1. Determine the degree of the Taylor polynomial $P_n(x)$ expanded about $c = 1$ that should be used to approximate $\ln(1.2)$ so that the error is less than 0.0005.

   (a) Find the first four derivatives for $f(z) = \ln(z)$

$$f(z) = \ln(z)$$
$$f'(z) = \frac{1}{z}$$
$$f''(z) = \frac{-1}{z^2}$$
$$f^{(3)}(z) = \frac{2}{z^3}$$
$$f^{(4)}(z) = \frac{-6}{z^4}$$

   (b) Find a formula for $\left|f^{(n+1)}(z)\right|$ where $z > 0$.

$$\left|f^{(n+1)}(z)\right| = \frac{n!}{z^{n+1}}$$

   (c) We are approximating the function $f(x) = \ln(x)$ at $x = 1.2$ using a Taylor polynomial centered at $c = 1$. If

$$|f^{(n+1)}(z)| \leq M$$

   for all z in the interval $1 \leq z \leq 1.2$. Then we wish to find the upper bound M. To do this find

$$\frac{d}{dz}\left(|f^{(n+1)}(z)|\right).$$

$$\frac{d}{dz}\left(|f^{(n+1)}(z)|\right) = \frac{-(n+1)!}{z^{n+2}}$$

      i. Does $|f^{(n+1)}(z)|$ have any critical points on $1 \leq z \leq 1.2$? No

      ii. Is $|f^{(n+1)}(z)|$ increasing or decreasing on $1 \leq z \leq 1.2$? decreasing

(d) Use part (c) to identify a good upper bound M for $|f^{(n+1)}(z)|$ on the interval $1 \leq z \leq 1.2$. (Notice that M will depend on n.)

Using the fact that $|f^{(n+1)}(z)|$ is decreasing we know that the maximum value will occur on the left endpoint. Therefore to find the maximum we evaluate at $x = 1$. Thus $M = n!$

(e) Make use of the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-c|^{n+1} < 0.0005$$

to find an appropriate value for n that gives accuracy within 0.0005.

Using the fact that $M = n!$, $x = 1.2$, and $c = 1$ we have,

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-c|^{n+1} = \frac{n!}{(n+1)!}|.2|^{n+1} < 10^{-3}$$

which by guessing and checking we see that for $n = 3$ this is true.

(f) Finally approximate $\ln(1.2)$ using the nth-degree Taylor polynomial centered at $c = 1$ with the n you found in part (e).

$$P_3(z) = \sum_{n=1}^{3} \frac{(-1)^{n-1}(x-1)^n}{n} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$$

Therefore when $x = 1.2$ we get $P_3(1.2) = 0.182667$.