MIDTERM 3
CALCULUS 2

MATH 2300
FALL 2018

Monday, December 3, 2018
5:15 PM to 6:45 PM

Name ____________________

PRACTICE EXAM

SOLUTIONS

Please answer all of the questions, and show your work.
You must explain your answers to get credit.
You will be graded on the clarity of your exposition!

Date: October 27, 2018.

1
8 points

1. Match the following functions with their corresponding Maclaurin series:

(a) $e^{x^2/2} =$ _______________ (VI)

(b) $\cos\left(\frac{x}{2}\right) =$ _______________ (II)

(c) $\frac{1}{(1-x)^2} =$ _______________ (III)

(d) $x\ \arctan(x) =$ _______________ (IV)

(I) $\sum_{n=0}^{\infty} x^{2n}$

(II) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(2n)!}$

(III) $\sum_{n=1}^{\infty} nx^{n-1}$

(IV) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2n+1}$

(V) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)!}$

(VI) $\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$

SOLUTION

Here are more details on the solutions:

1.(a). We know that the Maclaurin series for e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Thus, substituting $x^2 \mapsto x^2/2$, we obtain that the Macluarin series for $e^{x^2/2}$ is

$$\boxed{\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}.}$$

1.(b). We know that the Maclaurin series for $\cos(x)$ is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$. Thus, substituting $x \mapsto x/2$, we obtain that the Macluarin series for $\cos(x/2)$ is

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(2n)!}.}$$

1.(c). We have the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Using term-by-term differentiation,

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = \boxed{\sum_{n=1}^{\infty} nx^{n-1}}$$

1.(d). The Maclaurin series for $\arctan x$ is

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Multiplying by x,

$$x\ \arctan x = x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2n+1}}$$

2
12 points

2. Consider the power series $\sum_{n=1}^{\infty} \frac{(x-5)^n}{2^n n^2}$

2.(a). Find the *radius of convergence* of the power series. *Show all work in justifying your answer*. $\boxed{R = 2}$

2.(b). Find the *interval of convergence*. *Show all work in justifying your answer*. $\boxed{[3, 7]}$

SOLUTION

Here are more details on the solutions:

2.(a). Using the Ratio Test and applying limit laws,

$$L = \lim_{n\to\infty} \left| \frac{(x-5)^{n+1}}{2^{n+1}(n+1)^2} \cdot \frac{2^n n^2}{(x-5)^n} \right| = \lim_{n\to\infty} \frac{|x-5|}{2} \cdot \frac{n^2}{(n+1)^2}$$
$$= \frac{|x-5|}{2} \cdot \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \frac{|x-5|}{2} \cdot 1 = \frac{|x-5|}{2}$$

Setting $L < 1$,

$$\frac{|x-5|}{2} < 1$$
$$|x-5| < 2$$

Hence the radius of convergence is 2.

2.(b). The above inequality gives us an interval of radius 2 centered at $a = 5$. This interval has $x = 3$ and $x = 7$ as its endpoints so we must check for convergence at these points.

$x = 3$

$$\sum_{n=1}^{\infty} \frac{(3-5)^n}{2^n n^2} = \sum_{n=1}^{\infty} \frac{(-2)^n}{2^n n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

The above series converges by the Alternating Series Test; even better, it converges absolutely by the p-Test with $p = 2$.

$x = 7$

$$\sum_{n=1}^{\infty} \frac{(7-5)^n}{2^n n^2} = \sum_{n=1}^{\infty} \frac{2^n}{2^n n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

The above series converges by the p-Test with $p = 2$.

Since the power series converges on both endpoints, the interval of convergence is $[3, 7]$.

3
12 points

3. Find the solution of the differential equation

$$y(x+1) + y' = 0$$

that satisfies the initial condition $y(-2) = 1$. Show all your work.

SOLUTION

The solution is

$$\boxed{y = e^{-x^2/2-x}}$$

To see this, observe that the differential equation is separable.

$$\begin{aligned} y(x+1) + y' &= 0 \\ y' &= -y(x+1) \\ \frac{dy}{dx} &= -y(x+1) \\ \frac{dy}{y} &= -(x+1)dx \\ \int \frac{dy}{y} &= \int -(x+1)dx \\ \ln|y| &= -x^2/2 - x + C \\ e^{\ln|y|} &= e^{-x^2/2-x+C} \\ |y| &= e^{-x^2/2-x+C} \end{aligned}$$

Let $K = \pm e^C$. Then $y = Ke^{-x^2/2-x}$ and plugging in the initial condition,

$$1 = Ke^{-(-2)^2/2-(-2)} = Ke^{-2+2} = K$$

Hence the solution to the differential equation with the given initial condition is

$$y = e^{-x^2/2-x}$$

4
8 points

4. Given the following power series $\sum_{n=0}^{\infty} a_n (x-2)^n$ we know that at $x = 0$ the series converges and at $x = 8$ the series diverges. What do we know about the following values?

4.(a). At $x = 3$ the series $\sum_{n=0}^{\infty} a_n (x-2)^n$ is:

(i) Convergent ✓
(ii) Divergent
(iii) We cannot determine its convergence/divergence with the given information.

4.(b). At $x = -4$ the series $\sum_{n=0}^{\infty} a_n (x-2)^n$ is:

(i) Convergent
(ii) Divergent
(iii) We cannot determine its convergence/divergence with the given information. ✓

4.(c). At $x = 9$ the series $\sum_{n=0}^{\infty} a_n (x-2)^n$ is:

(i) Convergent
(ii) Divergent ✓
(iii) We cannot determine its convergence/divergence with the given information.

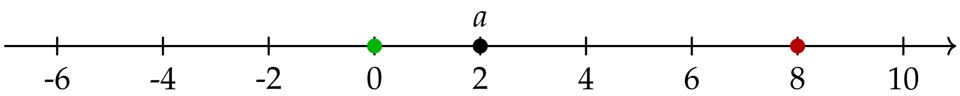
4.(d). The following series $\sum_{n=0}^{\infty} a_n$ is:

(i) Convergent ✓
(ii) Divergent
(iii) We cannot determine its convergence/divergence with the given information.

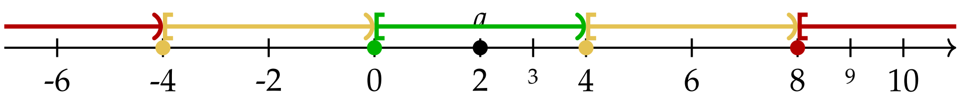
SOLUTION

In a little more detail:

The given power series is centered at $a = 2$, converges at $x = 0$, and diverges at $x = 8$. Graphically we have



Since the convergent point $x = 0$ is distance 2 from the center $a = 2$, the radius of convergence is **at least** 2. Similarly, since the divergent point $x = 8$ is distance 6 from the center $a = 2$, the radius of convergence is **at most** 6. Below we have the green interval $[0, 4)$ indicating points of guaranteed convergence, the red intervals $(-\infty, -4) \cup [8, \infty)$ indicating points of guaranteed divergence, and the yellow intervals indicating points of uncertainty, where we cannot determine convergence or divergence with the given information. Note that the points $x = -4, 4$ are in the yellow interval.



(a) $x = 3$ is in the green interval so the series converges there.
(b) $x = -4$ is in the yellow interval so we cannot determine convergence.
(c) $x = 9$ is in the red interval so the series is divergent there.
(d) Observe that $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_n(1)^n = \sum_{n=0}^{\infty} a_n(3-2)^n$. This means we are looking for convergence of the point $x = 3$. Clearly $x = 3$ is in the green interval so the series converges there.

5
12 points

5.(a). Write the definition for the nth degree Taylor polynomial of a function $f(x)$ centered at $x = a$.

5.(b). Find the second degree Taylor polynomial for $f(x) = \ln(\sec(x))$ centered at $\pi/4$.

SOLUTION

(a) This is just writing the formula

$$\boxed{T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots}$$

(b) The second degree Taylor polynomial has the formula

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

For the specific function $f(x) = \ln(\sec(x))$, we have to find $f(\pi/4)$, $f'(\pi/4)$, and $f''(\pi/4)$.

$$f(\pi/4) = \ln(\sec(\pi/4)) = \ln(2/\sqrt{2})$$
$$f'(x) = \frac{1}{\sec(x)} \cdot \sec(x)\tan(x) = \tan(x)$$
$$f'(\pi/4) = \tan(\pi/4) = 1$$
$$f''(x) = \sec^2(x)$$
$$f''(\pi/4) = \sec^2(\pi/4) = (2/\sqrt{2})^2 = 2$$

Hence the Taylor polynomial is

$$T_2(x) = \ln(2/\sqrt{2}) + 1(x-\pi/4) + \frac{2}{2!}(x-\pi/4)^2 = \boxed{\ln(2/\sqrt{2}) + (x-\pi/4) + (x-\pi/4)^2}$$

6
12 points

6.(a). Express the function $f(x) = \ln(1+x^3)$ as a power series centered about $x = 0$.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{3n}}{n}$$

6.(b). Express the definite integral $\int_0^1 \ln(1+x^3)\,dx$ as an infinite series.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(3n+1)}$$

SOLUTION

(a) The Maclaurin series for $\ln(1+x)$ is

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$$

Substituting $x \mapsto x^3$,

$$\ln(1+x^3) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x^3)^n}{n} = \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{3n}}{n}}$$

(b) Using the series from part (a) and applying term-by-term integration,

$$\begin{aligned}\int_0^1 \ln(1+x^3)\,dx = \int_0^1 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{3n}}{n}\,dx &= \sum_{n=1}^{\infty} \int_0^1 \frac{(-1)^{n-1}x^{3n}}{n}\,dx \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_0^1 x^{3n}\,dx \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left[\frac{x^{3n+1}}{3n+1}\right]_0^1 \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{1}{3n+1} \\ &= \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(3n+1)}}\end{aligned}$$

7
12 points

7.(a). Fill in the blanks to complete the statement of **Taylor's Inequality:**

If $|f^{(n+1)}(x)|$ $\leq M$ on the interval between the center, a, and the point of approximation x, then the remainder, $R_n(x)$, of the nth degree Taylor polynomial $T_n(x)$, satisfies the inequality:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

7.(b). Use Taylor's inequality to determine the number of terms of the Maclaurin series for e^x that should be used to estimate the number e with an error less than 0.6. Clearly justify your choice of M. 3 or more terms

SOLUTION

The Maclaurin series for $f(x) = e^x$ is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Since we want to approximate $e = e^1 = f(1)$, x is equal to 1. Taylor's Inequality for the above Maclaurin series gives us

$$|R_n(1)| \leq \frac{M}{(n+1)!}|1-0|^{n+1} = \frac{M}{(n+1)!}$$

To find M, note that $f^{(n+1)}(x) = e^x$ for all positive integers n. Then

$$M \geq |f^{(n+1)}(1)| = |e^1| = e$$

Ironically, finding a good choice for M requires us to guess how big e can be. Nevertheless, we will choose $M = 3$ to avoid any circular arguments. Lastly we bound the Taylor's Inequality by our error margin of 0.6:

$$|R_n(1)| \leq \frac{3}{(n+1)!} < 0.6 = \frac{3}{5}$$

Solving the inequality, we get

$$\frac{3}{(n+1)!} < \frac{3}{5}$$
$$5 < (n+1)!$$

Since $5 < 6 = (2+1)!$, choice of $n \geq 2$ guarantees that the nth degree Taylor polynomial $T_n(1)$ approximates e to within our error margin of 0.6. Since $T_n(x)$ contains $n+1$ terms, we need **at least three terms** of the Maclaurin series.

8
8 points

8. Each of the following slope fields represents one of the following differential equations. Match each slope field to the corresponding dfferential equation.

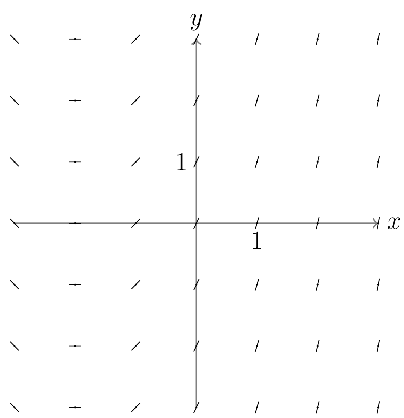
(a) $\dfrac{dy}{dx} = \dfrac{xy}{2}$ ________________ (II)

(b) $\dfrac{dy}{dx} = y - x - 2$ ________________ (III)

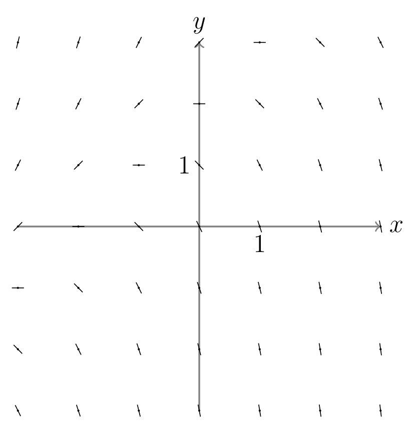
(c) $\dfrac{dy}{dx} = x + 2$ ________________ (I)

(d) $\dfrac{dy}{dx} = e^x$ ________________ (IV)

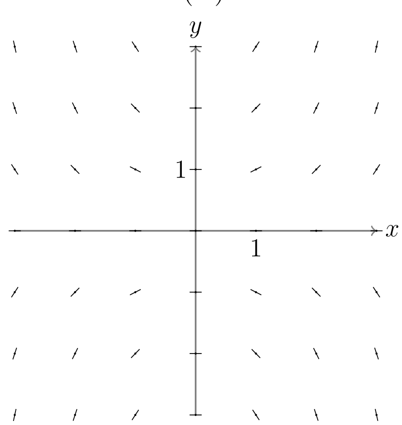
(I)



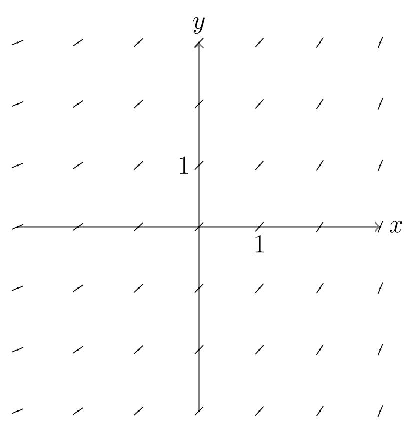
(III)



(II)



(IV)



SOLUTION

Here are more details on the solutions:

8.(a). When $x = 0$ or $y = 0$, $dy/dx = 0$ and we see that (II) has slope of 0 along the x and the y axis.

8.(b). Along the diagonal line $y = x$, $dy/dx = -2$ and we see that (III) has the slope fixed at -2.

8.(c). Along the vertical line $x = -2$, $dy/dx = 0$ and we see that (I) has a fixed slope of 0.

8.(d). Observe that the equation for dy/dx in (d) has no y. We see that the graphs (I) and (IV) have the same slope for a fixed value of x. Hence we are down to two choices (I) and (IV). But it can't be (I) since its slopes are 0 when x is a negative number. e^x is never 0 so we eliminate (I) as a possibility and we have (IV) as our answer.

9
6 points

9. Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

$$= \boxed{\pi/4}$$

SOLUTION

Recall that the Maclaurin series for arctan x is

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Plugging in $x = 1$, we get

$$\arctan 1 = \sum_{n=0}^{\infty} (-1)^n \frac{1^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

and $\boxed{\arctan 1 = \pi/4}$.

10
10 points

10. Assume we approximate the sum of the series

$$\sum_{n=1}^{\infty} \frac{2}{n^2}$$

by using the first 3 terms. Give an upper bound for the error involved in the approximation by using the Remainder Estimate for the Integral Test. $\boxed{R_3 \leq \frac{2}{3}}$

SOLUTION

Let $f(x) = \frac{2}{x^2}$. To apply the Remainder Estimate for the Integral Test, we first check the conditions necessary. Firstly, $\frac{2}{x^2}$ is differentiable since $f'(x) = \frac{-4}{x^3}$ and so it is continuous. $f(x)$ is also positive for any positive value of x and it is decreasing since it is a reciprocal of $x^2/2$, an increasing function. Lastly, we know that the series converges via the p-Test with $p = 2 > 1$.

We are using the first three terms so we want to estimate the error associated with s_3, the partial sum up to $n = 3$. Then the Integral Test gives us

$$\int_{3+1}^{\infty} f(x)\,dx \leq R_3 \leq \int_{3}^{\infty} f(x)\,dx$$

Since we are only interested in the upper bound, we compute the integral on the right side.

$$\begin{aligned} R_3 \leq \int_3^{\infty} \frac{2}{x^2}\,dx &= \lim_{t\to\infty} \int_3^t \frac{2}{x^2}\,dx \\ &= \lim_{t\to\infty} 2 \int_3^t \frac{1}{x^2}\,dx \\ &= \lim_{t\to\infty} 2 \left[-x^{-1}\right]_3^t \\ &= \lim_{t\to\infty} -2 \left[\frac{1}{t} - \frac{1}{3}\right] \\ &= -2\left[0 - \frac{1}{3}\right] = \frac{2}{3} \end{aligned}$$

Hence $\boxed{R_3 \leq \frac{2}{3}}$.