MIDTERM 2
CALCULUS 2

MATH 2300
FALL 2018

Monday, October 22,
5:15 PM to 6:45 PM.

Name

PRACTICE EXAM

SOLUTIONS

Please answer all of the questions, and show your work.
You must explain your answers to get credit.
You will be graded on the clarity of your exposition!

Date: October 22, 2018.

1
8 points

1. (4 points) Which of the following integrals gives the arc length of the function $f(x) = 3\cos(x)$ from $x = 0$ to $x = \pi/4$? You do not need to show any work for this problem.

(A). $\displaystyle\int_0^{\pi/4} \sqrt{1 + 9\cos^2(x)}\,dx$ (C). $\displaystyle\int_0^{\pi/4} \sqrt{1 - 3\sin(x)}\,dx$

(B). $\boxed{\displaystyle\int_0^{\pi/4} \sqrt{1 + 9\sin^2(x)}\,dx}$ (D). $\displaystyle\int_0^{\pi/4} \sqrt{1 - 9\sin^2(x)}\,dx$

2. (4 points) Which of the following integrals gives the average value of the function $g(x) = x\ln(x)$ between $x = 1$ and $x = 10$? You do not need to show any work for this problem.

(A). $\displaystyle\int_1^{10} x^2\ln(x)\,dx$ (C). $\boxed{\displaystyle\int_1^{10} \frac{x\ln(x)}{9}dx}$

(B). $\displaystyle\int_1^{10} \frac{x\ln(x)}{10}\,dx$ (D). $\displaystyle\int_1^{10} x\ln(x)\,dx$

3
8 points

3. Consider the region bounded by the x-axis, the y-axis, the line $x = 5$, and the curve $y = xe^{-x}$.

3.(a). (4 points) Which of the following expressions gives the x-coordinate of the center of mass of the region? You do not need to show any work for this problem.

(A). $\dfrac{\int_0^5 xe^{-x}\,dx}{\int_0^5 e^{-x}\,dx}$ (C). $\boxed{\dfrac{\int_0^5 x^2e^{-x}\,dx}{\int_0^5 xe^{-x}\,dx}}$ $\dfrac{1}{A}\displaystyle\int_a^b xf(x)\,dx$

(B). $\dfrac{\int_0^5 x^2e^{-2x}\,dx}{\int_0^5 2xe^{-x}\,dx}$ (D). $\dfrac{\int_0^5 xe^{-x}\,dx}{\int_0^5 x^2e^{-x}\,dx}$

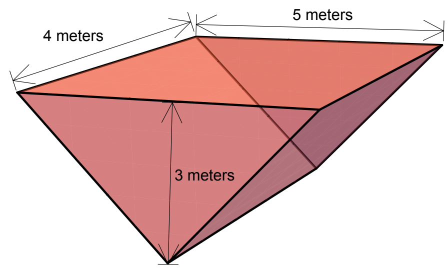
3.(b). (4 points) Which of the following epressions gives the y-coordinate of the center of mass of the region? You do not need to show any work for this problem.

(A). $\boxed{\dfrac{\int_0^5 x^2e^{-2x}\,dx}{\int_0^5 2xe^{-x}\,dx}}$ $\dfrac{1}{A}\displaystyle\int_a^b \frac{1}{2}f(x)^2\,dx$ (C). $\dfrac{\int_0^5 x^2e^{-x}\,dx}{\int_0^5 xe^{-x}\,dx}$

(B). $\dfrac{\int_0^5 x^2e^{-x}\,dx}{\int_0^5 xe^{-x}\,dx}$ (D). $\dfrac{\int_0^5 x^2e^{-x}\,dx}{2\int_0^5 xe^{-x}}\,dx$

4
10 points

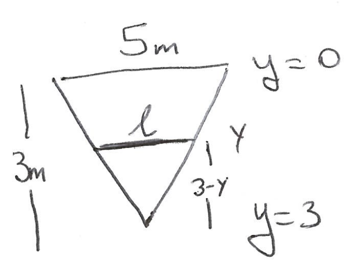
4. The solid pictured below shows a tank filled with water. Set up an integral that represents the work required to pump all of the water out of the the top of the tank. Use ρ for density of water and g for the acceleration due to gravity. Show any work that you use to arrive at your answer. *You do not need to evaluate the integral.*



SOLUTION

The solution is: $\boxed{\int_0^3 \frac{20}{3}\rho g(3-y)y\,dy}$

This is derived as follows. First we slice the tank into horizontal pieces, and estimate the volume of the slices. If we let y be the distance from the top of the tank,



then a 2-D horizontal slice at that distance from the top will be a rectangle with sides of length 4 and ℓ, where using similar triangles we have

$$\frac{\ell}{3-y} = \frac{5}{3}$$

In other words,

$$\ell = \frac{5}{3}(3 - y).$$

It follows that the area of the 2-D slice will be

$$\text{Area} = 4 \cdot \frac{5}{3}(3 - y) = \frac{20}{3}(3 - y)$$

We can then estimate the volume of the corresponding 3-D slice, as well as the mass of the 3-D slice, as follows:

$$\text{Volume of Slice} \approx \frac{20}{3}(3 - y)\Delta y$$

$$\text{Mass of Slice} \approx \rho \cdot \frac{20}{3}(3 - y)\Delta y$$

The force of gravity on that slice is then approximated as:

$$\text{Force on Slice} \approx g\rho\frac{20}{3}(3 - y)\Delta y$$

The work to move that slice to the top of the tank is then approximated as:

$$\text{Work for Slice} \approx y\rho g\frac{20}{3}(3 - y)\Delta y$$

Therefore, we conclude that the total work to pump all of the water out of the top of the tank is

$$\int_0^3 y\rho g\frac{20}{3}(3 - y)\, dy$$

as claimed.

5
20 points

5. Determine if each of the following series **converges conditionally, converges absolutely, or diverges**. Recall that a series is conditionally convergent if it converges but does not converge absolutely. Show all your work and carefully and fully justify your reasoning, including naming the convergence test.

5.(a). (10 points) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+2}}$

SOLUTION

The series $\boxed{\text{converges conditionally}}$

As the series is alternating, the computation

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+2}} = 0$$

and noting that $\left\{\frac{1}{\sqrt{n+2}}\right\}$ is decreasing (constant numerator and increasing denominator) shows that $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+2}}$ converges by the Alternating Series Test.

To show conditional convergence, we will show that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$ does not converge. Since $0 < \frac{1}{\sqrt{n+2}}, \frac{1}{\sqrt{n}}$ for all $n \geq 1$, we may use the Limit Comparison Test with the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$:

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+2}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+2}} = \lim_{n\to\infty} \sqrt{\frac{n}{n+2}} = \sqrt{\lim_{n\to\infty} \frac{n}{n+2}} = \sqrt{1} = 1 > 0.$$

Consequently, as $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges, so must $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$.

5.(b). (10 points) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

SOLUTION

The computation

$$\lim_{n\to\infty} \left| \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{(n+1)!}} \right| = \lim_{n\to\infty} \frac{2^n \cdot 2}{(n+1)\cdot n!} \cdot \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$$

shows that the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ $\boxed{\text{converges absolutely}}$ by the Ratio Test.

6
10 points

6. (10 points) Determine if the following series converges or diverges. Show all your work and carefully justify your reasoning. If the series converges, give the value of its sum.

$\sum_{n=1}^{\infty} \frac{3 \cdot 2^{n+1}}{5^{n-1}}$

SOLUTION

Manipulating the sum so that it is in the form of a geometric series:

$$\sum_{n=1}^{\infty} \frac{3 \cdot 2^{n+1}}{5^{n-1}} = \sum_{n=1}^{\infty} 3 \cdot 2^2 \cdot \frac{2^{n-1}}{5^{n-1}} = \sum_{n=1}^{\infty} 12 \cdot \frac{2^{n-1}}{5^{n-1}} = \sum_{n=1}^{\infty} 12 \cdot \left(\frac{2}{5}\right)^{n-1} = 12 \sum_{m=0}^{\infty} \left(\frac{2}{5}\right)^{m}$$

and using the formula

$$\sum_{m=0}^{\infty} r^m = \frac{1}{1-r}$$

for the sum of a geometric series when $-1 < r < 1$ yields, with $r = 2/5$, that the series converges to the value $\boxed{\frac{12}{1-2/5} = \frac{12}{3/5} = 20.}$

7
10 points

7. (10 points) The infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

converges to some real number s. We want to estimate s by computing the partial sum

$$s_N = \sum_{n=1}^{N} \frac{(-1)^n}{n^2} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \cdots + \frac{(-1)^N}{N^2}$$

How many terms should we use in the partial sum if we want the error to be less than or equal to $.0001 = 10^{-4}$?

SOLUTION

Because the series is alternating and the sequence $a_n = \dfrac{1}{n^2}$ is decreasing, an upper bound on the error can be computed by using the inequality

$$|s - s_N| \le a_{N+1} = \frac{1}{(N+1)^2}.$$

To get the right hand side of this inequality to be less than or equal to 10^{-4}, we need

$$\begin{aligned} \frac{1}{(N+1)^2} &\le 10^{-4} \\ \Rightarrow (N+1)^2 &\ge 10^4 \\ \Rightarrow N+1 &\ge 10^2 \\ \Rightarrow \boxed{N \ge 99} \end{aligned}$$

8
24 points

8. (24 points) For each of the following series, determine if it converges absolutely, converges conditionally or diverges. Recall that a series is conditionally convergent if it converges but does not converge absolutely. You do not need to show any work.

8.(a). $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

(A). converges absolutely (C). diverges
(B). converges conditionally

8.(b). $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$

(A). converges absolutely (C). diverges
(B). converges conditionally

8.(c). $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[5]{n^6}}$

(A). converges absolutely (C). diverges
(B). converges conditionally

8.(d). $\sum_{n=1}^{\infty} \frac{\arctan(n^2)}{n}$

(A). converges absolutely (C). diverges
(B). coverges conditionally

8.(e). $\sum_{n=1}^{\infty} \frac{1}{(-\pi)^n}$

(A). coverges absolutely (C). diverges
(B). converges conditionally

8.(f). $\sum_{n=1}^{\infty} \cos(\pi n)$

(A). converges absolutely (C). diverges
(B). converges conditionally

8.(g). $\sum_{n=1}^{\infty} \frac{(-n)^7}{2^n}$

(A). converges absolutely (C). diverges
(B). converges conditionally

8.(h). $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{5n+n^3}}{2n^2+3}$

(A). converges absolutely

(B). converges conditionally

(C). diverges

SOLUTION

Here are more details on the solutions:

8.(a). $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$. Converges by Alternating Series Test. Does not converge absolutely by direct comparison to $\sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series.

8.(b). $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$. Diverges since $\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = 1 \neq 0$ and the summand is always positive.

8.(c). $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[5]{n^6}}$. Converges absolutely by the p-test.

8.(d). $\sum_{n=1}^{\infty} \frac{\arctan(n^2)}{n}$. Diverges by direct comparison to the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$.

8.(e). $\sum_{n=1}^{\infty} \frac{1}{(-\pi)^n}$. Can be rewritten as a geometric series

$$\sum_{n=1}^{\infty} \frac{-1}{\pi} \cdot \left(\frac{-1}{\pi}\right)^{n-1}$$

with common ratio $r = -1/\pi$ between -1 and 1 and so converges absolutely.

8.(f). $\sum_{n=1}^{\infty} \cos(\pi n)$. The partial sums are $-1, 0, -1, 0, -1, 0, \ldots$ and so do not have a limit. The series diverges.

8.(g). $\sum_{n=1}^{\infty} \frac{(-n)^7}{2^n}$. Converges absolutely by the Ratio Test. The computation is:

$$\lim_{n\to\infty} \frac{\frac{-(n+1)^7}{2^{n+1}}}{\frac{-n^7}{2^n}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^7 \cdot \frac{2^n}{2\cdot 2^n} = \left(\lim_{n\to\infty} \frac{n+1}{n}\right)^7 \cdot \frac{1}{2} = 1^7 \cdot \frac{1}{2} = \frac{1}{2} < 1.$$

8.(h). $\sum_{n=1}^{\infty} \frac{\sqrt[3]{5n+n^3}}{2n^2+3}$. Can be seen to diverge by using the Limit Comparison Test on $\sum_{n=1}^{\infty} \frac{1}{n}$. The computation may be done as:

$$\lim_{n\to\infty} \frac{\frac{\sqrt[3]{5n+n^3}}{2n^2+3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sqrt[3]{5n+n^3}}{2n+\frac{3}{n}} = \lim_{n\to\infty} \frac{\frac{1}{n}\sqrt[3]{5n+n^3}}{2+\frac{3}{n^2}} = \lim_{n\to\infty} \frac{\sqrt[3]{\frac{5}{n^2}+1}}{2+\frac{3}{n^2}} = \frac{\sqrt[3]{1}}{2} = \frac{1}{2}.$$

9
10 points

9. (10 points) Select the best method for determining whether the following series converge or diverge. You do not need to show any work.

9.(a). $\sum \frac{10}{n \ln(n)}$

(A). alternating series test (C). integral test
(B). divergence test (D). ratio test

9.(b). $\sum \frac{3^{2n-1}}{(2n+1)!}$

(A). p-series (C). geometric series
(B). divergence test (D). ratio test

9.(c). $\sum \frac{5n^3}{\sqrt{n^7 - 4}}$

(A). alternating series test (C). ratio test
(B). divergence test (D). limit comparison test

9.(d). $\sum \frac{(-1)^n \sqrt{2n^2+3}}{n-3}$

(A). alternating series test (C). integral test
(B). divergence test (D). ratio test

9.(e). $\sum n^{-1/2}$

(A). alternating series test (C). p-series
(B). divergence test (D). ratio test