii) *F is generated by global sections.*

Proof. Suppose F is a nonzero subsheaf of E satisfying the hypothesis. By Lemma 5.6, if $e > e_3(d,g,r,0)$, $h^1(C,F) = 0$.

Let $x \in C$ be a point. We have the exact sequence

$$0 \to \mathfrak{m}_x F \to F \to F/\mathfrak{m}_x F \to 0.$$

There is a constant δ_x (see Corollary 2.3), depending only on r, g, d, and T such that

$$\dim_k F/\mathfrak{m}_x F < \delta_x.$$

Let $\delta = \max_{x \in C} \delta_x$. Note that the multirank of $\mathfrak{m}_x F$ is the same as F. We have

$$\chi(\mathfrak{m}_x F) + \delta > \chi(F).$$

By hypothesis,

$$\frac{\chi(\mathfrak{m}_x F) + \delta}{\sum s_i d_i} > \frac{\chi(E)}{rd}.$$

For $e > e_3(d,g,r,\delta)$, $h^1(C, \mathfrak{m}_x F) = 0$ by Lemma 5.6. In this case, F_x is generated by global sections for every point x. Thus, F is globally generated, so we may take $e_4(d,g,r,T) = e_3(d,g,r,\delta)$. □

5.5 Proof of Theorem 5.1

The proof of Theorem 5.1 is complete, but the pieces must be assembled. First, we explicitly state the degree bound determined by results in the previous section.

Definition 5.2. Let g, r and d be integers, and T be a class of singularity types. Let

- $e_0(d,g,r,T) = r(g-1)$
- $e_1(d,g,r,T) = (dg(rd+1)+1)(rd)^2 - r(1-g)$ (see Lemma 5.3)
- $e_2(d,g,r,T) = rd(2(dg(rd+1)+1) + g + rd - 1) - r(1-g)$ (see Lemma 5.4)
- $e_3(d,g,r,T) = (rd)^2(g + dg(rd+1) + 3) - r(1-g)$ (see Lemma 5.6)
- $e_4(d,g,r,T) = (rd)^2(g + \delta + 2) - r(1-g)$ (see Proposition 5.6),

where

$$\delta = rd \max_{A \in T}(1 + \dim_k \overline{A}/A).$$

Define

$$E(d, g, r, T) := \max_i e_i(d, g, r, T). \tag{5.1}$$

Let $\mathscr{T}$ denote the class of singularity types of α-stable curves and define

$$E(r, g) := \max_i e_i(\deg L, g, r, \mathscr{T}), \tag{5.2}$$

where because $\mathscr{T}$ is fixed, $E(\deg L, r, g)$ depends only on $\deg L$, r, and g. Lastly, define

$$t(d, g, r, e) = \max_{i=0,\ldots,5} t_i(d, g, r, e), \tag{5.3}$$

where for $i = 0, \ldots, 4$ the t_i are given by Propositions 5.2, 5.3, 5.4, 5.5, and 5.6, and $t_5 = \hat{t}(d, g, r, e)$.

Proof of Theorem 5.1. Let g, r, and d be integers. Define $\Phi(t) = e + r(1 - g) + rdt$, and let $n = \Phi(0)$. Let (C, L) be a polarized reduced Gorenstein curve such that $\deg L = d$. Let T be the class of singularity types of C, and let $E(d, g, r, T)$ be as in (5.1). Let $e > E(d, g, r, T)$ and $\xi \in \mathfrak{Quot}^{\Phi, L}_{\mathbb{C}^n \otimes \mathcal{O}_C / C}$ corresponding to

$$\mathbb{C}^n \otimes \mathcal{O}_C \to E \to 0,$$

and let

$$\psi : \mathbb{C}^n \otimes H^0(C, \mathcal{O}_C) \to H^0(C, E)$$

be the function on global sections. Let $t(d, g, r, e)$ be as given by (5.3). By Propositions 5.1, 5.2, 5.3, and 5.4, ξ is GIT unstable if any of the following are true:

- ψ is not an isomorphism,
- E is not torsion-free,
- E is slope-unstable.

Now we turn to the reverse implication. By Proposition 5.5, if E is torsion-free and slope-stable (slope-semistable) and ψ is an isomorphism, ξ is GIT stable (semi-stable).

Finally, by Proposition 5.6, if ξ is GIT semi-stable then ξ is stable if and only if E is slope-stable, and the theorem is proved. □

5.6 Construction of the GIT quotient over a fixed curve

In this section, we construct the fiberwise GIT quotient.

Definition 5.3. Let (C, L) be a polarized curve of genus g and let d be the degree of L. Let e, t, and r be integers such that $e > E(d, g, r, T(C))$ defined in (5.1) and $t > t(d, g, r, e)$ as defined in (5.3). Define $\Phi(t) = e + r(1 - g) + rdt$. Let

$$Q^r \subset \mathfrak{Quot}^{\Phi,L}_{\mathbb{C}^n \otimes \mathcal{O}_C/C}$$

be the locus of sheaves of uniform rank r and $\mathcal{O}_Q(1)$ the very ample line bundle determined by i_t (see Theorem 5.1 for the definition of i_t). By Lemma 3.1, Q^r is both closed and open. Define

$$U^L_{e,r}(C) := Q^r /\!\!/_{\mathcal{O}_{Q(1)}} SL_n.$$

Because stability and semi-stability are determined by slopestability and slope semi-stability for $t > t(d, g, r, e)$, we see that the quotient is independent of the choice of t. Observe that twisting by $L^{\otimes a}$ induces an isomorphism

$$\mathfrak{Quot}^{\Phi,L}_{\mathbb{C}^n \otimes \mathcal{O}_C/C} \cong \mathfrak{Quot}^{(e+ard)+r(1-g)+rdt, L^{\otimes a}}_{\mathbb{C}^n \otimes \mathcal{O}_C/C}.$$

Thus, for any e, we may take the smallest integer a with the property that $e + ard$ is greater than the e_i and define

$$U^L_{e,r}(C) := U^{L^{\otimes a}}_{e+ard,r}(C).$$

The following theorem is similar to [21, Thm. 1.21] which makes the same claim, but for $e \gg 0$. Our formulation includes an explicit lower bound on e and includes only sheaves of uniform rank. The explicit bound on e will allow us to ensure the description holds for families of curves.

Theorem 5.9. *Let* (C, L) *be a polarized curve of genus* g *with* $\deg L = d$. *Then for all* r *and* $e > E(d, g, r, T(C))$, *the scheme* $U^L_{e,r}(C)$ *is a projective variety containing the set of aut-equivalence classes of slope-semistable vector bundles as an open subset.*

Proof. The proof follows from the definition of E and results in this section, and is roughly equivalent to the proof of Theorem 1.1 above. We leave the details to the reader. □

Theorem 5.10. *Let* (C, L) *be a polarized curve. For any* e, r *the scheme* $U_{e,r}(C)$ *is the categorical moduli space for* $\mathcal{U}_{e,r}(C)$.

Proof. The proof is essentially identical to that of Theorem 3.2, and so for brevity, we leave the details in this case to the reader. □

Remark 5.4. The functor $\mathcal{U}_{e,r}(C)$ is the subfunctor of the moduli problem described in [21, p. 9] consisting of sheaves of uniform rank. Theorem 5.10 thus implies that Theorem 1.21 of [21] applies to $U_{e,r}(C)$ as a subspace of the Simpson moduli space. In particular,

(1) $U_{e,r}(C)$ is projective;

(2) The points of $U_{e,r}(C)$ represent the equivalence classes of semi-stable sheaves under aut-equivalence.

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