Worksheet Purpose: A few weeks ago we saw that a given improper integral converges if its integrand is less than the integrand of another integral known to converge (where all integrands are positive). Similarly a given improper integral diverges if its integrand is greater than the integrand of another integral known to diverge (where all integrands are positive). In problems 1-7 you'll apply a similar strategy to determine if certain series converge or diverge. Additionally, in problems 8 and 9 you'll apply a different method (using limits) to determine if a series converges or diverges.

1. For each of the following situations, determine if $\sum_{n=1}^{\infty} c_n$ converges, diverges, or if one cannot tell without more information.

   (a) $0 \leq c_n \leq \frac{1}{n}$ for all n, we can conclude that $\sum c_n$ ____________________________

   (b) $\frac{1}{n} \leq c_n$ for all n, we can conclude that $\sum c_n$ ____________________________

   (c) $0 \leq c_n \leq \frac{1}{n^2}$ for all n, we can conclude that $\sum c_n$ ____________________________

   (d) $\frac{1}{n^2} \leq c_n$ for all n, we can conclude that $\sum c_n$ ____________________________

   (e) $\frac{1}{n^2} \leq c_n \leq \frac{1}{n}$ for all n, we can conclude that $\sum c_n$ ____________________________

2. **Follow-up to problem 1:** For each of the cases above where you needed more information, give (i) an example of a series that converges and (ii) an example of a series that diverges, both of which satisfy the given conditions.

3. Fill in the blanks:

> **The Comparison Test** (also known as Term-size Comparison Test or Direct Comparison Test)
> Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
>
> - If $\sum b_n$ _______________ and $a_n \leq b_n$, then $\sum a_n$ also _______________.
> - If $\sum b_n$ _______________ and $a_n \geq b_n$, then $\sum a_n$ also _______________.

Note: in the above theorem and for the rest of this worksheet, we will use $\sum b_n$ to represent the series whose convergence/divergence we already know (p-series or geometric), and $\sum a_n$ will represent the series we are trying to determine convergence/divergence of.

Now we'll practice using the Comparison Test:

4. Let $a_n = \dfrac{1}{2^n + n}$ and let $b_n = \left(\dfrac{1}{2}\right)^n$ for $n \geq 1$, both sequences with positive terms.

   (a) Does $\displaystyle\sum_{n=1}^{\infty} b_n$ converge or diverge? Why?

   (b) How do the size of the terms a_n and b_n compare?

   (c) What can you conclude about $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + n}$?

5. Let $a_n = \dfrac{1}{n^2 + n + 1}$, a sequence with positive terms.
   Consider the rate of growth of the denominator. This hints at a choice of:

   $b_n =$ ________________, another positive term sequence.

   (a) Does $\sum b_n$ converge or diverge? Why?

   (b) How do the size of the terms a_n and b_n compare?

   (c) What can you conclude about $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + n + 1}$?

6. Use the Comparison Test to determine if $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n^4 + 1}}{n^3 - 2}$ converges or diverges.

7. Use the Comparison test to determine if $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{\sqrt{n^3+n}}$ converges or diverges.

8. Disappointingly, sometimes the Comparison Test doesn't work like we wish it would. For example, let $a_n = \dfrac{1}{n^2-1}$ and $b_n = \dfrac{1}{n^2}$ for $n \geq 2$.

   (a) By comparing the relative sizes of the terms of the two sequences, do we have enough information to determine if $\displaystyle\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converges or diverges?

   (b) Show that $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

   (c) Since $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, we know that $a_n \approx b_n$ for large values of n. Do you think that $\displaystyle\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{n^2-1}$ must converge?

When we have chosen a good series to compare to, but the inequalities don't work in our favor, we use the Limit Comparison Test instead of the Comparison Test.

The Limit Comparison Test
Suppose $a_n > 0$ and $b_n > 0$ for all n. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where c is finite and $c > 0$, then the two series $\sum a_n$ and $\sum b_n$ either both ______________ or both _____________.

Now we'll practice using the Limit Comparison Test:

9. Determine if the series $\displaystyle\sum_{n=2}^{\infty} \frac{n^3 - 2n}{n^4 + 3}$ converges or diverges.