Math 2300-007: Quiz 7

Name: Solutions 3/10/18 Score: ______

Collaborators:

Directions: This take-home quiz will be due at the beginning of class on Wednesday, March 7. You may use your notes, textbook, and colleagues from our class as resources, but your final write-up should be in your own words. If you work with collaborators from our class, please include their names on this quiz. You can earn 1 bonus point for discussing this quiz with me during my office hours.

1. (5 points) Determine if the **sequence** converges or diverges. If it converges, find the limit. Justify each answer in a way that would make sense to a colleague from class.

(a) $a_n = \left(1 + \frac{5}{n}\right)^{3n}$

$\lim_{n\to\infty} a_n$ has form 1^∞. Indeterminate. Try $\ln(a_n)$ and use l'Hopital's.

$$\lim_{n\to\infty} \ln(a_n) = \lim_{n\to\infty} 3n \ln\left(1+\tfrac{5}{n}\right) \quad \text{form: } \infty \cdot 0$$
$$= \lim_{n\to\infty} \frac{\ln(1+\frac{5}{n})}{\frac{1}{3n}} \quad \text{form: } \tfrac{0}{0}$$
$$\overset{H}{=} \lim_{n\to\infty} \frac{\frac{1}{1+5/n} \cdot \frac{-5}{n^2}}{\frac{-1}{3n^2}}$$
$$= \lim_{n\to\infty} \frac{15}{1+5/n}$$
$$= 15$$

We showed that $\lim_{n\to\infty} \ln(a_n) = 15$, so $\lim_{n\to\infty} a_n = e^{15}$.

[because $\ln$ is continuous, $\lim_{n\to\infty} \ln(a_n) = \ln\left(\lim_{n\to\infty} a_n\right)$, so $15 = \ln\left(\lim_{n\to\infty} a_n\right)$]

(b) $a_n = \frac{\ln(n)\cos(n)}{1+n^2}$

The squeeze theorem is a good choice here because $|\cos(n)| \le 1$.

We have

$$\frac{-\ln(n)}{1+n^2} \le \frac{\ln(n)\cos(n)}{1+n^2} \le \frac{\ln(n)}{1+n^2} \quad \text{since } -1 \le \cos(n) \le 1$$

$$\lim_{n\to\infty} \frac{-\ln(n)}{1+n^2} \le \lim_{n\to\infty} a_n \le \lim_{n\to\infty} \frac{\ln n}{1+n^2}$$

$$0 \le \lim_{n\to\infty} a_n \le 0$$

[It follows that $\lim_{n\to\infty} a_n = 0$ by the squeeze theorem.]

(c) $a_n = \sqrt[n]{3^n + 5^n}$

There are several methods for this one. The key is that 5^n "dominates" for large n, so you have to emphasize that (by e.g. dividing it out)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (3^n + 5^n)^{1/n}$$

$$= \lim_{n\to\infty} \left(5^n\left[\frac{3^n}{5^n}+1\right]\right)^{1/n}$$

$$= \lim_{n\to\infty} (5^n)^{1/n} \cdot \left(\left(\tfrac{3}{5}\right)^n+1\right)^{1/n}$$

$$= \lim_{n\to\infty} 5 \cdot \left((3/5)^n+1\right)^{1/n}$$

$$= 5 \cdot (0+1)^0$$

$$= 5$$

(d) $a_n = (-1)^n \dfrac{5n^2 - 7n}{\sqrt[3]{n^6 + 3n^2 - 1}}$

This one is an alternating sequence, so look at $|a_n|$.

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{5n^2 - 7n}{\sqrt[3]{n^6+3n^2-1}}$$

← top + bottom both have highest degree n^2, so divide by n^2 on both top + bottom

$$= \lim_{n\to\infty} \frac{5n^2-7n}{\sqrt[3]{n^6+3n^2-1}} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{5 - 7/n^4}{\sqrt[3]{\frac{n^6}{n^6} + \frac{3n^2}{n^6} - \frac{1}{n^6}}}$$

← ($\frac{1}{n^2}$ becomes $\sqrt[3]{\frac{1}{n^6}}$)

$$= \lim_{n\to\infty} \frac{5 - 7/n^4}{\sqrt[3]{1 + \frac{3}{n^4} - \frac{1}{n^6}}}$$

$$= \frac{5-0}{\sqrt[3]{1+0-0}}$$

$$= 5$$

2. (5 points) For each of the following **series**, determine if the series converges or diverges. If the series converges, find its sum. Justify each answer in a way that would make sense to a colleague from class.

(a) $\sum_{n=1}^{\infty} \sqrt[n]{4}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 4^{1/n} = 4^0 = 1 \neq 0$

Since the limit of the terms $a_n = \sqrt[n]{4}$ is not zero, the divergence test tells us that $\sum_{n=1}^{\infty} \sqrt[n]{4}$ diverges

(b) $\sum_{n=2}^{\infty} \frac{(-3)^n}{7^{n+3}}$ $= \frac{(-3)^2}{7^5} + \frac{(-3)^3}{7^6} + \frac{(-3)^4}{7^7} + \cdots$

This is a geometric series with

$a = \frac{(-3)^2}{7^5} = \frac{9}{7^5}$ ← First term

$r = \frac{-3}{7}$ ←ratio

Since $|r| = \frac{3}{7} < 1$, the series converges to

$\frac{a}{1-r} = \frac{\frac{9}{7^5}}{1-\frac{-3}{7}} = \cdots = \frac{9}{10 \cdot 7^4}$.

(c) $\sum_{n=3}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$ Try integral test. Let $f(x) = \frac{1}{x\sqrt{\ln x}}$.

Hypotheses:
- $f(x) \geq 0$ for $x \geq 3$ ✓
- $f(x)$ decreasing for $x \geq 3$ because x and $\sqrt{\ln x}$ are increasing and the product of increasing functions is increasing. $\frac{1}{\text{increasing}}$ = decreasing. Could also show $f' < 0$. ✓
- $f(x)$ continuous on $x \geq 3$ because the only problem spots are $x=0$ & $x=1$. ✓

Test.

$$\int_3^\infty \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{t\to\infty} \int_3^t \frac{1}{x\sqrt{\ln x}}\,dx$$

$$\left\{\begin{array}{l} u = \ln x \\ du = \frac{1}{x}dx \end{array}\right\} = \lim_{t\to\infty} \int_{\ln(3)}^{\ln(t)} \frac{1}{\sqrt{u}}\,du$$

$$= \lim_{t\to\infty} 2\sqrt{u}\Big|_{\ln(3)}^{\ln(t)}$$

$$= \lim_{t\to\infty} \left[2\sqrt{\ln t} - 2\sqrt{\ln(3)}\right]$$

$$= \infty$$

Conclusion: Since $\int_3^\infty \frac{1}{x\sqrt{\ln x}}\,dx$ diverges, the integral test says that the series $\sum_{n=3}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges, too.

(d) $\sum_{n=1}^{\infty} \ln\left|\frac{\cos\left(\frac{1}{n}\right)}{\cos\left(\frac{1}{n+1}\right)}\right|$ Telescoping series

$$= \lim_{N\to\infty} \sum_{n=1}^{N} \left[\ln\left|\cos\left(\tfrac{1}{n}\right)\right| - \ln\left|\cos\left(\tfrac{1}{n+1}\right)\right|\right]$$

$$= \lim_{N\to\infty} \Big[\left(\ln|\cos(\tfrac{1}{1})| - \cancel{\ln|\cos(\tfrac{1}{2})|}\right) + \left(\cancel{\ln|\cos(\tfrac{1}{2})|} - \cancel{\ln|\cos(\tfrac{1}{3})|}\right) + \left(\cancel{\ln|\cos(\tfrac{1}{3})|} - \cancel{\ln|\cos(\tfrac{1}{4})|}\right)$$
$$+ \cdots + \left(\cancel{\ln|\cos(\tfrac{1}{N-1})|} - \cancel{\ln|\cos(\tfrac{1}{N})|}\right) + \left(\cancel{\ln|\cos(\tfrac{1}{N})|} - \ln|\cos(\tfrac{1}{N+1})|\right)\Big]$$

$$= \lim_{N\to\infty} \left[\ln|\cos(1)| - \ln\left|\cos\left(\tfrac{1}{N+1}\right)\right|\right]$$

$$= \ln|\cos(1)| - \ln|\cos(0)|$$

$$= \ln|\cos(1)| - \ln(1)$$

$$= \ln|\cos(1)|$$

$\left[\text{The series converges to } \ln|\cos(1)|\right]$