MATH 2300 – review problems for Exam 3, part 1

1. Find the radius of convergence and interval of convergence for each of these power series:

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{(x+5)^n}{2^n \ln n}$

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{n(x-1)^n}{4^n}$

   (c) $\displaystyle\sum_{n=0}^{\infty} n!(3x+1)^n$

   (d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^{n+1}x^n}{n^3+1}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n x^n}{n!}$

2. Let
$$f(x) = \sum_{n=1}^{\infty} \frac{(x+4)^n}{n^2}$$
Find the intervals of convergence of f and f'.

3. If $\sum b_n(x-2)^n$ converges at $x = 0$ but diverges at $x = 7$, what is the largest possible interval of convergence of this series? What's the smallest possible?

4. The power series $\sum c_n(x-5)^n$ converges at $x = 3$ and diverges at $x = 11$. What are the possibilities for the radius of convergence? What can you say about the convergence of $\sum c_n$? Can you determine if the series converges at $x = 6$? At $x = 7$? At $x = 8$? at $x = 2$? At $x = -1$? At $x = -2$? At $x = 12$? At $x = -3$?

5. The series $\sum c_n(x+2)^n$ converges at $x = -4$ and diverges at $x = 0$. What can you say about the radius of convergence of the power series? What can you say about the convergence of $\sum c_n$? What can you say about the convergence of the series $\sum c_n 2^n$? What can you say about the convergence/divergence of the series at $x = -1$? At $x = -3$? At $x = 1$? At $x = -10$?

6. Say that $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. Find $f'(x)$ by differentiating termwise.

7. Use any method to find a power series representation of each of these functions, centered about $a = 0$. Give the interval of convergence (Note: you should be able to give this interval based on your derivation of the series, not by using the ratio test.)

   (a) $\displaystyle\frac{1}{1+x}$

   (b) $\displaystyle\frac{1}{1+x^2}$

   (c) $\arctan x$

   (d) $xe^x - x$

   (e) $\ln(1+x)$

(f) $x \ln(1+3x^2)$

(g) $\dfrac{\sin(-2x^2)}{x}$

(h) $\dfrac{1}{(1-x)^2}$

(i) $\displaystyle\int \frac{1}{1+x^5}\, dx$

8. Determine the function or number represented by the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} n x^{n-1}$

(b) $\displaystyle\sum_{n=1}^{\infty} n x^{n}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{5^{2n} n!}$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n+1}}{(2n+1)!}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n}}{n}$

(f) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n}}{(2n)!}$

9. A car is moving with speed 20 m/s and acceleration 2 $\mathrm{m/s^2}$ at a given instant. Using a second degree Taylor polynomial, estimate how far the car moves in the next second.

10. Estimate $\int_0^1 \frac{\sin t}{t} dt$ using a 3rd degree Taylor Polynomial. What degree Taylor Polynomial should be used to get an estimate within 0.005 of the true value of the integral? (Hint: use the alternating series estimate).

11. Calculate the Taylor series of $\ln(1+x)$ by two methods. First calculate it "from scratch" by finding terms from the general form of Taylor series. Then calculate it again by starting with the Taylor series for $f(x) = \frac{1}{1-x}$ and manipulating it. Determine the interval of convergence each time.

12. Express the integral as an infinite series.

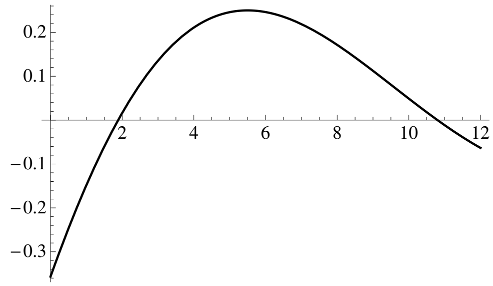
$$\int \frac{e^x - 1}{x}\, dx$$

13. Let $f(x) = \dfrac{1}{1-x}$.

(a) Find an upper bound M for $|f^{(n+1)}(x)|$ on the interval $(-1/2, 1/2)$.

(b) Use this result to show that the Taylor series for $\dfrac{1}{1-x}$ converges to $\dfrac{1}{1-x}$ on the interval $(-1/2, 1/2)$.

14. Consider the function $y = f(x)$ sketched below.



Suppose $f(x)$ has Taylor series

$$f(x) = a_0 + a_1(x-4) + a_2(x-4)^2 + a_3(x-4)^3 + ...$$

about $x = 4$.

(a) Is a_0 positive or negative? Please explain.
(b) Is a_1 positive or negative? Please explain.
(c) Is a_2 positive or negative? Please explain.

15. How many terms of the Taylor series for $\ln(1+x)$ centered at $x = 0$ do you need to estimate the value of $\ln(1.4)$ to three decimal places (that is, to within .0005)?

16. (a) Find the 4th degree Taylor Polynomial for $\cos x$ centered at $a = \pi/2$.

(b) Use it to estimate $\cos(89°)$.
(c) Use Taylor's inequality to determine what degree Taylor Polynomial should be used to guarantee the estimate to within .005.

17. (a) Find the 3rd degree Taylor Polynomial $P_3(x)$ for $f(x) = \sqrt{x}$ centered at $a = 1$ by differentiating and using the general form of Taylor Polynomials.
(b) Use the Taylor Polynomial in part (a) to estimate $\sqrt{1.1}$.
(c) Use Taylor's inequality to determine how accurate is your estimate is guaranteed to be.

18. Use Taylor's inequality to find a reasonable bound for the error in approximating the quantity $e^{0.60}$ with a third degree Taylor polynomial for e^x centered at $a = 0$.

19. Consider the error in using the approximation $\sin\theta \approx \theta - \theta^3/3!$ on the interval $[-1, 1]$. Where is the approximation an overestimate? Where is it an underestimate?

20. Write down from memory the Taylor Series centered around $a = 0$ for the functions e^x, $\sin x$, $\cos x$ and $\frac{1}{1-x}$.

21. (a) Find the 4th degree Taylor Polynomial for $f(x) = \sqrt{x}$ centered at $a = 1$ by differentiating and using the general form of Taylor Polynomials.
(b) Use the previous answer to find the 4th degree T.P. for $f(x) = \sqrt{1-x}$ centered at $x = 0$.
(c) Use the previous answer to find the 3rd degree T.P. for $f(x) = \frac{1}{\sqrt{1-x}}$.
(d) Use the previous answer to find the 3rd degree T.P. for $f(x) = \frac{1}{\sqrt{1-x^2}}$.
(e) Use the previous answer to find the 3rd degree T.P. for $f(x) = \arcsin x$.